Courant algebroid connections and Einstein-Hilbert type actions

Jan Vysoký

Czech Technical University in Prague - Jacobs University Bremen

May 29 - June 1 2015, Bayrischzell Worskhop

Motivation

- In our recent work, generalized geometry proved to be useful tool to naturally describe the objects in open string low energy effective actions.

$$S \propto \int d^n x \sqrt{\det(g + B + F)}.$$

- Low energy effective closed string action gives Einstein-Hilbert type action:

$$S \propto \int d^n x \sqrt{g} e^{-2\Phi} (\mathcal{R}(g) - \frac{1}{12} dB^2 + 4\nabla\Phi^2).$$

- Is there a way how to obtain actions of this type from generalized geometry objects?
- Can we use this tools to find higher order corrections to effective actions?
- Collaboration with Branislav Jurčo

Generalized metric

- Let g be a Riemannian metric on a manifold M, $B \in \Omega^2(M)$ a B-field on M.
- A Hamiltonian density of Polyakov string moving in $M(g, B)$ can conveniently be described in terms of $2n \times 2n$ matrix $\mathbf{G}$

$$\mathbf{G} = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

- It can be viewed as a fiberwise metric on vector bundle

$$E = TM \oplus T^*M.$$

- $\mathbf{G}$ is called a **generalized metric**
- Vector bundle E is equipped with a signature (n, n) fiberwise metric $g_E \equiv \langle \cdot, \cdot \rangle_E$ defined as a canonical pairing

$$\langle V + \xi, W + \eta \rangle_E = \eta(V) + \xi(W).$$

- $\mathbf{G}$ is equivalent to choosing rank n positive definite subbundle $V_+ \subseteq E$ with respect to the metric $\langle \cdot, \cdot \rangle_E$.

Courant algebroids

- A way of promoting quadratic Lie algebra to the realm of vector bundles. Courant algebroid = (Quadratic Lie algebra)-oid.

Definition

Let E be a vector bundle with base manifold M. Define the *anchor* $\rho \in \mathrm{Hom}(E, TM)$, fiberwise metric $g_E \equiv \langle \cdot, \cdot \rangle_E$ on E, and $\mathbb{R}$-bilinear bracket $[\cdot, \cdot]_E : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$.
Then $(E, \rho, g_E, [\cdot, \cdot]_E)$ is a **Courant algebroid** if

1. $(\Gamma(E), [\cdot, \cdot]_E)$ is a Leibniz algebra, that is
$$[e, [e', e'']_E]_E = [[e, e']_E, e'']_E + [e', [e, e'']_E]_E.$$
2. It satisfies the Leibniz rule: $[e, fe']_E = f[e, e']_E + (\rho(e).f)e'$.
3. Pairing g_E is invariant with respect to $[\cdot, \cdot]_E$:
$$\rho(e).\langle e', e'' \rangle_E = \langle [e, e']_E, e'' \rangle_E + \langle e', [e, e'']_E \rangle_E.$$
4. Skew-symmetry is reasonably broken: $\langle [e, e]_E, e'' \rangle_E = \rho(e').\langle e, e \rangle_E$.

- Courant algebroid bracket is not skew-symmetric, $[\cdot,\cdot]_E$ is not a Lie algebra bracket.
- In fact, only Lie algebroid satisfying first 3 axioms has $\rho = 0$, it is then a collection of quadratic Lie algebras on each fiber.
- Canonical non-trivial example is the (twisted) **Dorfman bracket**

Example

Consider $E = TM \oplus T^*M$, $g_E = \langle \cdot,\cdot \rangle_E$ be the canonical pairing defined previously, and $\rho(X+\xi) = X$. The H-twisted Dorfman bracket $[\cdot,\cdot]_D^H$ is defined as

$$[X+\xi, Y+\eta]_D^H = [X,Y] + \mathcal{L}_X \eta - i_Y d\xi - H(X,Y,\cdot),$$

where $H \in \Omega^3(M)$.
Then $(E, \rho, g_E, [\cdot,\cdot]_D^H)$ is a Courant algebroid, iff $dH = 0$.

- For any $B \in \Omega^2(M)$, define the map $e^B(X+\xi) = X + \xi + B(X)$. Then there is a relation

$$e^B([X+\xi, Y+\eta]_D^{H+dB}) = [e^B(X+\xi), e^B(Y+\eta)]_D^H.$$

Connections on Courant algebroids

- On every vector bundle, a linear connection ∇ is a map $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$, satisfying

$$\nabla_{fX} e' = f \nabla_X e',$$
$$\nabla_X (fe') = f \nabla_X e' + (X.f) e'.$$

- Consider Courant algebroid $(E, \rho, g_E, [\cdot,\cdot]_E)$. Having $\rho \in \mathrm{Hom}(E, TM)$, we can generalize. $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ is a **Courant algebroid connection**, if

$$\nabla_e e' = f \nabla_e e',$$
$$\nabla_e (fe') = f \nabla_e e' + (\rho(e).f) e',$$

for all $e, e' \in \Gamma(E)$, and it is compatible with $\langle \cdot,\cdot \rangle_E$:

$$\rho(e).\langle e', e'' \rangle_E = \langle \nabla_e e', e'' \rangle_E + \langle e', \nabla_e e'' \rangle_E.$$

- We say that ∇ is an **induced Courant algebroid connection**, if there is an ordinary connection ∇', such that $\nabla_e = \nabla'_{\rho(e)}$.

- Courant algebroid axioms imply $\rho([e, e']_E) = [\rho(e), \rho(e')]_E$. It is this property which is important in the following.
- Having a bracket $[\cdot, \cdot]_E$, we can try to generalize the torsion operator. A following naive guess fails:

$$T_N(e, e') = \nabla_e e' - \nabla_{e'} e - [e, e']_E.$$

- It is not tensorial, and it is not skew-symmetric.
- There are two correct definitions of torsion in the literature.
  1. M. Gualtieri:

$$T_G(e, e', e'') = \langle T_N(e, e'), e'' \rangle_E + \frac{1}{2}(\langle \nabla_{e''} e, e' \rangle_E - \langle \nabla_{e''} e', e \rangle_E)$$

  2. P. Xu & A. Alekseev:

$$C(e, e', e') = \frac{1}{3}(\langle [e, e']_E, e'' \rangle_E - \frac{1}{2} \langle \nabla_e e' - \nabla_{e'} e, e'' \rangle_E + cyclic(e, e', e'').$$

- Both are well defined based on Leibniz rule for $[\cdot, \cdot]_E$, but using the other axioms, one can see that in fact $T_G \in \Omega^3(E)$, and

$$T_G(e, e', e'') = -C(e, e', e'').$$

- One can write this as $T_G(e, e', e'') = \langle T(e, e'), e'' \rangle_E$, where

$$T(e, e') = \nabla_e e' - \nabla_{e'} e - [e, e']_E + \langle \nabla_{e_\lambda} e, e' \rangle_E g_E^{-1}(e^\lambda),$$

where we view $\mathbf{K}(e, e') = \langle \nabla_{e_\lambda} e, e' \rangle_E g_E^{-1}(e^\lambda)$ as a *correcting term*, which kills the non-tensorial behavior of $T_N(e, e')$ in e:

$$\mathbf{K}(fe, e') = f\mathbf{K}(e, e') + \langle e, e' \rangle_E \mathcal{D}f,$$
$$\mathbf{K}(e, fe') = f\mathbf{K}(e, e').$$

Here $\mathcal{D}f$ is a map defined by $\langle \mathcal{D}f, e \rangle_E = \rho(e).f$.

- One can also attempt to generalize a curvature operator:

$$R_N(e, e')e'' = [\nabla_e, \nabla_{e'}]e'' - \nabla_{[e, e']_E} e''.$$

- This defines a tensorial object in (e', e''), but not in e. It works sufficently well only on isotropic subbundles, and for induced connections. Is there a way arround?

- Kind of. The idea is similar - correct the wrong behaviour in e.

$$R(e,e')e'' = [\nabla_e, \nabla_{e'}]e'' - \nabla_{[e,e']_E} e'' + \nabla_{\mathbf{K}(e,e')} e''.$$

- This spoils the tensoriality in e'', unless $\rho(\mathbf{K}(e,e')) = 0$. The $\mathbf{K}$ defined above does not satisfy this.
- On the other hand, for ρ with locally constant rank, one can always find $\mathbf{K}$ which works correctly. This choice, however, is not canonical.
- In fact, let $\mathfrak{d}f = \rho^T(df) \in \Gamma(E^*)$. One can find $\mathbf{L} \in \mathcal{T}_2^1(E)$, such that

$$[fe, e']_E = f[e,e']_E - (\rho(e').f)e + \mathbf{L}(\mathfrak{d}f, e, e'),$$

It satisfies $\rho(\mathbf{L}(\mathfrak{d}f, e, e')) = 0$. Note that $\mathfrak{d}f$ span $\text{Ann}(\ker \rho)$. If one can extend this property to $\Gamma(E^*)$, then

$$\mathbf{K}(e,e') = \mathbf{L}(e^\lambda, \nabla_{e_\lambda} e, e').$$

- One can see that $\mathbf{L}$ for Courant algebroid has to satisfy

$$\mathbf{L}(\mathfrak{d}f, e, e') = \langle e, e' \rangle_E \mathcal{D}f.$$

- Let $E = TM \oplus T^*M$. Then one can extend the $\mathbf{L}$ trivially to some *isotropic* complement to the subbundle $\text{Ann}(\ker \rho)$. This is as canonical as we can get. We obtain a class of maps parametrized by two-form $C \in \Omega^2(M)$:

$$\mathbf{L}_C((\zeta + Z), (X + \xi), (Y + \eta)) = \langle X + \xi, Y + \eta \rangle_E (\zeta - C(Z)).$$

- Let $\mathbf{K}_C$ be a corresponding class of correcting maps. We tend to work with $\mathbf{K}_0$.
- This works similarly also on exact Courant algebroids.
- Note that for Courant algebroid connections, the curvature operator R defined in this way still has some remarkable properties! We get

$$R(e, e')e'' = -R(e', e)e'',$$

It still posseses the following convenient symmetry:

$$\langle R(e, e')f', f'' \rangle_E = -\langle R(e', e)f'', f' \rangle_E.$$

LC connections

- Let $E = TM \oplus T^*M$. Recall that we have generalized metric $\mathbf{G}$. We impose the generalized metric compatibility condition:

$$\rho(e).\mathbf{G}(e', e'') = \mathbf{G}(\nabla_e e', e'') + \mathbf{G}(e', \nabla_e e''),$$

  for the Courant algebroid connection ∇.
- Further, we impose a torsion-free condition: $T(e, e') = 0$.
- Usual formulas for Levi-Civita connections do not work.
- This cannot stop one from examining all possibilities. One arrives to the conclusion that

$$\nabla_e = e^{B} \widehat{\nabla}_{e^{-B}(e)} e^{-B}(e'),$$

  where $\widehat{\nabla}$ is also a Courant algebroid connection, but compatible with $\mathbf{G}(B = 0)$. The explicit form of $\widehat{\nabla}$ is ... on the next slide

$$\widehat{\nabla}_X = \begin{pmatrix} \nabla^{LC}_X + g^{-1}\mathcal{D}(X,\star,\cdot) & -\frac{1}{3}g^{-1}H(X,g^{-1}(\star),\cdot) - \mathcal{J}(g(X),\star,\cdot) \\ -\frac{1}{3}H(X,\star,\cdot) - g\mathcal{J}(g(X),g(\star),\cdot) & \nabla^{LC}_X + \mathcal{D}(X,g^{-1}(\star),\cdot) \end{pmatrix}$$

$$\widehat{\nabla}_\xi = \begin{pmatrix} \frac{1}{6}g^{-1}H(g^{-1}(\xi),\star,\cdot) - \mathcal{J}(\xi,g(\star),\cdot) & g^{-1}\mathcal{D}(g^{-1}(\xi),g^{-1}(\star),\cdot) \\ \mathcal{D}(g^{-1}(\xi),\star,\cdot) & \frac{1}{6}H(g^{-1}(\xi),g^{-1}(\star),\cdot) - g\mathcal{J}(\xi,\star,\cdot) \end{pmatrix}$$

- ∇^{LC} is the Levi-Civita connection corresponding to the metric g, and $H = dB$.
- $\mathcal{D} \in \Omega^1(M) \otimes \Omega^2(M)$ satisfies $\mathcal{D}(X,Y,Z) + cyclic(X,Y,Z) = 0$.
- $\mathcal{J} \in \mathfrak{X}^1(M) \otimes \mathfrak{X}^2(M)$ satisfies $\mathcal{J}(\xi,\eta,\zeta) + cyclic(\xi,\eta,\zeta) = 0$.
- This shows that torsion-free ∇ is not determined uniquelly by metric compatibility conditions. The freedom lies exactly in the choice of tensors $\mathcal{D}$ and $\mathcal{J}$.
- What about the curvature operator R and scalar curvatures of such connections? It is a straightforward calculation...

... two weeks later ...

- Curvature operator R certainly depends on $(g, B, \mathcal{J}, \mathcal{D})$, and moreover on the choice of map $\mathbf{K}_C$.
- We can define a Ricci tensor Ric in a usual way as

$$\mathrm{Ric}(e, e') = \langle e^\lambda, R(e_\lambda, e')e \rangle.$$

- Finally, we may use the two available metrics $\mathbf{G}$, and g_E, to produce two scalars:

$$\mathcal{R} = \mathrm{Ric}(\mathbf{G}^{-1}(e^\lambda), e_\lambda),$$
$$\mathcal{R}_E = \mathrm{Ric}(g_E^{-1}(e^\lambda), e_\lambda).$$

- Let $\mathcal{D}'(X) = \mathcal{D}(g^{-1}(dy^k), \partial_k, X)$, and $\mathcal{J}'(\xi) = \mathcal{J}(g(\partial_k), dy^k, \xi)$.
- The final result for curvatures is

$$\mathcal{R} = \mathcal{R}(g) - \frac{1}{12} H_{ijk} H^{ijk} + 4 \,\mathrm{div}\, \mathcal{D}' - 4\|\mathcal{D}'\|_g^2 - 4\|\mathcal{J}'\|_g^2,$$
$$\mathcal{R}_E = -4 \,\mathrm{div}\, \mathcal{J}' + 8\langle \mathcal{J}', \mathcal{D}' \rangle.$$

Applications

- It is not clear yet, how to choose between the connections. Clearly, the most natural choice is $\mathcal{J} = \mathcal{D} = 0$. In this case, we have

$$\mathcal{R} = \mathcal{R}(g) - \frac{1}{12} H_{ijk} H^{ijk}, \; \mathcal{R}_E = 0.$$

- $\mathcal{R}$ is thus (without the dilaton Φ) exactly the closed string effective action, including the correct factor $1/12$.
- The fields (g, B) are coming from the string backgrounds, whereas $\mathcal{J}$ and $\mathcal{D}$ from the freedom in the connection. Can we fix them for example by equations of motion?
- Many interesting field redefinitions, e.g. T-duality or Seiberg-Witten open-closed relations, can be written as orthogonal transformations of the generalized metric $\mathbf{G}$.
- Assume that $\mathcal{O} \in O(E)$ is an orthogonal automorphism, that is

$$\langle \mathcal{O}(e), \mathcal{O}(e') \rangle_E = \langle e, e' \rangle_E.$$

- Then $\mathbf{G}' = \mathcal{O}^T \mathbf{G} \mathcal{O}$ is again a generalized metric. This time corresponding to a different pair (g', B').
- Having a connection ∇, define new connection ∇' as

$$\nabla'_e e' = \mathcal{O}^{-1} \nabla_{\mathcal{O}(e)} \mathcal{O}(e').$$

- This is Courant algebroid connection metric compatible with $\mathbf{G}'$.
- But we have to twist also the Courant algebroid bracket,

$$[e, e]'_E = \mathcal{O}^{-1} [\mathcal{O}(e), \mathcal{O}(e')]_E.$$

  This twist is usually quite unpleasant.
- However, there always exist two stabilizing maps $\mathcal{N}_\pm$, that is

$$\mathcal{N}_\pm^T \mathbf{G}' \mathcal{N}_\pm = \mathbf{G}',$$

  making this transformation into nicer one: $\mathcal{F}_\pm = \mathcal{O} \mathcal{N}_\pm$.

- There are two automorphisms of TM, denoted as $\mathbf{\Phi}_{\pm}$, induced by $\mathcal{O}$. The Lie algebroid on TM is then $[X,Y]_L^{\pm} = \mathbf{\Phi}_{\pm}[\mathbf{\Phi}_{\pm}^{-1}(X), \mathbf{\Phi}_{\pm}^{-1}(Y)]$. The bracket $[\cdot,\cdot]_E^{\pm}$ is then

$$[X+\xi, Y+\eta]_E^{\pm} = [X,Y]_L^{\pm} + \mathcal{L}_X^{L\pm}\eta - i_\eta d_{L\pm}(\xi) - (d_{L\pm}\omega_{\pm})(X,Y,\cdot),$$

for $\omega_{\pm} \in \Omega^2(M)$.

- One can now define $\nabla_e^{\pm} e' = \mathcal{F}_{\pm}^{-1}\nabla_{\mathcal{F}_{\pm}(e)}\mathcal{F}_{\pm}(e')$. By definition $\nabla_e^{\pm} e'$ is torsion-free with respect to $[\cdot,\cdot]_E^{\pm}$.
- Their (both) scalar curvatures, calculated using the respective Courant brackets, coincide.
- One can use this approach to simply prove certain equalities, e.g.

$$\mathcal{R}(g) - \frac{1}{12} H_{ijk} H^{ijk} = \mathcal{R}_\theta(G) - \frac{1}{12} S_{ijk} S^{ijk},$$

where $G = -Bg^{-1}B$, $\theta = B^{-1}$ is H-twisted Poisson tensor, $S = [\theta,\theta]_S$, and $\mathcal{R}_\theta(G)$ corresponds to LC-connection with respect to Koszul Lie algebroid on T^*M.

Outlooks

- So far, we have worked only with Dorfman bracket on $TM \oplus T^*M$.
- With minimum of modifications, everything works for exact Courant algebroids, because in fact $E \cong (TM \oplus T^*M)$.

$$0 \longrightarrow T^*M \stackrel{g_E^{-1} \circ \rho^T}{\longrightarrow} E \stackrel{\rho}{\longrightarrow} TM \longrightarrow 0.$$

- We would like to re-itroduce the dilaton Φ. This corresponds to working with Atiyah-Lie algebroids coming from $U(1)$-bundles.
- We can thus think of $E = L \oplus L^*$, where L is a transitive Lie algebroid, E equipped with the Dorfman bracket corresponding to L.
- There are many more interesting Courant algebroids to try our approach on, e.g. heterotic Courant algebroids

$$E \cong TM \oplus (P \times_{Ad} \mathfrak{g}) \oplus T^*M,$$

where $(\mathfrak{g}, c)$ is a quadratic Lie algebra.

Thank you for your attention!