The Moyal Sphere

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Joint project with
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Constant curvature metrics on $\mathbb{R}^2$

- Consider the conformally rescaled metric on the plane:

$$ds^2 = k(x,y)^2(dx^2 + dy^2).$$

- We assume that $k = k(r)$, then the scalar curvature is:

$$R(k) = 2k(r)^{-4}\left(k'(r)k'(r) - \frac{k(r)}{r}k'(r) - k(r)k''(r)\right).$$

- The equation $R(k) = \text{const.}$ admits a family of solutions:

$$k(r) = \frac{Ar^{a-1}}{b + r^{2a}}, \qquad \text{for } A > 0,\ a > 1/2 \text{ and } b > 0,$$

- with the scalar curvature, the volume and the Gauss–Bonnet term:

$$R(k) = \frac{8a^2b}{A^2}, \qquad V(k) = \pi\frac{A^2}{ba}, \qquad \int\sqrt{g}R = 8\pi a.$$

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What is a *curved* noncommutative space?

Spectral triple approach

Scalar curvature à la Connes

Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be a n-summable spectral triple. Its scalar curvature is a functional R on $\mathcal{A}$ defined by

$$R(a) := ⨍ a\,|\mathcal{D}|^{-n+2} = \operatorname*{Res}_{z=0}\ \operatorname{Tr} a\,|\mathcal{D}|^{-n+2-s}\,. \tag{1}$$

Conformally rescaled Dirac $\mathcal{D}_h^2 = h\mathcal{D}^2 h$ and its problems:

- $h[\mathcal{D},h] \neq [\mathcal{D},h]h \Rightarrow$ no 'order expansion' of $|\mathcal{D}_h|^{-s}$.
- $\mathcal{D}_h^2$ is not, in general, a minimal operator.
  [A. Sitarz, *J. Pseudo-Diff. Oper. Appl.* (2014) 5:305–317]

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Alternative curvatures on noncommutative 2-torus

- Curvature à la Rosenberg [J. Rosenberg, SIGMA **9** (2013) 071]
  - Define a Riemannian metric: $\langle \delta_i, \delta_j \rangle = g_{ij} \in \mathcal{A}_\theta$.
  - Levi-Civita theorem $\Rightarrow$ Riemann curvature R_{ijkl}.
  - But R is not unique, as $g^{11} g^{22} R_{1212} \neq g^{11} R_{1212} g^{22}$!
- Orthonormal frame approach [L. Dąbrowski, A. Sitarz, JMP **54** (2013) 013518]
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The Moyal plane

The Moyal plane $A_\theta = (\mathcal{S}(\mathbb{R}^2), *)$, $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$

$$(f * g)(x) := (2\pi)^{-2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\xi(x-y)} f(x - \tfrac{1}{2}\Theta\xi)\, g(y)\, d^n y\, d^n \xi.$$

- $\partial_{x_1}, \partial_{x_2}$ are still derivations on the deformed algebra.
- The matrix basis for A_θ:

$$f_{0,0}(r,\phi) = 2e^{-\frac{1}{\theta}r^2},$$

$$f_{m,n}(r,\phi) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i\phi(m-n)} \left(\sqrt{\frac{2}{\theta}}r\right)^{(n-m)} L_m^{n-m}\left(\frac{2r^2}{\theta}\right) e^{-\frac{r^2}{\theta}}.$$

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- ONB approach
  - Take $h = h^* \in \mathcal{M}(A_\theta)$ and define $e_i = h\delta_i$.
  - The curvature then reads:

$$R = 2h^2 * (\delta_{ii} h) * h_*^{-1} - 2h^2 * \delta_i(h) * h_*^{-1} * \delta_i(h) * h_*^{-1}.$$

  - $\Rightarrow$ need to find h such that

$$\Delta(h) - \delta_i(h) * h_*^{-1} * \delta_i(h) = C h_*^{-1}, \qquad \Delta = \delta_i \delta_i.$$

- Rosenberg approach
  - With $g_{11} = g_{22} = H$, $R_{1212} = -\frac{1}{2}\left(\delta_{ii}(H) - \delta_i(H) * H^{-1} * \delta_i(H)\right)$.
  - $R_{1212} \Rightarrow$ an *unambigious* equation for H:

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- Note: $\widetilde{R}(H = h^{-2}) \neq R(h)$, unless $\theta = 0$!

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Conformal rescaling of the Moyal plane

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  - Take $h = h^* \in \mathcal{M}(A_\theta)$ and define $e_i = h\delta_i$.
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- We look for radial solutions $h = \sum_{n=0}^{\infty} \phi_n f_{n,n}$.
- $R(h) = C \Rightarrow$ a recurrence relation for ϕ_n:

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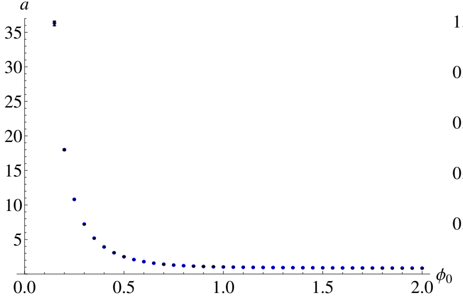

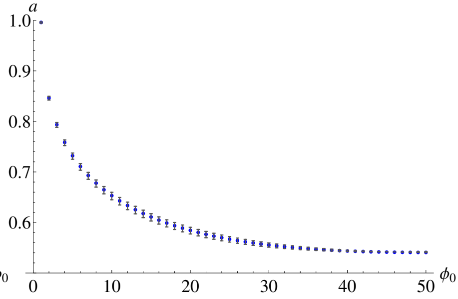

Moyal–Fubini–Study

- We need $\phi_n^{-2} = o(n^{-1})$, i.e. $a > 1/2$, to have a finite volume.
- With $\phi_n \sim A n^a$ we obtain

$$\tau(h_*^{-2} * R) = \int_{\mathbb{R}^2} \sqrt{g} R = 8\pi a.$$

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$$h = \sum_{n=0}^{\infty} \phi_n f_{n,n}, \qquad \text{with } \phi_n \sim n.$$

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$$\phi_n = n + \frac{1}{2}(r+1) + \frac{1}{8}\frac{1}{n} - \frac{13r+9}{144}\frac{1}{n^2} + \frac{1}{32}\left(-\frac{1}{4} + \frac{26}{9}r + \frac{29}{18}r^2\right)\frac{1}{n^3}.$$

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Perturbative solution

- Perturbative expansion of the Moyal product of two radial functions:

$$f * g = f\,g - \tfrac{\theta^2}{8r}\left(f''g' + f'g''\right) + o(\theta^2).$$

- Take $h(r) = h_{FS}(r) + \theta^2\epsilon(r) + o(\theta^2)$, with $h_{FS}(r) = (1 + r^2)$.

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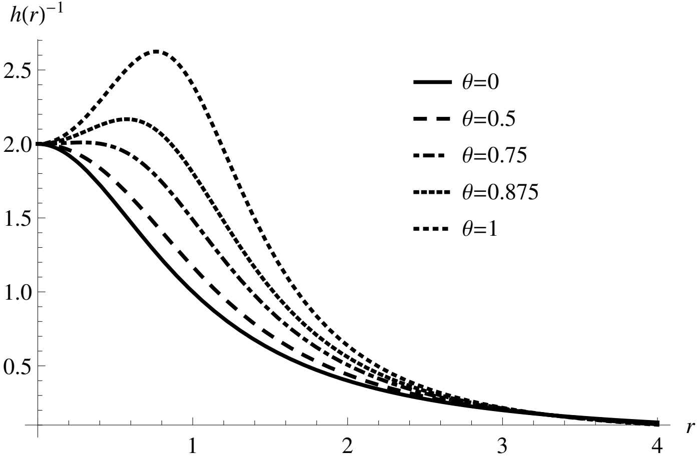

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Thank you for your attention!

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- M. Eckstein, A. Sitarz, R. Wulkenhaar: *The Moyal sphere*, arXiv:1601.05576 [math.QA]