Open/closed string theory and operads

joint work of
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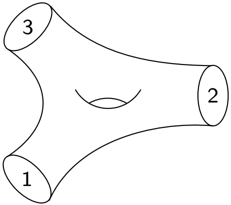
30.4.2016

Closed string worldsheets

$\mathcal{Q}$uantum $\mathcal{C}$losed string operad:

$$\mathcal{QC}(n, G) :=$$
$$\{\text{homeomorphism classes of compact orientable } 2D \text{ surfaces of genus } G \text{ with } n \text{ boundary components labelled } 1, 2, \ldots, n\}$$

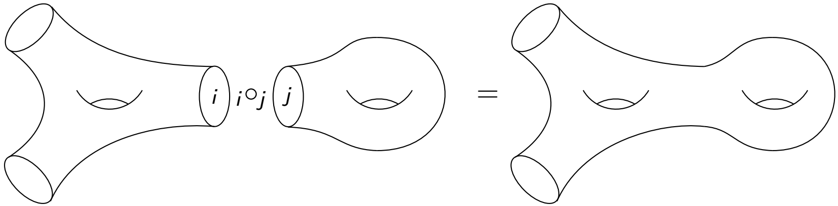


$|\mathcal{QC}(n, G)| = 1$

Closed string worldsheets - continued

Gluing of surfaces along boundary components:

$$_i\circ_j : \mathcal{QC}(n_1, G_1) \times \mathcal{QC}(n_2, G_2) \to \mathcal{QC}(n_1 + n_2 - 2, G_1 + G_2)$$



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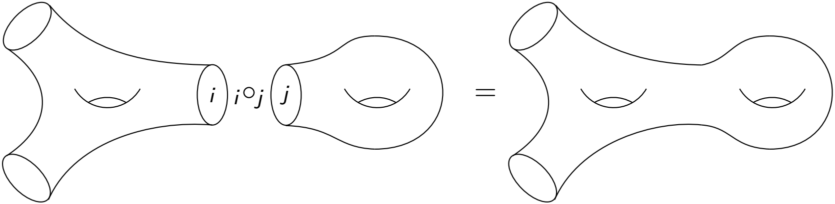
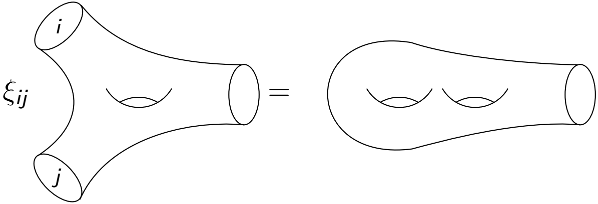


$$\xi_{ij} : \mathcal{QC}(n, G) \to \mathcal{QC}(n - 2, G + 1)$$



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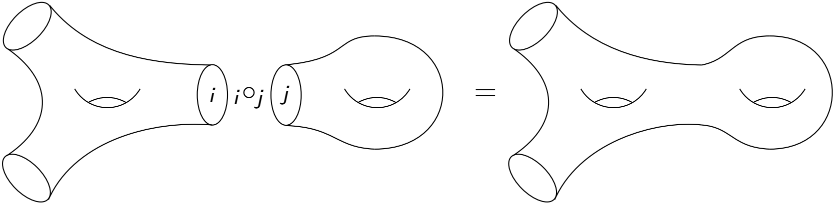

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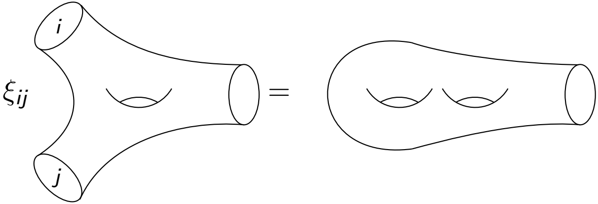
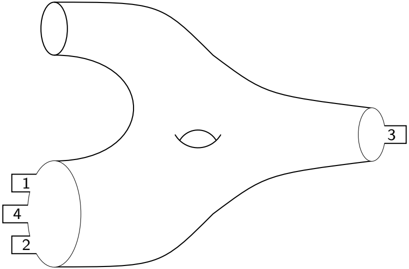


Σ_n acts on $\mathcal{QC}(n, G)$ by permuting the labels of the boundary components - trivial action.

Open string worldsheets

$\mathcal{Q}$uantum $\mathcal{O}$pen string operad:

$$\mathcal{QO}(n, G) :=$$
$$\{\text{homeomorphism classes of compact orientable } 2D \text{ surfaces}$$
$$\text{of genus } g \text{ with } n \text{ distinguished intervals (open inputs)}$$
$$\text{labelled } 1, 2, \ldots, n \text{ on } b \text{ boundary components,}$$
$$G = 2g + b - 1\}$$

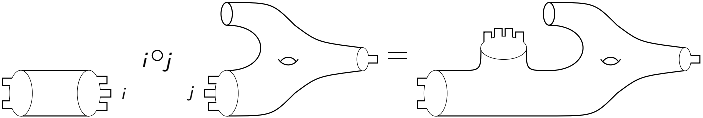


$|\mathcal{QO}(n, G)| > 1$

Open string worldsheets - continued

Gluing of surfaces using ribbons:

$$_i\circ_j : \mathcal{QO}(n_1, G_1) \times \mathcal{QO}(n_2, G_2) \to \mathcal{QO}(n_1 + n_2 - 2, G_1 + G_2)$$



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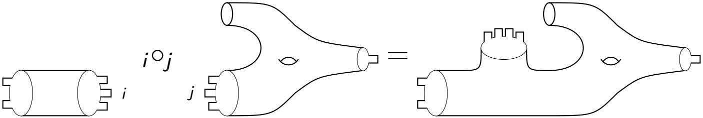

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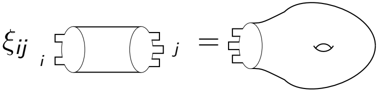

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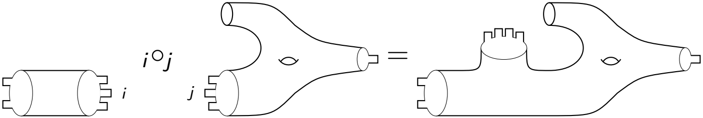

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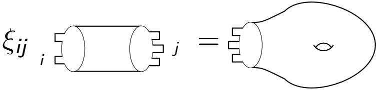

Σ_n acts on $\mathcal{QO}(n, G)$ by permuting the labels of the open inputs.

Modular operad

Symmetric monoidal category $\mathsf{C} = (\mathsf{C}, \otimes, I, s)$
Examples: $(\mathsf{Set}, \times, \{*\}, s)$, $(\mathsf{dgVec}_{\mathbb{K}}, \otimes, \mathbb{K}, s)$, $\mathsf{SSet}, \mathsf{Top}, \ldots$

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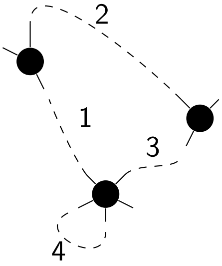
Symmetric monoidal category $\mathsf{C} = (\mathsf{C}, \otimes, I, s)$
Examples: $(\mathsf{Set}, \times, \{*\}, s)$, $(\mathsf{dgVec}_{\mathbb{K}}, \otimes, \mathbb{K}, s)$, $\mathsf{SSet}, \mathsf{Top}, \ldots$

Modular operad $\mathcal{P}$ in C consists of a collection $\{\mathcal{P}(n, G)\}_{n,G\geq 0}$ of objects of C with Σ_n actions by morphisms of C, and collections

$$ {}_i\circ_j : \mathcal{P}(n_1, G_1) \otimes \mathcal{P}(n_2, G_2) \to \mathcal{P}(n_1 + n_2 - 2, G_1 + G_2) $$
$$ \xi_{ij} : \mathcal{P}(n, G) \to \mathcal{P}(n-2, G+1) $$

of equivariant morphisms of C satisfying the axioms:

1. ${}_i\circ_j = {}_j\circ_i\, s, \quad \xi_{ij} = \xi_{ji}$
2. $\xi_{ij}\xi_{kl} = \xi_{kl}\xi_{ij}$
3. $\xi_{ij}\ {}_k\circ_l = \xi_{kl}\ {}_i\circ_j$
4. ${}_i\circ_j(\xi_{kl} \otimes 1) = \xi_{kl}\ {}_i\circ_j$
5. ${}_i\circ_j(1 \otimes {}_k\circ_l) = {}_k\circ_l({}_i\circ_j \otimes 1)$



Operad variants

$\mathcal{C}om \subset \mathcal{QC}$: surfaces with genus $G = 0$
$\mathcal{A}ss \subset \mathcal{QO}$: surfaces with genus $g = 0$ and a single boundary

Cyclic operad - no ξ_{ij}

Ordinary operad - one output, several inputs

Properad - several inputs, several outputs

Endomorphism operad

V is a vector space over $\mathbb{K}$, $\dim_{\mathbb{K}} V =: N < \infty$

$B : V \otimes V \to \mathbb{K}$ a scalar product

$\{a_i\}_{i=1}^{N}$ a basis, B^{ij} is the inverse matrix of $B_{ij} = B(a_i, a_j)$

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$$\mathcal{E}nd_V(n, G) := \mathrm{Hom}_{\mathbb{K}}(V^{\otimes n}, \mathbb{K})$$

$$_i\circ_j : \mathcal{E}nd_V(n_1, G_1) \otimes \mathcal{E}nd_V(n_2, G_2) \to \mathcal{E}nd_V(n_1 + n_2 - 2, G_1 + G_2)$$

$$(f\ _i\circ_j\ g)(v_1, \dots, v_{n_1+n_2-2}) :=$$

$$= \sum_{k,l} f(v_1, \dots, \underbrace{a_k}_{i-th}, \dots, v_{n_1-1}) B^{kl} g(v_{n_1}, \dots, \underbrace{a_l}_{j-th}, \dots, v_{n_1+n_2-2})$$

$$\xi_{ij} : \mathcal{E}nd_V(n, G) \to \mathcal{E}nd_V(n-2, G+1)$$

$$(\xi_{ij} f)(v_1, \dots, v_{n-2}) := \sum_{k,l} f(v_1, \dots, \underbrace{a_k}_{i-th}, \dots, \underbrace{a_l}_{j-th}, \dots, v_{n-2}) B^{kl}$$

$\sigma \in \Sigma_n$ acts by $(\sigma \cdot f)(v_1, \dots, v_n) := f(v_{\sigma(1)}, \dots, v_{\sigma(n)})$

Algebra over an operad

Morphism $\alpha : \mathcal{P} \to \mathcal{Q}$ of modular operads is a collection of equivariant maps

$$\alpha : \mathcal{P}(n, G) \to \mathcal{Q}(n, G), \quad n, G \geq 0,$$

satisfying

$$\alpha(p_1) \, {}_i\circ_j \, \alpha(p_2) = \alpha(p_1 \, {}_i\circ_j \, p_2) \quad \text{and} \quad \xi_{ij}\alpha(p) = \alpha(\xi_{ij} p)$$

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Algebra over $\mathcal{P}$ (or $\mathcal{P}$ algebra) on a vector space V is a modular operad morphism $\mathcal{P} \to \mathcal{E}nd_V$.
compare: representation of group, $G \to \mathrm{End}(V)$

Examples of algebras over operads

Algebras over

- $\mathcal{C}om$: non-unital commutative Frobenius algebra
- $\mathcal{QC}$: essentially 2D TQFT
  (compare: monoidal functor $\mathrm{Cob}_2 \to \mathrm{Vec}_{\mathbb{C}}$)
- $\mathcal{A}ss$: non-unital Frobenius algebra
- $\mathcal{QO}$: essentially 2D open TQFT

Modular envelope

$$\mathsf{CycOp} \underset{\text{forget}}{\overset{\text{Mod}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{ModOp}$$

$$\begin{array}{ccc}
\mathcal{C} & \hookrightarrow & \mathrm{Mod}(\mathcal{C}) \\
& {}_{\substack{\forall F \\ \text{cyclic}}}\searrow & \Big\downarrow {\scriptstyle \substack{\exists ! \tilde{F} \\ \text{modular}}} \\
& & \mathcal{P}
\end{array}$$

Modular envelope

$$\mathsf{CycOp} \underset{\text{forget}}{\overset{\text{Mod}}{\rightleftarrows}} \mathsf{ModOp} \quad (\text{Mod} \dashv \text{forget})$$

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In particular ($\mathcal{P} = \mathcal{E}nd_V$):

$$\{\text{algebras over } \mathcal{C}\} \leftrightarrow \{\text{algebras over } \mathrm{Mod}(\mathcal{C})\}$$

Examples:

- $\mathrm{Mod}(\mathcal{C}om) \cong \mathcal{QC}$, explains 2D TQFT
- $\mathrm{Mod}(\mathcal{A}ss) \cong \mathcal{QO}$
- $\mathrm{Mod}(\mathcal{OC})/(\text{Cardy relation}) \cong \mathcal{QOC}$

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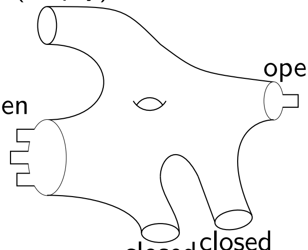
2-coloured modular operad:

$$\mathcal{QOC} := \{2D \text{ surfaces with both open and closed inputs}\}$$



2-coloured cyclic suboperad of $\mathcal{QOC}$:

$\mathcal{OC} := \{2D$ surfaces of genus 0 with both open and closed inputs$\}$

$\mathrm{Mod}(\mathcal{OC})/(\text{Cardy relation}) \cong \mathcal{QOC}$

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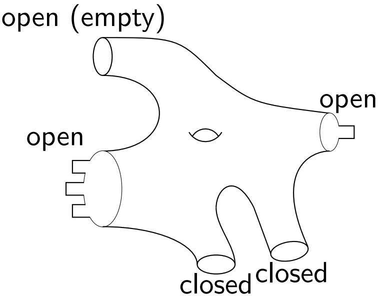

2-coloured cyclic suboperad of $\mathcal{QOC}$:

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$\mathcal{QOC}$ algebras: A, B Frobenius algebras, A commutative, $A \to B$ a morphisms of algebras, $f(A) \subset Z(B)$, Cardy condition

Homotopy algebras

$$h \circlearrowright (V, d_V) \underset{i}{\overset{p}{\rightleftarrows}} (H, d_H)$$

Transfer problem: Given $\mathcal{P}$ algebra structure on (V, d_V), given dg maps p, i such that $ip \sim 1_V$ via homotopy h, find a $\mathcal{P}$ algebra structure on (H, d_H) and extend p, i to morphisms of $\mathcal{P}$ algebras.

Example: Gauge fixing

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Example: Gauge fixing

$$\{\mathcal{P} \text{ algebras}\} \subset \{\mathcal{P}_\infty \text{ algebras}\}$$

E.g. cofibrant replacement $\mathcal{P}_\infty \xrightarrow{\sim} \mathcal{P}$

Theorem: The transfer problem is always solvable for algebras over cofibrant operads.

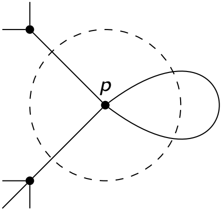
Ordinary operads: $\mathcal{P}_\infty = \Omega B(\mathcal{P})$ cobar-bar construction
$\mathcal{P}_\infty = \Omega(\mathcal{P}^{¡})$ Koszul resolution

Feynman transform

Let $\mathcal{P}$ be a modular operad.

$$\mathbf{F}(\mathcal{P})(n, G) \approx \mathbb{K}\{\text{graphs of genus } G \text{ with } n \text{ legs} \\ \text{and vertices decorated by elements of } \mathcal{P}\}$$

${}_i\circ_j$ and ξ_{ij} are defined by grafting of graphs

$G =$ 

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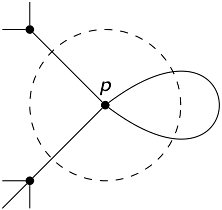
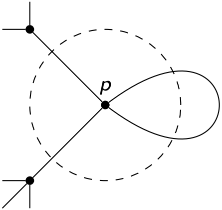
$G =$ 

$$\partial(G) := \sum_{v \text{ vertex of } G} \pm G_v$$

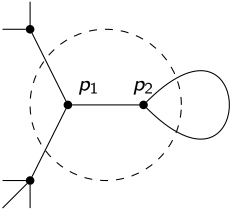
where G_v is obtained from G by replacing a small neighborhood of vertex v decorated by $p \in \mathcal{P}$ by a graph with a single inner edge and vertices decorated using duals of ${}_i\circ_j$ and ξ_{ij} on $\mathcal{P}$.

Feynman transform - continued

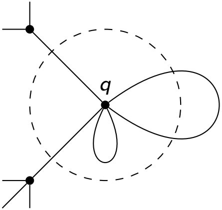
$G =$



$G_v = \sum_{p_1 \, {}_i\circ_j \, p_2 = p}$



$+ \sum_{\xi_{ij} q = p}$



${}_i\circ_j$ and ξ_{ij} refers to $\mathcal{P}$

Homotopy algebras - cyclic and modular examples

Algebras over

- $\Omega(\mathcal{C}om)$: cyclic L_∞ algebras
- $\mathbf{F}(\mathcal{QC})$: quantum L_∞ algebras (a.k.a. loop homotopy algebras)
- $\Omega(\mathcal{A}ss)$: cyclic A_∞ algebras
- $\mathbf{F}(\mathcal{QO})$: quantum A_∞ algebras
- $\mathbf{F}(\mathcal{QOC})$: quantum open/closed homotopy algebras

Master equation

$$P := \prod_{n,g} (\mathcal{P}(n,g) \otimes \mathcal{E}nd_V(n,g))^{\Sigma_n} \quad \text{(invariants)}$$

with operations $d, \Delta : P \to P,\ \{\} : P \otimes P \to P$:

$$d(p \otimes e) := d_{\mathcal{P}}(p) \otimes e \pm p \otimes d(e)$$

$$\Delta(p \otimes e) := \xi_{ij} p \otimes \xi_{ij} e$$

$$\{p \otimes e, p' \otimes e'\} := \sum_{\rho} \pm(\rho \otimes \rho)((p\ {}_i\circ_j\ p') \otimes (e\ {}_i\circ_j\ e'))$$

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$$\Delta(p \otimes e) := \xi_{ij} p \otimes \xi_{ij} e$$
$$\{p \otimes e, p' \otimes e'\} := \sum_{\rho} \pm(\rho \otimes \rho)((p\, {}_i\circ_j\, p') \otimes (e\, {}_i\circ_j\, e'))$$

Theorem: Algebras over $\mathbf{F}(\mathcal{P})$ on (V, d) are in bijection with degree 0 solutions of the master equation in P:

$$dS + \Delta S + \frac{1}{2}\{S, S\} = 0, \quad S \in P.$$

Moreover, $(P, d, \Delta, \{\})$ is a generalized Batalin-Vilkovisky algebra.

Master equation for $\mathbf{F}(\mathcal{QC})$

$$P \cong \prod_{n,g} \mathcal{QC}(n,g) \otimes_{\Sigma_n} V^{\# \otimes n} \cong \mathbb{K}[[\phi^1, \ldots, \phi^N]] \quad \text{(power series)}$$

with operations

$$\Delta(S) = \sum_{i,j} \pm B^{ij} \frac{\partial^2 S}{\partial \phi^i \partial \phi^j}$$

$$\{S, T\} = \sum_{i,j} \pm B^{ij} \frac{\partial S}{\partial \phi^i} \cdot \frac{\partial T}{\partial \phi^j}$$

for $S, T \in \mathbb{K}[[\phi^1, \ldots, \phi^N]]$.

$(P, d, \Delta, \{\}, \cdot)$ is an honest Batalin-Vilkovisky algebra.

Homotopy transfer for $\mathbf{F}\mathcal{QC}$ algebras

$$h \circlearrowright (V, d_V) \underset{i}{\overset{p}{\rightleftarrows}} (H_*(V), 0)$$

$$\Downarrow$$

$$h \circlearrowright (P_V, \{S_0, -\}) \underset{i}{\overset{p}{\rightleftarrows}} (P_H, 0)$$

The following are equivalent:

1. $d_V S + \Delta_V S + \frac{1}{2}\{S, S\}_V = 0$
2. $\Delta_V S' + \frac{1}{2}\{S', S'\}_V = 0$ for $S' = S_0 + S$
3. $(\Delta_V(-) + \{S', -\}_V) \circ (\Delta_V(-) + \{S', -\}_V) = 0$, i.e. $(\Delta_V(-) + \{S, -\}_V)$ is a perturbation of $\{S_0, -\}_V$

Homological perturbation lemma $\Rightarrow$ explicit perturbation of 0 on P_H of the form $\Delta_H(-) + \{S_H, -\}_H$, i.e. a $\mathbf{F}(\mathcal{QC})$ algebra on $H_*(V)$

Homology of Feynman transform

- What are $\mathbf{F}(\mathcal{QC}), \mathbf{F}(\mathcal{QO}), \mathbf{F}(\mathcal{QOC})$ cofibrantly replacing? Calculate homology!
- $H_*(\mathbf{F}(\underline{\mathcal{C}om}))$ is classifying Vassiliev invariants
- $H_*(\mathbf{F}(\mathcal{QO})(n, G)) \cong H^{\mathrm{top}(e),e}(\overline{M}_{G,n})$ (moduli space of complex stable curves)
- graph complexes...

Conclusion

	closed strings	open strings	open and closed
classical theory	$\mathcal{C}om$	$\mathcal{A}ss$	$\mathcal{OC}$
modular envelope	$\downarrow$	$\downarrow$	
quantum theory	$\mathcal{QC}$	$\mathcal{QO}$	$\mathcal{QOC}$
tree level Feyn. d.	$\Omega(\mathcal{C}om)$	$\Omega(\mathcal{A}ss)$	$\Omega(\mathcal{OC})$
full Feyn. diagrams	$\mathbf{F}(\mathcal{QC})$	$\mathbf{F}(\mathcal{QO})$	$\mathbf{F}(\mathcal{QOC})$

generalizations: D-branes, different cyclic suboperads...

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Thank you for your attention!