SUPERSYMMETRIC GAUGE THEORIES ON ALE SPACES

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Quantization, Geometry and Mathematical Physics
Bayrischzell 2015

Instantons and framed sheaves

- Moduli spaces of finite-energy charge-n instantons (ASD connections $F = -*F$) in $U(r)$ gauge theory on $X = \mathbb{R}^4 \simeq \mathbb{C}^2$
  $\subset$ moduli spaces $\mathcal{M}_{r,n}$ of rank r framed torsion-free sheaves $E\big|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus r}$ on $\mathbb{P}^2 = \mathbb{C}^2 \cup \ell_\infty$ with $c_2(E) = n$ (Donaldson '84)

- $\mathcal{M}_{r,n}$ smooth, quasi-projective, $\dim(\mathcal{M}_{r,n}) = 2\,r\,n$
  $\mathcal{M}_{1,n} \simeq \mathrm{Hilb}^n(\mathbb{C}^2)$

- $\mathcal{N} = 2$ gauge theories in 4 dimensions of class $\mathcal{S}$ arise by compactification from 6 dimensions in 2 ways:
  - Reduction of $\mathcal{N} = 1$ gauge theory on flat X-bundle $M \longrightarrow T^2$ with flat $(\mathbb{C}^*)^2$-bundle on T^2 (Ω-deformation)
  - Compactification of $(2,0)$ superconformal theory on $M_6 = \Sigma \times X$, Seiberg–Witten curve branched cover of Σ (AGT and BPS/CFT correspondences)

Instanton counting

- Instanton part of Nekrasov's partition function for pure $\mathcal{N} = 2$ $U(r)$ gauge theory:

$$\mathcal{Z}_{\mathbb{R}^4} = \sum_{n\geq 0} q^n \int_{\mathcal{M}_{r,n}} 1 = \sum_{n\geq 0} q^n \left\langle \left[\mathcal{M}_{r,n}\right]_T , \left[\mathcal{M}_{r,n}\right]_T \right\rangle$$

$T = (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^r, \ (\mathbb{C}^*)^r \subset GL(r, \mathbb{C})$

- $\mathcal{M}^T_{r,n}$ parameterized by Young diagrams $\boldsymbol{Y} = (Y_1, \dots, Y_r)$, localization theorem in T-equivariant cohomology gives combinatorial expression in q and in equivariant parameters $(\varepsilon_1, \varepsilon_2, \vec{a})$

- Extension to $\mathcal{N} = 2$ quiver gauge theories on $\mathbb{R}^4$: Matter fields represent Euler classes of universal vector bundles on $\mathcal{M}_{r,n}$

AGT duality

- Highest weight rep of $\mathcal{W}(\mathfrak{gl}(r))$ on $\mathbb{H}_r = \bigoplus_{n \geq 0} H_T^*(\mathcal{M}_{r,n})_{\mathrm{loc}}$ with Whittaker vector (Gaiotto state) $\sum_{n \geq 0} \left[\mathcal{M}_{r,n}\right]_T$
  (Schiffmann & Vasserot '12; Maulik & Okounkov '12)

- Generalises Nakajima–Vasserot construction of reps of Heisenberg algebra $\mathfrak{h}$ for $r = 1$

- $\mathcal{N} = 2$ quiver gauge theory partition functions agree with conformal blocks of A_{r-1} Toda CFTs
  (Alday, Gaiotto & Tachikawa '09; Wyllard '09; ...)

- Geometrically defined $\mathcal{W}(\mathfrak{gl}(r))$-primary vertex operator on $\mathbb{H}_r$? Construction for $r = 1$ in terms of Nakajima correspondences
  (Carlsson & Okounkov '08)

Quantum integrability

- Ω-deformed gauge theory $\equiv$ quantized version of integrable system associated to $\mathcal{N} = 2$ gauge theory

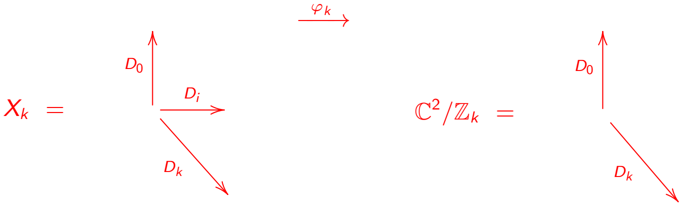
- $\lim_{\varepsilon_1,\varepsilon_2\to 0} -\varepsilon_1\,\varepsilon_2 \log \mathcal{Z}_{\mathbb{R}^4} =$ Seiberg–Witten prepotential (Nekrasov '02), captures geometry of Seiberg–Witten spectral curve of Hitchin system on Σ (Martinec & Warner '95; Donagi & Witten '95)

- Vector multiplet scalars $\boldsymbol{I}_p = \mathrm{Tr}\,\phi^p$ commuting Hamiltonians, realised geometrically on $\mathbb{H}_r$ as $\boldsymbol{I}_p = (c_{p-1})_T(\boldsymbol{V}_r)$ of "natural vector bundle" $\boldsymbol{V}_r$ on $\coprod_{n\geq 0} \mathcal{M}_{r,n}$

- $\mathbb{H}_r \simeq$ symmetric functions:
  Fixed point basis $[\boldsymbol{Y}]_T \Longleftrightarrow$ generalized Jack symmetric functions $J_{\boldsymbol{Y}}$, eigenfunctions of quantum Calogero–Moser–Sutherland model
  (Alba, Fateev, Litvinov & Tarnopolsky '10; Smirnov '14)

A_{k-1} ALE spaces

- $\varphi_k : X_k \longrightarrow \mathbb{C}^2/\mathbb{Z}_k$ minimal resolution of A_{k-1} toric singularity, $\omega \cdot (z, w) = (\omega\, z, \omega^{-1}\, w)$, $\omega^k = 1$
- ALE space $X \simeq_{\mathrm{diff}} X_k$ with ALE Kähler metric, i.e. Euclidean on $X \setminus K$
- $\mathbb{C}^2/\mathbb{Z}_k = \{x\, y - z^k = 0 \text{ in } \mathbb{C}^3\} \xrightarrow{\text{deform}} \{x\, y - z^k + \mathrm{Poly}_{<k} = 0\}$



Smooth $(\mathbb{C}^*)^2$-fixed points $p_1, \ldots, p_k \longrightarrow$ singular 0 order k

Instantons on ALE spaces

- $U(r)$ instantons on X_k asymptotically flat: Approach flat connections on $X_k \setminus K$ with holonomy ρ in $\pi_1(X_k \setminus K) \simeq \mathbb{Z}_k$, so correspond to sum of irreps $\rho_0, \rho_1, \dots, \rho_{k-1} : \mathbb{Z}_k \longrightarrow U(1)$
- Tautological line bundles: $\mathcal{R}_0 = \mathcal{O}_{X_k}, \mathcal{R}_1, \dots, \mathcal{R}_{k-1}$ with $\mathcal{R}_i|_{X_k \setminus K} \Longleftrightarrow \rho_i$ (McKay correspondence)

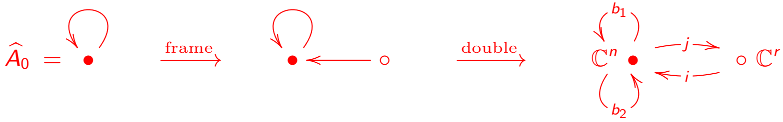
$$\int_{X_k} c_1(\mathcal{R}_i) \cdot c_1(\mathcal{R}_j) = (C^{-1})^{ij}$$

  Cartan matrix of A_{k-1} Dynkin diagram (Kronheimer & Nakajima '90)

- Kronheimer–Nakajima construction: Moduli spaces $\mathcal{M}(\vec{v}, \vec{w})$ of $U(r)$ instantons on X_k with $\vec{v}, \vec{w} \in \mathbb{N}^k$ parameterizing $\mathrm{ch}(E)$, $\rho = \rho_{\vec{w}} = \bigoplus_{i=0}^{k-1} w_i \, \rho_i$

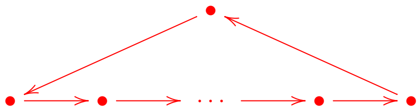
Quiver varieties

- ADHM data: $\mathcal{M}_{r,n} \ = \ \mu^{-1}(0)^{\xi} \big/ GL(n,\mathbb{C}), \ \ \mu \ = \ [b_1,b_2] + i\,j$



- $\widehat{A}_{k-1}$ Nakajima quiver varieties $\mathcal{M}_{\xi}(\vec{v},\vec{w}\,)$:
  Smooth, quasi-projective, carry rep of $\widehat{\mathfrak{gl}}(k)_r$ via geometric Hecke correspondences (Nakajima '94)

$\widehat{A}_{k-1} \ =$



- (Semi)stable $\xi \ \in \ \mathbb{R}^k = H^2(X_k;\mathbb{R})$ subdivided into chambers with isomorphic $\mathcal{M}_{\xi}(\vec{v},\vec{w}\,)$

AGT on ALE

- $\mathcal{N} = 4$ gauge theory partition function (Euler chars of instanton moduli spaces) computes characters of $\widehat{\mathfrak{gl}}(k)_r$ (Vafa & Witten '94)
- For suitable $\xi_0 \in C_0$, $\mathcal{M}_{\xi_0}(\vec{v}, \vec{w}) \subset \{\mathbb{Z}_k$-equivariant framed torsion free sheaves $E\big|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty} \otimes \rho_{\vec{w}}$ on $\mathbb{P}^2\}$; enables computation of $\mathcal{N} = 2$ gauge theory partition functions over $\mathcal{M}_{\xi_0}(\vec{v}, \vec{w})$, T-fixed points are $\vec{w}$-coloured Young diagrams $\boldsymbol{Y}$ with k colours (Fucito, Morales & Poghossian '04; Fujii & Minabe '05)
- Equivariant cohomology carry reps of coset construction: (Belavin & Feigin '11; Nishioka & Tachikawa '11; Belavin, Belavin & Bershtein '11; Wyllard '11; ...)

$$\mathcal{A}(r,k) = \mathfrak{h} \oplus \widehat{\mathfrak{sl}}(k)_r \oplus \frac{\widehat{\mathfrak{sl}}(r)_k \oplus \widehat{\mathfrak{sl}}(r)_{n-k}}{\widehat{\mathfrak{sl}}(r)_n}$$

- Vertex algebra realization of $\mathfrak{h} \oplus \widehat{\mathfrak{sl}}(k)_1$ for $r = 1$ on equivariant cohomology of quiver varieties with $\vec{w} = (1, 0, \ldots, 0)$ (Nagao '07)

AGT on other quiver varieties?

- For suitable $\xi_\infty \in C_\infty$, $\mathcal{M}_{\xi_0}((1,\ldots,1),(1,0,\ldots,0)) \simeq X_k$
  (Kronheimer '89)
- $\mathcal{M}_{\xi_0}(\vec{v},\vec{w}) \not\approx \mathcal{M}_{\xi_\infty}(\vec{v},\vec{w})$; $\mathcal{M}_{\xi_0}(\vec{v},\vec{w})^T \not\approx \mathcal{M}_{\xi_\infty}(\vec{v},\vec{w})^T$
- Conjectural factorization: $\mathcal{Z}_{X_k} =$ product of k copies of $\mathcal{Z}_{\mathbb{R}^4}$, equivariant parameters of torus action on affine patches of X_k, leg-factor depending on $\varphi_k^{-1}(0)$, generalizing Nakajima–Yoshioka blowup formulas
  (Bonelli, Maruyoshi & Tanzini '11; Bonelli, Maruyoshi, Tanzini & Yagi '12)
- AGT relations for $k = 2$ (Bonelli, Maruyoshi & Tanzini '11)

Orbifold compactification

(Bruzzo, Pedrini, Sala & RS '13)

- $\mathcal{X}_k = X_k \cup \mathcal{D}_\infty$ smooth projective toric orbifold
  $\mathcal{D}_\infty = \mathbb{Z}_k$-gerbe over a football
- $\mathcal{D}_\infty \simeq \left[\frac{\mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^* \times \mathbb{Z}_k}\right]$ as toric DM stack with DM torus $\mathbb{C}^* \times B\mathbb{Z}_k$
  Hence $\mathrm{Pic}(\mathcal{D}_\infty) \simeq \mathbb{Z} \oplus \mathbb{Z}_k$ generated by $\mathcal{L}_1, \mathcal{L}_2$
- $\pi_1(\mathcal{D}_\infty) \simeq \mathbb{Z}_k$; $\mathcal{O}_{\mathcal{D}_\infty}(i) = \mathcal{L}_2^{\otimes i}$ endowed with unitary flat connection associated to $\rho_i : \mathbb{Z}_k \longrightarrow U(1)$ (Eyssidieux & Sala '14)
- $\mathrm{Pic}(\mathcal{X}_k)$ generated by $\mathcal{O}_{\mathcal{X}_k}(\mathcal{D}_\infty)$ and $\mathcal{R}_1, \dots, \mathcal{R}_{k-1}$
  $\mathcal{R}_i|_{X_k}$ = tautological line bundles, $\mathcal{R}_i|_{\mathcal{D}_\infty} \simeq \mathcal{O}_{\mathcal{D}_\infty}(i)$

Framed sheaves

- $\mathcal{F}_\infty^{\vec{w}} \ := \ \bigoplus_{i=0}^{k-1} \mathcal{O}_{\mathcal{D}_\infty}(i)^{\oplus w_i} \ , \qquad \vec{w} \ = \ (w_0, w_1, \dots, w_{k-1}) \ \in \ \mathbb{N}^k$

- $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ = fine moduli space parameterizing framed sheaves $\mathcal{E}\big|_{\mathcal{D}_\infty} \ \simeq \ \mathcal{F}_\infty^{\vec{w}}$ on $\mathcal{X}_k$ with

  $$r \ = \ \mathrm{rk}(\mathcal{E}) \ = \ \sum_{i=0}^{k-1} w_i \ , \qquad c_1(\mathcal{E}) \ = \ \sum_{i=1}^{k-1} u_i \, c_1(\mathcal{R}_i) \ ,$$

  $$\Delta \ = \ \Delta(\mathcal{E}) \ = \ \int_{\mathcal{X}_k} \Big(c_2(\mathcal{E}) - \frac{r-1}{r} \, c_1(\mathcal{E})^2 \Big)$$ (Bruzzo & Sala '13)

- $\mathcal{U}_{\vec{w}} \ := \ \vec{u} \ \in \ \mathbb{Z}^{k-1}$ corresponds to sum of weight $\sum_{i=1}^{k-1} w_i \, \omega_i$ and $\gamma_{\vec{u}} \ \in \ \mathcal{Q} \ = \ A_{k-1}$ root lattice

Moduli spaces

- $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ smooth, quasi-projective

- $$\dim\left(\mathcal{M}_{\vec{u},\Delta,\vec{w}}\right) = 2\,r\,\Delta - \frac{1}{2}\sum_{j=1}^{k-1}\left(C^{-1}\right)^{jj}\,\vec{w}\cdot\vec{w}(j)$$
  $$\vec{w}(j) = \left(w_j,\dots,w_{k-1},w_0,w_1,\dots,w_{j-1}\right)$$

- Contains moduli space of $U(r)$ instantons on X_k with $c_1 = \sum_i u_i\,c_1\big(\mathcal{R}_i|_{X_k}\big)$, holonomy at infinity $\rho = \bigoplus_j w_j\,\rho_j$
  (Eyssidieux & Sala '14)

- Birational morphism $\mathcal{M}_{\vec{u},\Delta,\vec{w}} \longrightarrow \mathcal{M}_{\xi_\infty}(\vec{v},\vec{w}\,)$
  for some $\vec{v} \in \mathbb{N}^k$

- Isomorphism for rank one: $\mathcal{M}_{\vec{u},n,\vec{w}} \simeq \mathrm{Hilb}^n(X_k)$ $\forall\vec{u},\vec{w}$
  Hence is a Nakajima quiver variety (Kuznetsov '01)

Ext-bundles (hypermultiplets)

- $\boldsymbol{\mathcal{E}} \ = \ $ universal sheaf on $\mathcal{M}_{\vec{u},\Delta,\vec{w}} \times \mathcal{X}_k$, $\boldsymbol{\mathcal{E}}\big|_{\{\mathcal{E}\}\times\mathcal{X}_k} \ = \ \mathcal{E}$

- Natural bundle on $\mathcal{M}_{\vec{u},\Delta,\vec{w}}$ (fundamental matter):

$$\boldsymbol{V} \ = \ R^1 p_{1*}\big(\boldsymbol{\mathcal{E}} \otimes p_2^*(\mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty))\big)$$

- $\boldsymbol{V}$ locally free, $\mathrm{rk}(\boldsymbol{V}) \ = \ \Delta + \frac{1}{2r}\,\vec{u}\cdot C^{-1}\vec{u} - \frac{1}{2}\sum_{j=1}^{k-1}\big(C^{-1}\big)^{jj}\,w_j$

- More generally, Carlsson–Okounkov type Ext-bundles (bifundamental hypermultiplets) $\boldsymbol{E}$ in $K\big(\mathcal{M}_{\vec{u},\Delta,\vec{w}} \times \mathcal{M}_{\vec{u}\,',\Delta',\vec{w}\,'}\big)$ with

$$\boldsymbol{E}\big|_{(\mathcal{E},\mathcal{E}'\,)} \ = \ \mathrm{Ext}^1\left(\mathcal{E}\,,\,\mathcal{E}'\otimes\mathcal{O}_{\mathcal{X}_k}(-\mathcal{D}_\infty)\right)$$

Torus fixed points $\mathcal{M}^{T}_{\vec{u},\Delta,\vec{w}}$

- Parameterized by $\vec{\boldsymbol{Y}} = (\vec{Y}_1, \ldots, \vec{Y}_r)$, $\vec{Y}_\alpha = \{Y^i_\alpha\}_{i=1,\ldots,k}$, $\vec{\boldsymbol{u}} = (\vec{u}_1, \ldots, \vec{u}_r)$

- $\vec{u} = \sum_{\alpha=1}^{r} \vec{u}_\alpha$, $\vec{u}_\alpha \Longleftrightarrow \gamma_{\vec{u}_\alpha} + \omega_i$

  for $i = 1, \ldots, k-1$, $\sum_{j=0}^{i-1} w_j < \alpha \leq \sum_{j=0}^{i} w_j$

- $\Delta = \sum_{\alpha=1}^{r} |\vec{Y}_\alpha| + \frac{1}{2} \sum_{\alpha=1}^{r} \vec{u}_\alpha \cdot C^{-1} \vec{u}_\alpha - \frac{1}{2r} \sum_{\alpha,\beta=1}^{r} \vec{u}_\alpha \cdot C^{-1} \vec{u}_\beta$

Representations

(Pedrini, Sala & RS '14)

- For $r = 1$: $\mathbb{H}^{\infty}_{\vec{u},j} := \bigoplus_{n\geq 0} H^*_T\big(\mathcal{M}_{\vec{u},n,\vec{e}_j}\big)_{\mathrm{loc}} \quad (j = 0,1,\ldots,k-1)$
$$\simeq \mathbb{C}(\varepsilon_1,\varepsilon_2)[\mathcal{Q}] \otimes \bigoplus_{n\geq 0} H^*_T\big(\mathrm{Hilb}^n(X_k)\big)_{\mathrm{loc}}$$

- $\mathbb{H}^{\infty}_{j} := \bigoplus_{\vec{u}\in\mathcal{U}_{\vec{e}_j}} \mathbb{H}^{\infty}_{\vec{u},j} =$ j-th fundamental rep of $\widehat{\mathfrak{gl}}(k)_1$
$\mathbb{H}^{\infty}_{\vec{u},j} =$ weight spaces with weights $\gamma_{\vec{u}} + \omega_j$
(use Nakajima correspondences for $\mathfrak{h}\oplus\mathfrak{h}_{\mathcal{Q}}$ on $\mathrm{Hilb}^n(X_k)$,
Frenkel–Kac construction on $\mathfrak{h}_{\mathcal{Q}}$)

- $\mathbb{H}^{\infty}_{\vec{u},j} =$ highest weight rep of Virasoro algebra with conformal dimension $\Delta_{\vec{u}} = \frac{1}{2}\,\vec{u}\cdot C^{-1}\vec{u}$

- $\sum_{n,\vec{u}} \big[\mathcal{M}_{\vec{u},n,\vec{e}_j}\big]_T =$ Whittaker vector
(eigenvector for >0 operators $\mathfrak{h}\oplus\mathfrak{h}_{\mathcal{Q}} \subset \widehat{\mathfrak{gl}}(k)$)

Quantum integrability

- $\boldsymbol{V}_j = $ natural vector bundle on $\coprod_{\vec{u}\in\mathcal{U}_{\vec{e}_j}} \coprod_{n\geq 0} \mathcal{M}_{\vec{u},n,\vec{e}_j}$
- $\boldsymbol{I}_1 = \mathrm{rk}(\boldsymbol{V}_j)$, $\boldsymbol{I}_{p\geq 2} = (c_{p-1})_T(\boldsymbol{V}_j)$ infinite system of commuting multiplication operators diagonalized in fixed point basis $[\vec{Y},\vec{u}]_T$
- $\boldsymbol{I}_1 = $ Virasoro operator L_0 for $\widehat{\mathfrak{gl}}(k)$

  $\boldsymbol{I}_2 = $ sum of k non-interacting quantum Calogero–Sutherland Hamiltonians with prescribed couplings
- $\mathbb{H}_j^\infty \simeq$ symmetric functions:
  $[\vec{Y},\vec{u}]_T \Longleftrightarrow J_{Y^1}\big(\beta^{(1)}\big) \otimes \cdots \otimes J_{Y^k}\big(\beta^{(k)}\big) \otimes (\gamma_{\vec{u}} + \omega_j)$
  $J_{Y^i}\big(\beta^{(i)}\big) = $ Jack symmetric function with parameter
  $\beta^{(i)} \in \mathbb{C}(\varepsilon_1,\varepsilon_2)$ given by fixed point $p_i \in X_k^T$ and
  $H_T^*(\mathrm{pt}) = \mathbb{C}[\varepsilon_1,\varepsilon_2]$

Comparison with $\mathcal{M}_{\xi_0}(\vec{v}, \vec{e}_j)$

- $\mathbb{H}_j^0 \;:=\; \bigoplus_{\vec{v}\in\mathbb{N}^k} H_T^*\big(\mathcal{M}_{\xi_0}(\vec{v}, \vec{e}_j)\big)_{\rm loc} \;=\;$ j-th fundamental rep of $\widehat{\mathfrak{gl}}(k)_1$
  (Nakajima '94)

- $\mathbb{H}_j^0 \;\simeq\;$ symmetric functions:
  $[Y]_T \Longleftrightarrow$ rank k Uglov symmetric function $U_Y(\beta; k)$ for coloured Young diagram Y, with parameter $\beta \in \mathbb{C}(\varepsilon_1, \varepsilon_2)$ given by unique fixed point $0 \in \big(\mathbb{C}^2/\mathbb{Z}_k\big)^T$; eigenfunctions of "spin" Calogero–Sutherland models

- Look for relations between:
  - Different bases and quantum integrable systems
  - Different representations of $\widehat{\mathfrak{gl}}(k)_1$
  - $\mathbb{H}_j^\infty$ and $\mathbb{H}_j^0$ geometrically