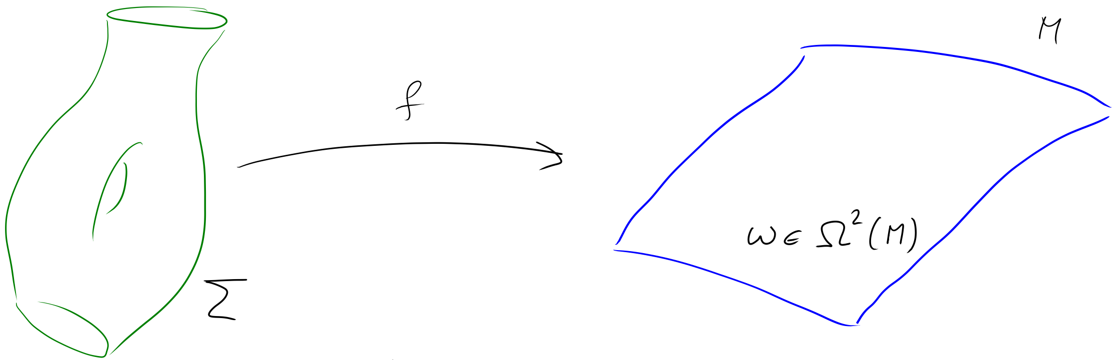
# Poisson-Lie T-duality and Courant algebroids

(or: the geometry of non-Abelian conservation laws)

# (baby) σ-models (De Donder – Weyl)




action functional $S(f) = \int_\Sigma f^*\omega$

$f$ is critical iff $f^*(i_v d\omega) = 0$ $\forall$ vect. field $v$ on $M$

Euler-Lagrange equation

$H := d\omega \in \Omega^3(M)^{\text{closed}}$ $f^*(i_v H) = 0$

- in place of $\omega$ we can use an arbitrary closed 3-form $H$

(for quantization: $H \rightsquigarrow$ differential character)

## (baby) Noether theorem

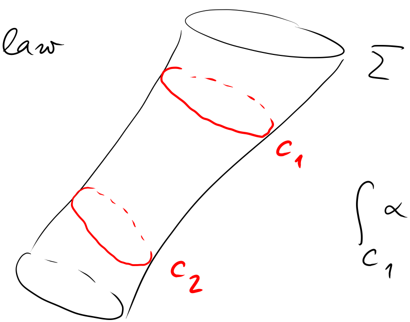
symmetry $\rightsquigarrow$ conservation law

if $v \in \mathfrak{X}(M)$ is a symmetry $\quad ( L_v \omega = 0 = i_v d\omega + d\, i_v \omega )$

and $f : \Sigma \longrightarrow M$ a critical map $\quad ( f^*(i_u d\omega) = 0 \quad \forall u )$

then $\quad \alpha := f^* i_v \omega$ is a closed 1-form on $\Sigma$

$\rightsquigarrow$ conservation law




$$\int_{c_1} \alpha = \int_{c_2} \alpha$$

# Non-abelian Noether theorem?

a flat connection on $\Sigma$ in place of a closed 1-form on $\Sigma$

simplest case: $\mathfrak{g}$ a Lie algebra $\rightsquigarrow$ we want $\alpha \in \Omega^1(\Sigma, \mathfrak{g})$

$$d\alpha + [\alpha, \alpha]/2 = 0 \qquad (*)$$

we need: $\mathfrak{g}^*$ a Lie algebra, action $\rho$ of $\mathfrak{g}^*$ on $M$

$$\langle \xi, \alpha \rangle := f^* i_{\rho(\xi)} \omega \qquad \forall \xi \in \mathfrak{g}^*$$

$(*)$ is satisfied for critical $f$'s: $L_{\rho(\xi)}\omega + (i_{\rho(\xi_{(1)})}\omega) \wedge (i_{\rho(\xi_{(2)})}\omega) = 0$

($\xi \mapsto \xi_{(1)} \otimes \xi_2$ dual to the Lie bracket on $\mathfrak{g}$)

+ integrability condition: $\mathfrak{g}$ is a Lie bialgebra

# Non-abelian Noether theorem 2

the meaning of $\quad L_{\rho(\xi)}\omega + (i_{\rho(\xi_{(1)})}\omega)\wedge(i_{\rho(\xi_{(2)})}\omega) = 0 \qquad (*)$

$C \subset (T\oplus T^*)M$ : the graph of $\omega : TM \to T^*M$

$G^*$ is a Poisson-Lie group, $\quad G^* \underset{\rho}{\circlearrowright} M$

lift to

$$G^* \circlearrowright (T\oplus T^*)M \qquad (v,\alpha) \longmapsto (v + \langle \rho_*(\pi),\alpha\rangle, \alpha)$$

$(*) \iff C$ is $G^*$-invariant

but: we want $H$ ($= d\omega$ locally) in place of $\omega$

a non-trivial $G$-bundle – and a duality!

# (baby) Noether theorem revisited

## or : $(T \oplus T^*)M$ strikes again

symmetry up to "total divergence":

$$(v,\beta) \in \Gamma((T \oplus T^*)M), \qquad L_v \omega + d\beta = 0 \qquad (*)$$

$$\alpha := f^*(i_v \omega + \beta) \in \Omega^1(\Sigma) \qquad f \text{ critical} \Rightarrow d\alpha = 0$$

meaning of $(*)$: $C$ ($=$ graph of $\omega: TM \to T^*M$) is invariant under a flow of $(T \oplus T^*)M$

the flow: $\left[(v,\beta), (\tilde{v},\tilde{\beta})\right] = \left([v,\tilde{v}], L_v \tilde{\beta} - i_{\tilde{v}} d\beta\right)$

the (standard) Courant bracket

# Courant algebroids

Definition: CA is: vector bundle $E \to M$,

$a : E \to TM$ (anchor map), inner product $\langle \cdot , \cdot \rangle$ on $E$,

a bracket $[\cdot,\cdot]$ on $\Gamma(E) \Leftrightarrow s \longmapsto Z_s$ ($s \in \Gamma(E)$, $Z_s \in \mathcal{X}(E)$, $a_* Z_s = a(s)$)

s.t. $Z_s$ is an infinitesimal automorphism & $[s,s] = \frac{1}{2} a^t d\langle s, s\rangle$

(here $a^t : T^*M \longrightarrow E^* \xrightarrow[\sim]{\langle,\rangle} E$)

examples: $M = $ point, $E = (\mathfrak{g}, \langle\,\rangle)$

$E$ is exact if $0 \longrightarrow T^*M \xrightarrow{a^t} E \xrightarrow{a} TM \longrightarrow 0$ is exact

classification of exact CAs: by $H^3(M, \mathbb{R})$

- if $H \in \Omega^3(M)^{\text{closed}}$ $\quad \langle (v,\beta), (\hat{v},\hat{\beta})\rangle = \hat{\beta}(v) + \beta(\hat{v})$

$$[(v,\beta),(\tilde{v},\tilde{\beta})] = \left([v,\tilde{v}],\ L_v\tilde{\beta} - i_{\tilde{v}}d\beta + H(v,\tilde{v},\cdot)\right)$$

# Curvature:

if $C \subset E$, $C^\perp = C$: $F_C : \Lambda^2 C \longrightarrow E/C \cong C^*$

$F_C(s_1, s_2) = [s_1, s_2] \mod C$

equivalently: $H_C : \Lambda^3 C \longrightarrow \mathbb{R}$, $H_C(s_1, s_2, s_3) = \langle [s_1, s_2], s_3 \rangle$

( $C$ is a Dirac structure if $H_C = 0$ )

if $E$ is exact and $C \subset E$ splits $0 \longrightarrow T^*M \to E \to TM \to 0$

then $H_C \in \Omega^3(M)$ closed $\quad (C \cong TM)$

# $\sigma$-models and exact CAs

setup: exact CA $E \to M$ + splitting $C \subset E$

(equivalently: closed 3-form $H = H_C$ on $M$)

$TM \cong C$

**Euler-Lagrange = zero curvature:**

$f$ is critical ($f^* i_v H = 0 \ \forall v$) $\iff$ $f^* F_C = 0$

$\in \Omega^2(\Sigma, f^* E/C)$

Noether: if $s \in \Gamma(E)$ s.t. $Z_s$ preserves $C$ then

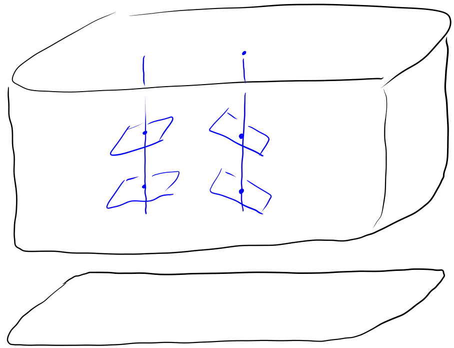
$f^* \langle s, \cdot \rangle \in \Omega^1(\Sigma)$ is closed for critical $f$

# non-Abelian Noether, or $\sigma$-models with Poisson-Lie T-duality

$P$

$\downarrow$

$M = P/G$



principal $G$-bundle

connection $A$

$H \in \Omega^3(M)^{\text{closed}}$, or $C \subset E$

we want: $f^*A$ flat for $f$ critical

i.e. $f^* F_C = 0 \Rightarrow f^* F_A = 0$

what is needed: Lie group $D \supset G$, invariant $\langle \cdot , \cdot \rangle$ on $\mathfrak{d} = \mathrm{Lie}\, D$

s.t. $\mathfrak{g}^\perp = \mathfrak{g}$ (Manin pair) + action $D \circlearrowright P$

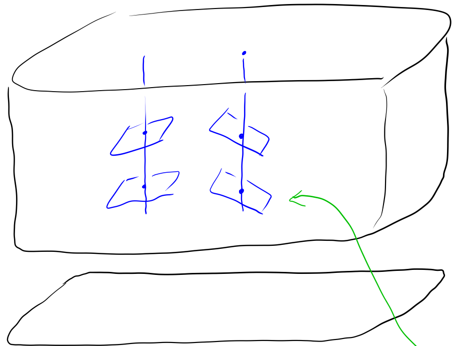
duality ...

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ M' = P/G' & & M = P/G \end{tikzcd}$$

$$D \circlearrowright P$$

$$\downarrow$$

$$M = P/G$$



$E_P \to P$ $D$-equivariant exact CA

$E_M \to M$ reduction of $E_P$

$C_P \subset E_P$ $D$-invariant

horizontal spaces of the connection $= a(C_P)$ $(\cong C_P)$

$C_M \subset E_M$ comes from $C_P$

$\rightsquigarrow$ closed 3-form on $M = P/G$

# Equivariant Courant algebroids

$E \to X$ a C.A., $\rho : \mathfrak{d} \to \Gamma(E)$ s.t. $[\rho(\xi), \rho(\eta)] = \rho([\xi, \eta])$

$\langle \rho(\xi), \rho(\eta) \rangle = \langle \xi, \eta \rangle$ $(\xi, \eta \in \mathfrak{d})$ $\quad \mathfrak{d} \overset{\xi}{\circlearrowright} E, X$

**Classification of exact $E$'s:**

if $D \circlearrowright X$ is free & proper ... $X \longrightarrow X/D$ princ. $D$-bundle

equiv. exact $E \to X$ exists iff the 1st Pontryagin class

$[\langle F, F \rangle] \in H^4(X/D, \mathbb{R})$ vanishes (affine space/over $H^3(X/D, \mathbb{R})$)

**we need:** $D$-equivariant exact CA $\quad E_P \to P$

# Reduction and curvature

$E_P \to P \quad \rightsquigarrow$ how to get an exact $E_M \to M \quad (M = P/G)$ ?

reduction by $G$: $\quad (E_M)_x = \rho_y(\mathfrak{g})^\perp / \rho(\mathfrak{g})_y \cong \rho_y(\partial)^\perp \subset E_P \quad (\rho_y : \partial \to E_y)$

$$y \in P \longmapsto x \in M = P/G$$

i.e. $\quad \rho(\partial)^\perp = p^* E_M \qquad (p : P \to M)$

How to get $C_M \subset E_M$ (and thus to $H_M \in \Omega^3(M)^{\text{closed}}$) ?

we need $D$-invariant $C_P \subset \rho(\partial)^\perp \subset E_P$

(s.t. $C_P^\perp = C_P$ in $\rho(\partial)^\perp$)

the curvature of $F_{C_P}$ of $C_P$ $([s_1, s_2] \bmod C_P)$ is the pullback of the curvature of $C_M$

$C_P \rightsquigarrow a(C_P) \subset TP$ $\quad$ $F_{C_P}$ projects to the curvature of $A$

a connection $A$ on $G \circlearrowright P \longrightarrow M = P/G$

Non-abelian Noether: If $f : \Sigma \longrightarrow M$ is $H_M$-critical ($f^* i_v H_M = 0$) then $f^* A$ is a flat connection on $f^* P \to \Sigma$

# Poisson – Lie T–duality

(C. Klimčík, P. Š. 1995 ; this formulation 1998)

$\tilde{f} : \Sigma \to P$ is critical if $T\tilde{f} : T\Sigma \to a(C_p) \cong C_p$

and if $\tilde{f}^* F_{C_p} = 0$

non-Abelian Noether $\Rightarrow$

$f : \Sigma \to M$ critical $\Leftrightarrow$ $\exists\, \tilde{f} : \Sigma \to P$ critical s.t. $f = p \circ \tilde{f}$

($\tilde{f}$ unique up to action of $G$)

## Poisson – Lie T–duality

if $G, G' \subset D$ $\to$ equivalence between critical

$f : \Sigma \to M = P/G$ and $f' : \Sigma \to M' = P/G'$

(both equivalent to $\tilde{f} : \Sigma \to P$)

# Some random things that were left out

- "D-branes": given by Dirac structures $L \subset E$

  $(L^\perp = L, \quad F_L = 0)$

- global point of view, on gerbes:

  multiplicative gerbe over $D$, acting on a gerbe over $P$

- quantization $\rightsquigarrow$ quantum groups?

  $\exists$ discrete version (Kramers-Wannier duality)

  with finite-dim. Hopf algebra

  – on the boundary of a 3dim TQFT