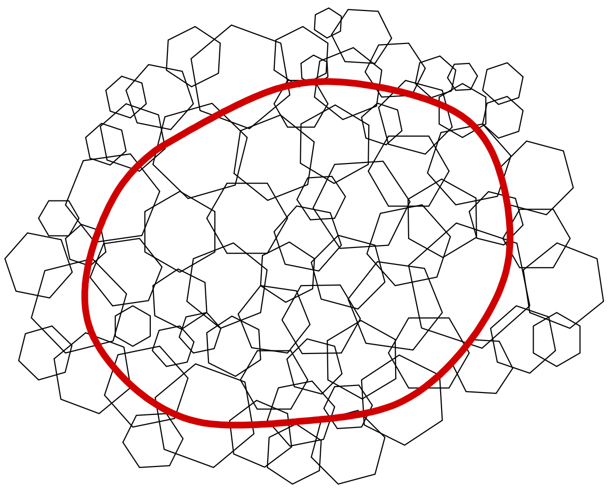
Perspectives on Nonassociative Geometry

Ralph Blumenhagen

Max-Planck-Institut für Physik, München



Introduction

Introduction

Recall:

- Closed string theory is described by 2D non-linear sigma model

$$\mathcal{S} = \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2 z \left(G_{ab} + B_{ab} \right) \partial X^a \, \overline{\partial} X^b + \ldots \ ,$$

- Classical vacua of string theory are described by 2D conformal field theories
- There exist conformal field theories which cannot be identified with such simple large radius geometries.

Examples are left-right asymmetric like asymmetric orbifolds.

Introduction

Introduction

Applying T-duality leads to the chain of fluxes (Shelton, Taylor, Wecht)

$$H_{abc} \leftrightarrow f_{ab}{}^{c} \leftrightarrow Q_{a}{}^{bc} \overset{?}{\leftrightarrow} R^{abc} \, ,$$

- Q and R are non-geometric fluxes. What is their nature?
- Need to better understand this regime of string theory $\rightarrow$ Generalized Geometry and Double Field Theory (DFT)
- DFT provides a formal definition of the non-geometric fluxes. In a non-geometric frame $B_{ij} \rightarrow \beta^{ij}$ and

$$Q_{k}{}^{ij} = \partial_k \beta^{ij}$$

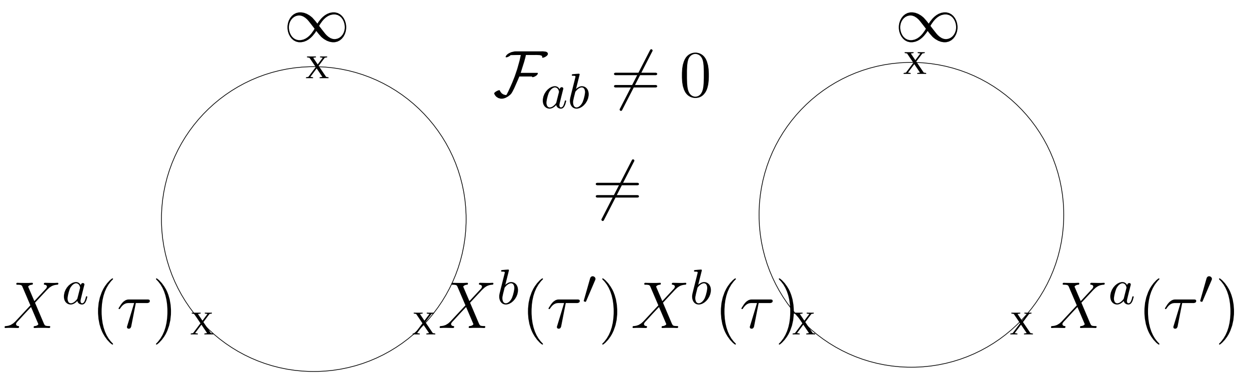
$$R^{ijk} = \tilde{\partial}^{[i} \beta^{jk]}$$

Introduction

Introduction

Open string theory:

For non-vanishing $\mathcal{F}_{ab}$ flux on a D-brane, the two disc 2-point functions turn out to be different.



Introduction

Introduction

- Applying a left-right asymmetric rotation to a D-brane turns on the magnetic flux.
- D-brane with $F_{ij} \to$ non-commutative geometry with

$$[x^i, x^j] = x^i \star x^j - x^j \star x^i = i\,\theta^{ij}$$

with a non-commutative product on function space
(Schomerus + Seiberg, Witten)

$$f \star g = f \cdot g + \frac{i}{2}\theta^{ij}\,\partial_i f\,\partial_j g + \ldots\,.$$

Does there exist a generalization to the closed string?

Fluxed closed string

Fluxed closed string

Issues:

- On the world-sheet sphere, an $SL(2,\mathbb{Z})$ transformation can change the order of two points: $\{z_1, z_2, \infty\} \to \{z_2, z_1, \infty\} \Rightarrow$ NC not visible in CFT
- Choosing 3 points $\{z_1, z_2, z_3, \infty\}$, one can have for the cross-ratio $|X| = \left|\frac{z_{12}\, z_{34}}{z_{13}\, z_{24}}\right| <> 1$
- A background 3-flux θ^{ijk} distinguishing these two configurations might potentially lead to nonassociativity

However, in CFT one requires crossing symmetry, $X \to 1/X$, of correlation functions $\to$ on-shell there should be no nonassociativity present.

Review of CFT results

Review of CFT results

This question was analyzed by performing genuine CFT computations? (see also work of (Bouwknegt, Hannabuss, Mathai))

- Compute cyclic double commutator

$$\big[X^\mu, X^\nu, X^\rho\big] := \lim_{\sigma_i \to \sigma} \big[[X^\mu(\sigma_1,\tau), X^\nu(\sigma_2,\tau)], X^\rho(\sigma_3,\tau)\big] + \text{cyclic}$$

  for WZW-model (Bhg, Plauschinn, arXiv:1010.1263)

- Compute commutators by direct quantization of closed strings in linear B-field (Lüst, arXiv:1010.1361)(Condeescu, Florakis, Lüst,arXiv:1202.6366)(Andriot, Larfors, Lüst, Patalong, arXiv:1211.6437)

- Conformal perturbation theory around flat geometry with constant H-flux + CFT T-duality (Bhg, Deser, Lüst, Plauschinn, Rennecke, arXiv:1106.0316)

Review of CFT results

Review of CFT results

- Non-commutativity: Wilson line of Q-flux.
- Non-associativity: local R-flux .

A tri-bracket for the coordinates x^a appeared as

$$[x^a, x^b, x^c] = R^{abc},$$

and the precursor noncommutative algebra (Lüst, arXiv:1010.1361)

$$[x^i, x^j] = R^{ijk}\ p_k\ , \qquad [x^i, p_j] = \delta^i{}_j\ .$$

Can be captured by a nonassociative $\star$-product on phase space $T^*(M)$:

$$f \star g = f\, g + R^{ijk} p_k\, \partial_i f \partial_j g + (\tilde{\partial}^i f \partial_i g - \partial_i f \tilde{\partial}_i g) + \dots$$

(Mylonas, Schupp, Szabo, arXiv:1207.0926+1312.1621) (Bakas, Lüst, arXiv:1309.3172).

Review of CFT results

Review of CFT results

Keep in mind: these results were derived for a flat space with a constant flux and constant dilaton.
This does satisfy the closed string EOM only up to linear order in H

$$
\begin{aligned}
0 &= \Big(R_{ab} - \frac{1}{4}\, H_a{}^{cd}\, H_{bcd} + 2\nabla_a \nabla_b \Phi\Big) + O(\alpha'^2)\,, \\
0 &= \Big(-\frac{1}{2}\nabla_c H^c{}_{ab} + \alpha' H_{ab}{}^c \nabla_c \Phi\Big) + O(\alpha'^2)\,, \\
0 &= \frac{1}{4}(d - d_{\rm crit}) + \alpha'\Big((\nabla\Phi)^2 - \frac{1}{2}\nabla^2\Phi - \frac{1}{24}H^2\Big) + O(\alpha'^2)\,.
\end{aligned}
$$

- star-product was derived from correlation functions of tachyon vertex operators
- we were not able to construct the H-deformed graviton vertex operator

Tri-products

Tri-products

One defines tri-products for $f_i \in C^\infty(M)$ (Bhg, Fuchs, Hassler, Lüst, Sun),(Aschieri,Szabo).

$$f_1 \triangle f_2 \triangle \ldots \triangle f_N = f_1 \star (f_2 \star (\ldots (f_{N-1} \star f_N) \ldots))\Big|_{p_0=0}$$

$$= \cdot \Big[\exp\Big(-\frac{l_s^4}{12} \sum_{1 \le a < b < c \le N} R^{ijk}\, \partial_i^a \otimes \partial_j^b \otimes \partial_k^c \Big) (f_1 \otimes \ldots \otimes f_N) \Big] \,.$$

- Note that one is restricting to the $p_0 = 0$ leaf. This implies e.g. $f_1 \triangle f_2 = f_1\, f_2$.
- Peculiar property: N-cyclicity

$$\int d^d x\, f_1 \triangle f_2 \triangle \ldots \triangle f_N = \int d^d x\, f_1\, f_2\, \ldots f_N \,.$$

Issues

Issues

Two puzzling issues:

- On-shell string theory is described by a 2D QFT $\Rightarrow$ associative operator algebra
- What is the microscopic description of Q and R-flux?

On-shell associativity

On-shell associativity

For the open string we have:

1. For on-shell string scattering amplitudes, the conformal $SL(2,\mathbb{R})$ symmetry group leaves the cyclic order of fields invariant.
2. In CFT on the disc we have crossing symmetry of N-point functions $\rightarrow$ on-shell associativity.

(Herbst, Kling, Kreuzer),(Cornalba,Schiappa)

For the closed string we have:

1. In CFT on the sphere we have crossing symmetry of N-point functions $\rightarrow$ on-shell associativity. Could be guaranteed by N-cyclicity.

Definition of R-flux

Definition of R-flux

- Double Field Theory provides a unified description of non-geometric fluxes

$$R^{ijk} = 3\,\tilde{\partial}^{[i}\beta^{jk]}\,.$$

- No notion of a nonassociative, background dependent deformation of the geometry is visible.

Reviews on DFT: (Aldazabal, Marques, Nunez, arXiv: 1305.1907), (Berman, Thompson, arXiv: 1306.2643), (Hohm, Lüst, ,Zwiebach, arXiv: 1309.2977).

Comment: Attempt to formulate a nonassociative quantum mechanics in (Mylonas, Schupp, Szabo),(Bojowald, Brahma, Buyukcam, Strobl)

Basics of DFT

Basics of DFT

Doubled coordinates

$$X^M = (\tilde{x}_m, x^m)$$

One imposes symmetries:

- Generalized diffeomorphisms
- Generalized frame rotations
- Global $O(D,D)$ symmetry, including T-duality

Doubled diffeomorphisms, $X^M \to X^M + \xi^M$ acts via a generalized Lie-derivative, e.g.

$$\mathcal{L}_\xi V^M = \xi^N \partial_N V^M + (\partial^M \xi_N - \partial_N \xi^M) V^N \,.$$

Strong constraint: Stronger version of $L_0 - \overline{L}_0 = 0$

$$\partial_M \partial^M = 0 \,, \qquad \partial_M f \, \partial^M g = 0$$

DFT tri-poduct

DFT tri-poduct

In DFT, the tri-product takes the form: (Bhg, Fuchs, Hassler, Lüst, Sun, arXiv:1312.0719).

$$f \triangle g \triangle h = f\, g\, h + \tilde{\partial}^{[i}\beta^{jk]}\partial_i f\, \partial_j g\, \partial_k h + \ldots .$$

- It vanishes by the strong constraint!
- For a non-trivial tri-product the strong constraint (between background and fluctuations) needs to be weakened.

Change perspective: Make an attempt to construct a nonassociative differential geometry/gravity theory based on the R-star-product.

Towards nonassociative gravity

Towards nonassociative gravity

Idea: Extend the formalism of (Aschieri, Blohmann, Dimitrijevic, Meyer, Schupp, Wess) to nonassociative $\star$-products $\rightarrow$ quasi Hopf-algebras (Aschieri, Barnes, Mylonas, Schenkel, Schupp, Szabo),(Bhg, Fuchs)

Starting point: $\star$-product to realize the precurcor nonassociative algebra:

$$
\begin{aligned}
f \star g &= \cdot \left[\mathcal{F}^{-1}(f,g)\right] \\
&= \cdot \Big[\exp\Big(\tfrac{1}{2} i\hbar(\partial_i \otimes \tilde{\partial}_p^i - \tilde{\partial}_p^i \otimes \partial_i) \\
&\qquad + \tfrac{i l_s^4}{12\hbar} R^{ijk} \big(p_k \partial_i \otimes \partial_j - \partial_j \otimes p_k \partial_i\big)\Big) f \otimes g \Big]
\end{aligned}
$$

Note, it contains the momentum $p \rightarrow$ rather defined on functions $f(x,p)$ on T^*M.

Towards nonassociative gravity

Towards nonassociative gravity

The star-product still has two special features:

- Controlled permutation (talk by A. Schenkel)

$$f \star g = \overline{\mathcal{R}}(g) \star \overline{\mathcal{R}}(f).$$

with the universal $\mathcal{R}$-matrix $\mathcal{R} = \mathcal{F}^{-2}$.

- Reordering of brackets controlled by the associator ϕ

$$(f \star g) \star h = f^{\phi} \star (g^{\phi} \star h^{\phi})$$

with

$$\phi(f, g, h) = \exp\left(\frac{l_s^4}{6} R^{ijk} \partial_i \otimes \partial_j \otimes \partial_k\right)(f \otimes g \otimes h).$$

Towards nonassociative gravity

Towards nonassociative gravity

Tensor calculus: $\star$-scalar f transforms under $\star$-diffeos as

$$\delta_\xi f = \mathcal{L}_\xi f = \xi^m \star \partial_m f$$

The $\star$-product of two scalars is again a scalar

$$\begin{aligned} \delta_\xi (f \star g) &= \mathcal{L}_\xi (f \star g) = \xi^m \star \partial_m (f \star g) \\ &= \xi^m \star (\partial_m f \star g) + \xi^m \star (f \star \partial_m g) \\ &= \left(\mathcal{L}_{\xi^{\overline{\phi}}} f^{\overline{\phi}} \right) \star g^{\overline{\phi}} + \overline{\mathcal{R}}^\alpha (f^{\overline{\phi\phi}}) \star \left(\mathcal{L}_{\overline{\mathcal{R}}_\alpha (\xi^{\overline{\phi\phi}})} g^{\overline{\phi\phi}} \right) . \end{aligned}$$

This defines the $\star$-Leibniz rule.

Can be continued to the structure of star-tensors, star-covariant derivative, star-torsion, star-curvature.

(talk by Szabo)

Towards nonassociative gravity

Towards nonassociative gravity

Metric as

- scalar product
- object to raise and lower indices

A star-metric g_{ij} as the duality map $\mathcal{G} : T\mathcal{M} \xrightarrow{g} T^*\mathcal{M}$, whose inverse should satisfy

$$\left(v^k \star g_{kj}\right) \star (g^{-1\star})^{ji} = v^i \,.$$

Therefore, an inverse satisfying

$$g_{ij} \star (g^{-1\star})^{jk} = \delta_i{}^k$$

does not satisfy the above relation.

Towards nonassociative gravity

Towards nonassociative gravity

Up to linear order, one finds explicitly

$$g_{L/R}^{\star-1ij} = g^{ij} - \tfrac{il_s^4}{6\hbar} R^{abc}\, p_c\, g^{im}\, \partial_a g_{mn}\, \partial_b g^{nj} \\ \pm \tfrac{l_s^4}{12}\, R^{abc}\, \partial_a g^{im}\, \partial_b g_{mn}\, \partial_c g^{nj}\,,$$

with $\phi(g_{ij}, g^{jk}, \,.\,) \neq 1$.

Properties:

- not symmetric, with $g_{L/R}^{\star-1(ij)} = g^{ij}$
- momentum dependent
- left- and right-inverse differ by an associator

Towards nonassociative gravity

Towards nonassociative gravity

One cannot solve $\nabla g = \partial g - \Gamma \star g = 0$ in terms of a LC-connection Γ.

Possible way out: define its covariant derivative on a covector as

$$\begin{aligned}\nabla^{\mathrm{LC}}\omega &= \partial\omega - \Big(\big(\textstyle\sum\partial g\big) \circ g_L^{\star-1}\Big)(\omega) \\ &= \partial\omega - \big(\textstyle\sum\partial g\big) \star \big(g_L^{\star-1} \star \omega\big)\,.\end{aligned}$$

Open questions:

- Is this a mathematically consistent possibility?
- Can one define a star-Einstein-Hilbert action so that linear corrections in R are total derivatives?

Other approach

Other approach

Alternative recent approach:

There exists an α' deformed C-bracket (Hohm, Siegel, Zwiebach, arXiv: 1306.2970)

$$[\xi,\eta]_C = [\xi,\eta]_C^{(0)} - \alpha'\,[\xi,\eta]_C^{(1)}$$

with

$$[\xi,\eta]_C^{(1),\,N} = \frac{1}{2}\left(\partial^N\partial_Q\xi^P\partial_P\eta^Q - (\xi\leftrightarrow\eta)\right)\,.$$

Idea: Write this as a result of a $\star$-product, defining a deformation of a Courant algebroid structure (Deser, arXiv: 1412.5966).

Suggests: (Asymmetric) heterotic string theory to be the framework for non-associative gravity?

Sorry! – an open end

Sorry! – an open end

- What can we expect from such a nonassociative gravity theory?
    - Could be a trivial deformation
    - Inclusion of α' corrections (like for the open string)
- It is not clear whether this ansatz leads to something related to string theory. How are star-diffeomorphisms related to standard diffoes?
- In DFT it is not clear which form of the constraint is the right one. Related to the question whether Scherk-Schwarz reductions and gauged supergravities are parts of string theory

Stay tuned!