Fundamental Interactions from Strict Deformations

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Bayrischzell Workshop 2016

Outline

Aims for the talk

- Predict quantum physical effects from Deformation

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- Study Non-Abelian Deformational extensions of QM and QFT

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- Predict quantum physical effects from Deformation
- Study Non-Abelian Deformational extensions of QM and QFT
- Investigate Emergent Gravity from Deformation of Space-time

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Q_μ self-adjoint and abelian $[Q_\mu, Q_\nu] = 0$,

$$\Longrightarrow \boxed{\exists \text{ strongly continuous unitary group } U(k) := e^{ik^\mu Q_\mu}}$$

Buchholz, Lechner, Summers '10:

Definition of Deformation

Let Θ be a skew-symmetric matrix, then the warped convolution $A_{\Theta,Q}$ of $A \in \mathcal{C}^\infty$ is

$$A_{\Theta,Q}\Phi := \lim_{\epsilon\to 0} \iint dx\, dy\, \chi(\epsilon x, \epsilon y)\, e^{-ixy} U(\Theta x)\, A\, U(-\Theta x) U(y)\Phi, \qquad \Phi \in \mathcal{D} \subset \mathscr{H}$$

Deforming the Hamiltonian with $\mathbf{Q}(\mathbf{X})$*

Free Hamiltonian:

$$\boxed{H = -P_j P^j/(2m) = -\Delta/(2m)}$$

Proposition

The scalar product $\langle \Psi, H_{\Theta,\mathbf{Q}(\mathbf{X})}\Phi\rangle$ is bounded,

$$|\langle \Psi, H_{\Theta,\mathbf{Q}(\mathbf{X})}\Phi\rangle| \leq C_{\Theta}\,||\Psi||, \qquad \forall \Psi \in \mathscr{H},\ \ \Phi \in \mathcal{E} \subseteq \mathscr{S}(\mathbb{R}^3).$$

Therefore, the deformation for H is well-defined, self-adjoint** and the result is

$$H_{\Theta,\mathbf{Q}(\mathbf{X})}\Phi = -\frac{1}{2m}\Big(P_j + i(\Theta Q)^k[Q_k, P_j]\Big)\Big(P^j + i(\Theta Q)^r[Q_r, P^j]\Big)\Phi.$$

* AM., JMP. Vol. 55, 022302 (2014)

**AM., JMP. Vol. 56, Issue 9, (2015)

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Example $Q = X$

Result for $H_{\Theta,X}$

$$H_{\Theta,X}\Psi = -\frac{1}{2m}(P_j + \Theta_{jk}X^k)(P^j + \Theta^{jr}X_r)\Psi = -\frac{1}{2m}P_j^{\Theta,X}P^j_{\Theta,X}\Psi$$

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$$H_{\Theta,X}\Psi = -\frac{1}{2m}(P_j + \Theta_{jk}X^k)(P^j + \Theta^{jr}X_r)\Psi = -\frac{1}{2m}P_j^{\Theta,X}P^j_{\Theta,X}\Psi$$

Lemma

*Let the **deformation matrix** Θ_{ij} be given as,*

$$\Theta^1_{ij} = -(e/2)\,\varepsilon_{ijk}B^k, \qquad \Theta^2_{ij} = m\,\varepsilon_{ijk}\,\Omega^k$$

*where B^k is a **magnetic field** (MF) and $\Omega^k = (2GM/r_{hs})\omega^k$ is a **gravitomagnetic field** (GMF). Then, $H_{\Theta^1,\mathbf{X}}$ becomes the Hamiltonian of the **Landau quantization** and $H_{\Theta^2,\mathbf{X}}$ of a QM-particle coupled to a **GMF**.*

Quantum Plane from QM

In lowest LL motion described by $Q_i = X_i + (\Theta^{-1})_{ik} P^k$,

$$\boxed{[Q_i, Q_j] = 2i(\Theta^{-1})_{ij}.}$$

Can one obtain the noncommuting coordinates from warped convolutions?

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Lemma

$X_i^{\theta,P}$ satisfies the commutation relations of the Moyal-Weyl plane $\mathbb{R}^3_{-2\theta}$,

$$[X_i^{\theta,P}, X_j^{\theta,P}] = -2i\theta_{ij}.$$

If $-\theta_{ij}$ is $(\Theta^{-1})_{ij}$, then $X_i^{\theta,P}$ are equal to the guiding center coordinates Q_i.

$\Longrightarrow$ Idea of lemma can be used for Gravitomagnetism and QFT!

Take $X_{\Theta^{-1},P}$ to be GCC of an electron in LL,

$$\boxed{[X_i^{\Theta^{-1},P}, X_j^{\Theta^{-1},P}] = 2i(\Theta^{-1})_{ij}}$$

with $\Theta_{ij} = m\varepsilon_{ijk}\Omega^k = m\varepsilon_{ijk}(2GM/r_{hs})\omega^k$ where Ω^k is a **gravitomagnetic field** .

Uncertainty relations for coordinates in GMF

$$(\Delta X_2^{\Theta^{-1},P})(\Delta X_3^{\Theta^{-1},P}) \geq \hbar/(m\Omega).$$

$\Longrightarrow$ Physical effect deduced from deformation can be experimentally verified!

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Proposition

Let τ_α be inf. generators of $SU(m)$,

$$[\tau_\alpha, \tau_\beta] = if_{\alpha\beta\gamma}\tau^\gamma \qquad f_{\alpha\beta\gamma} \in \mathbb{C}, \quad \alpha, \beta, \gamma = 1, \cdots, m^2 - 1.$$

Define s.a. operator $Q(X)$ on $\mathcal{D}_Z \otimes \mathbb{C}^m \subset \mathscr{H} = \mathscr{L}^2(\mathbb{R}^d) \otimes \mathbb{C}^m$ as

$$Q(X)_\mu := Z(X)_\mu \otimes Y^\alpha \tau_\alpha, \qquad \mu = 0, 1, \cdots, n$$

$\Rightarrow \exists$ str. con. group on $\mathscr{H}$:

$$U^\tau(p) := \exp(ip^\mu Q_\mu(X)) = \sum_{r=1}^{m} U(\lambda_r p) \otimes W B_r W^{-1}, \qquad \forall p \in \mathbb{R}^d.$$

Definition of Deformation

Let Θ be a skew-symmetric matrix and $\Psi \in \mathcal{D} \subset \mathscr{H}$. Then A^{Θ_τ} of $A \in C^\infty$ is

$$A^{\Theta_\tau}\Psi := \lim_{\epsilon\to 0} \iint dx\, dy\, \chi(\epsilon x, \epsilon y)\, e^{-ixy}\, U^\tau(\Theta x)\, A\, U^\tau(-\Theta x)\, U(y)\Psi$$

*AM: Arxiv: 1511.07891

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Proposition

NA def. momentum op. (using $Q_k = Z(\mathbf{X})_k \otimes Y_\alpha \tau^\alpha$), is well-defined on $D_\mathcal{E}$:

$$P_i^{\Theta_\tau} = P_i - qA_{i,\alpha}\tau^\alpha,$$

where the gauge field $\vec{A}_\alpha$ is given as

$$-q\vec{A}_\alpha := (\Theta Z(\mathbf{X}))_k \vec{\partial} Z(\mathbf{X})^k \otimes Y_\alpha.$$

Non-Abelian Moyal-Weyl Plane

The def. coord. op. $\vec{X}^{\Theta_\tau}$ (using $Q_k = P_k \otimes Y_\alpha \tau^\alpha$) is well-defined on $D_\mathcal{E}$ and given

$$X^{i,\Theta_\tau} = X^i - (\Theta P)^i \otimes Y_\alpha \tau^\alpha.$$

Moreover, the non-abelian Moyal-Weyl plane is generated by the algebra of $\vec{X}^{\Theta_\tau}$

$$[X^{i,\Theta_\tau}, X^{j,\Theta_\tau}] = -2i\Theta^{ij} \otimes Y_\alpha \tau^\alpha.$$

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Deformation of QF with P corresponds to QF on Moyal-Weyl!

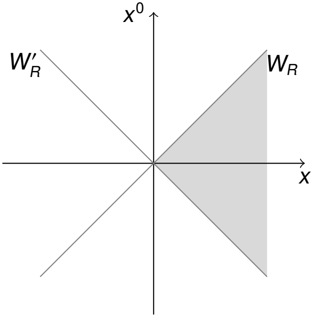
$$\phi_\theta(f)\Psi := (2\pi)^{-d} \iint dy\, dk\, e^{-iyk} U(\theta y)\phi(f)U(-\theta y + k)\Psi, \qquad \Psi \in \mathcal{D}$$

- Deformed field fulfills the same bounds as undeformed field
- $\phi_\theta(f)$ satisfies all Wightman axioms, except for covariance and locality.

Covariance and locality replaced by modified versions!

*Grosse, Lechner: JHEP, 0711:012, 2007

$\phi_{\theta,P}(f)$ defined as QF's on wedge by map $Q : W \mapsto Q(W)$



Definition

Let $\phi = \{\phi_W : W \in \mathcal{W}_0 := \mathcal{L}_+^\uparrow W_1\}$ denote fields satisfying the Wightman axioms. Then, ϕ is defined to be a wedge-local QF if the following holds:

- **Covariance:** For any $W \in \mathcal{W}_0$ and $f \in \mathscr{S}(\mathbb{R}^d)$ the following holds

$$U(y,\Lambda)\phi_W(f)U(y,\Lambda)^{-1} = \phi_{\Lambda W}(f \circ (y,\Lambda)^{-1}), \qquad (y,\Lambda) \in \mathcal{P}_+^\uparrow$$

- **Wedge-locality:**

$$[\phi_{W_1}(f), \phi_{-W_1}(g)]\Psi = 0, \quad \Psi \in \mathcal{D},$$

  for all $f, g \in C_0^\infty(\mathbb{R}^d)$ with supp $f \subset W_1$ and supp $g \subset -W_1$.

Deformed Field $\phi_{\theta,P}$ is a wedge-local QF $\Longrightarrow$ Calculate two-particle scattering matrix!

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Wedge-Local QF's on Non-abelian and NC Space-time.*

Deformation of QF with $Q(\mathbf{P})_\mu = -P_\mu \otimes Y_\alpha \tau^\alpha$ corresponds to QF on NA Moyal-Weyl!

$$\phi_{\Theta_\tau}(f)\Psi := (2\pi)^{-d} \iint dy\, dk\, e^{-iyk}\, U^\tau(\theta y)\phi(f)U^\tau(-\theta y)U(k)\Psi, \qquad \Psi \in \mathcal{D} \otimes \mathbb{C}^m$$

- Deformed field fulfills the same bounds as undeformed field
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Proposition

The deformed fields $\phi_{\Theta_\tau}(f)$ transform under the adjoint action of $\mathcal{P}_+^\uparrow$ as follows,

$$U(x,\Lambda)\phi_{\Theta_\tau}(f)U(x,\Lambda)^{-1} = \phi_{(\Lambda\Theta\Lambda^T)_\tau}(f \circ (x,\Lambda)^{-1}), \qquad (y,\Lambda) \in \mathcal{P}_+^\uparrow,$$

$$U(0,j)\phi_{\Theta_\tau}(f)U(0,j)^{-1} = \phi_{(-\Lambda\Theta\Lambda^T)_\tau}(\bar{f} \circ (0,j)^{-1}).$$

Therefore, the field ϕ_{Θ_τ} is a wedge-covariant field.

Proposition

By choosing Y to be matrix valued such that $Y_\alpha\tau^\alpha$ has positive eigenvalues, the family of QF's $\phi = \{\phi_W : W \in \mathcal{W}_0\}$ defined by $\phi_W(f) := \phi(Q(W), f) = \phi_{\Theta_\tau}(f)$ are **wedge-local** on the Bosonic Fockspace $\mathscr{H} \otimes \mathbb{C}^m$.

Outline

Commutative Algebra

Let $\mathcal{A}_C$ be generated by $\hat{x}^\mu$ fulfilling,

$$[\hat{x}^\mu, \hat{x}^\nu] = 0.$$

Define $\wedge^r(\mathcal{A}_C)$ r-forms and a $\mathbb{C}$-linear $d : \wedge^r(\mathcal{A}_C) \to \wedge^{r+1}(\mathcal{A}_C)$ satisfying,

$$d^2 = 0, \qquad d(ww') = (dw)w' + (-1)^r w dw',$$

By applying d on the CR of $\mathcal{A}_C$

$$[d\hat{x}^\mu, \hat{x}^\nu] + [\hat{x}^\mu, d\hat{x}^\nu] = 0,$$

we find two possible solutions. The commutative one and a more general solution

$$[\hat{x}^\mu, d\hat{x}^\nu] = \sum_{\sigma=0}^{n} C^{\mu\nu}_{\ \ \sigma}\, d\hat{x}^\sigma.$$

*AM, in preparation

Define flat metric $\eta \in \wedge^1(\mathcal{A}_C) \otimes_{\mathcal{A}_C} \wedge^1(\mathcal{A}_C)$ as tensor product of one forms,

$$\eta = \eta_{\mu\nu} d\hat{x}^\mu \otimes_{\mathcal{A}_C} d\hat{x}^\nu.$$

Deform the flat metric with abelian warped convolutions

$$\eta_\Theta = \int U(\Theta p)(\eta_{\mu\nu} d\hat{x}^\mu \otimes_{\mathcal{A}_C} d\hat{x}^\nu) U(-\Theta p)\, dE(p),$$

where $U(p) = \exp(ip_\mu \hat{x}^\mu)$.

Choose algebra $[\hat{x}^i, d\hat{x}^j] = i\delta^{ij} a\, e^j d\hat{x}^j$ and deform $\Longrightarrow$ by choosing $\Theta_{ik} = 0$, $\Theta_{0i} = \Theta e_i$ and $\Theta a = H$

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Deform the flat metric with abelian warped convolutions

$$\eta_{\Theta} = \int U(\Theta p)(\eta_{\mu\nu} d\hat{x}^{\mu} \otimes_{\mathcal{A}_C} d\hat{x}^{\nu}) U(-\Theta p)\, dE(p),$$

where $U(p) = \exp(i p_{\mu} \hat{x}^{\mu})$.

Choose algebra $[\hat{x}^i, d\hat{x}^j] = i\delta^{ij} a\, e^j d\hat{x}^j$ and deform $\Longrightarrow$ by choosing $\Theta_{ik} = 0$, $\Theta_{0i} = \Theta e_i$ and $\Theta a = H$

Metric of Inflationary Phase (FRW)

Curved Metric obained by a strict deformation of the flat space-time.

$$\eta_{\Theta} = d\hat{x}^0 \otimes_{\mathcal{A}_C} d\hat{x}^0 - \exp(-H\hat{x}^0) d\hat{x}^k \otimes_{\mathcal{A}_C} d\hat{x}_k$$

Outline

Conclusion

- From deformation $\Rightarrow$ well-known QM effects and predicted new ones

- Non-Abelian deformations give new class of wedge-local fields

- (Quantum) space-time generated by deformation

Work in Progress

- Extend Deformations to SUSY QM
- Investigate how far program of deformation of ST extends
- Study deformation in Non-relativistic QFT
- . . .

Thank you for your Attention!