"Universal" group cocycles

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Joint work with

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Suppose that G is a group acting strictly on a category $\mathcal{C}$ enriched over the category of bimodules over a ring R.

Basic assumptions

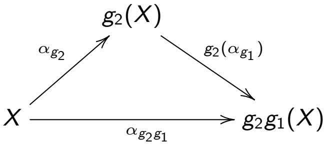
- For any object X of $\mathcal{C}$, there exists a non-zero morphism in $Mor_{\mathcal{C}}(X, g(X))$
- For any object X of $\mathcal{C}$, $Aut_{\mathcal{C}}(X, X) = R^*$.

Fix an object X of $\mathcal{C}$ and, for any $g \in G$, choose an invertible morphism $\alpha_g \in Mor_{\mathcal{C}}(X, g(X))$.

Definition

$$c(g_1, g_2) = \alpha_{g_2 g_1}^{-1} \circ g_2(\alpha_{g_1}) \circ \alpha_{g_2} \in Aut(X) = R^*.$$

The picture is as follows



Theorem

$c(g_1, g_2)$ is an R^*-valued cocycle on G and its class in $H^2(G, R^*)$ is independent of the choices made.

A version of this construction is due to Brylinski.

Example: Loop group extension.

Suppose that (H, H_+) is a polarised Hilbert space, and $GL_{res}(H)$ is the corresponding group of automorphisms. If P denotes the orthogonal projection onto the subspace H_+, $GL_{res}(H)$ consists of bounded invertible linear transformations u of H satisfying

$$u^{-1}[u, P] \in \mathcal{L}^2(H).$$

Recall that the determinant of a Fredholm operator T is the complex line

$$det(T) = \Lambda^{top} Ker(T) \otimes \Lambda^{top} Coker(T)^*$$

The category $\mathcal{C}$ is defined as follows

- Objects of $\mathcal{C}$ are given by group elements $u \in G$;
- $Mor(u, v) = det(vPv^{-1}uPu^{-1})$, where $vPv^{-1}uPu^{-1}$ is considered a Fredholm operator from uH_+ to vH_+.
- The composition of morphisms is given by the composition of maps

$$det(wPw^{-1}vPv^{-1}) \otimes det(vPv^{-1}uPu^{-1}) \to det(wPw^{-1}vPv^{-1}uPu^{-1})$$

and

$$det(wPw^{-1}vPv^{-1}uPu^{-1}) \to det(wPw^{-1}uPu^{-1})$$

Some more comments on this. The top map is a natural isomorphism, corresponding to the fact that Index map is additive on Fredholm operators. For the bottom map note that

$$wPw^{-1}PuPu^{-1} - wPw^{-1}uPu^{-1} = -wP[w^{-1}, P][P, u]Pu^{-1}$$

which, by our assumption, is in $\mathcal{L}^1(H)$. Given that, we use the perturbation isomorphism, given by the following theorem.
First a definition. A Fredholm complex is a finite complex $\mathcal{C} = \{\mathcal{H}_i, T_i\}$ of Hilbert spaces with finite dimensional cohomology and where all the boundary maps have closed range. The determinant of such a complex is given by

$$det(\mathcal{C}) = \bigotimes_{n=0 mod(2)} \Lambda^{top}(H^n(\mathcal{C})) \otimes \bigotimes_{n=1 mod(2)} \Lambda^{top}(H^n(\mathcal{C}))^*$$

Theorem

Suppose that T is a Fredholm complex and that $T + \delta$, $\delta \in \mathcal{L}^1(H)$, its perturbation. There exists a canonical isomorphism

$$det(T) \to det(T + \delta)$$

compatible with composition of perturbations and the mapping cone construction of intertwiners of Fredholm complexes.

The group 2-cocycle constructed above leads to

1. The universal loop group extension

$$1 \to \mathbb{T} \to \widetilde{LK} \to LK \to 1$$

via the standard construction (for a compact Lie group K).

Choose a finite dimensional unitary representation h of K and set

$$H = L^2(\mathbb{T}, h), H_+ = H^2(\mathbb{T}, h) \subset H,$$

where H^2 is the Hardy space of functions of the form $\sum_{n\geq 0} a(n)e^{in\theta}$, θ the coordinate on $\mathbb{T}$. The natural action of $LK = C^\infty(\mathbb{T}, K)$ on H gives a homomorphism $LK \subset GL_{res}(H)$.

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2. Given an algebra $A \subset B(H)$ such that $A^{inv} \subset GL_{res}(H)$, any two commuting invertible elements $u, v \in A^{inv}$ produces an element

$$\{u, v\} \in K_2^{alg}(A).$$

The map from $K_*^{alg}(A) \to H_*(GL(A), \mathbb{Z})$ produces a group cycle

$$ch\{u, v\} = u \otimes v - v \otimes u$$

and the pairing with the group two cocycle constructed above produces the Tate symbol $[u, v] \in \mathbb{C}^*$.
In another disguise this is the universal Connes-Karoubi cocycle associated to a two-summable Fredholm module

$$(A, H, F).$$

Here $H = H_+ \oplus H_-$, F is a partial isometry with domain H_+ and range H_- and A is the subalgebra of bounded operators on H satisfying $[a, F], [a, F^*] \in \mathcal{L}^2$. This is more or less immediate consequence of the definitions.

Suppose that G is a group acting strictly on a category $\mathcal{C}$ enriched over a the category of bimodules over a ring R.

Basic assumptions

We will assume that $\mathcal{C}$ satisfies the following conditions.

- For any two objects X and Y of $\mathcal{C}$, $Mor_1(X, Y)$ contains an invertible element.
- For any 1-isomorphism $\alpha \in Mor_1(X, Y)$, $Mor_2(\alpha, \alpha) = R$.
- For any two 1-morphisms α and β in $Mor_1(X, Y)$, $Mor_2(\alpha, \beta)$ contains an invertible element .

We fix an object $X \in \mathcal{C}$ and, for any $g \in G$, a 1-isomorphism

$$\alpha_g : X \to g(X).$$

For a pair $g_1, g_2 \in G$, above choice produces a pair of 1-morphisms

$$\alpha_{g_1 g_2} : X \to g_1 g_2(X)$$

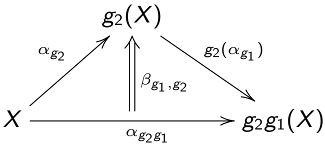
and

$$g_1(\alpha_{g_2}) \circ_1 \alpha_{g_1} : X \xrightarrow{g_1} g_1(X) \xrightarrow{g_1(\alpha_{g_2})} g_1(g_2(X)).$$

By assumption, we can choose a 2-isomorphism

$$\beta_{g_1,g_2} : \alpha_{g_1 g_2} \Longrightarrow g_1(\alpha_{g_2}) \circ_1 \alpha_{g_1}.$$

The β's are the two morphisms associated to the 2-simplices according to the rule

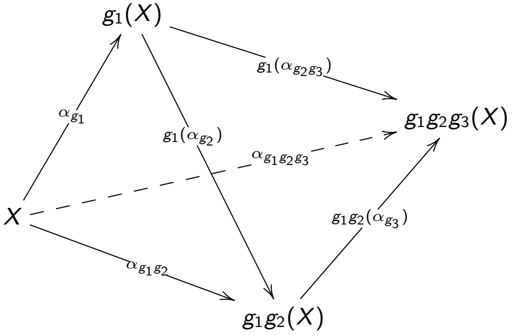


Definition

Given elements g_1, g_2, g_3 of G, we set

$$c(g_1, g_2, g_3) = \beta^{-1}_{g_1, g_2 g_3} \circ_2 g_1(\beta^{-1}_{g_2, g_3}) \otimes 1 \circ_2 (1 \otimes \beta_{g_1, g_2}) \circ \beta_{g_1 g_2, g_3} \in Aut(\alpha_{g_1 g_2 g_3})$$

The following picture might help.



Theorem

c is an R^*-valued cocycle on the group G and its class in $H^3(G, R^*)$ is independent of the choices made above.

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The same kind of construction works in higher dimensions. Given an action of a group G on an n-category with the same type of properties as above, there is an associated group (n+1)-cocycle. This, in particular, applies to Tate spaces.

The associated group cocycles are *conjecturally* the same as Beilinson regulators on the K-theory of the associated field of fractions. The first non-trivial case is the field $\mathbb{C}((s))((t))$.

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The main question is whether one can construct analogous higher cocycles in an analytic context. These should exist since, by work of Connes and Karoubi, $\mathcal{L}^p$-summable Fredholm modules produce functionals on K_n^{alg}. The first interesting case is that of a two-category.

As noted above, a polarisation of a Hilbert space H leads to a central extension of the restricted general linear group of H. Our higher analogue of a polarisation will be as follows.

Basic assumptions

P and Q are two idempotents in $\mathcal{L}(H)$ and G is a subgroup of $GL(H)$ satisfying the following conditions.

For any two elements g and h of G,

1. $[gPg^{-1}, hQh^{-1}] \in \mathcal{L}^2(H)$;
2. $[g, P][h, Q] \in \mathcal{L}^2(H)$.

A quick comment. In distinction to the one-dimensional case, given two idempotents in general position, there is no "universal" group G satisfying the conditions above.

Definition

The two category $\mathcal{C}$ has

1. Objects

$$Obj(\mathcal{C}) = G$$

Alternatively, idempotents $Q_u, u \in G$, where $Q_u = uQu^{-1}$.

2. 1-morphisms from u to v are given by finite sequences of the form

$$\{u \xrightarrow{g_1} w_1 \xrightarrow{g_2} w_2 \longrightarrow \ldots\ldots \xrightarrow{g_k} w_k \xrightarrow{g_{k+1}} v\}$$

where $w_1, \ldots w_k j, g_1, g_2, \ldots, g_{k+1} \in G$. The composition of one-morphisms is given by concatenation of paths.

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where $w_1, \ldots w_k j, g_1, g_2, \ldots, g_{k+1} \in G$. The composition of one-morphisms is given by concatenation of paths.

The definition (construction) of the two-morphisms will occupy us for most of the rest of this talk.

First a regularised version of the K-theory class $[Q_u] - [Q_v]$. Set

$$F = uQu^{-1}vQv^{-1} + u(1-Q)v^{-1},$$

$$G = vQv^{-1}uQu^{-1} + v(1-Q)u^{-1}$$

and

$$E = \begin{pmatrix} (2-FG)FG & (1-FG)(2-FG)F \\ G(1-FG) & (1-GF)^2 \end{pmatrix}, E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

E and E_0 are both bounded idempotents, acting on the direct sum $H \oplus H$.

We will think of $[E] - [E_0]$ as a representative of $[Q_u] - [Q_v]$ in the appropriate K(I)-group, where I is the ideal generated by the products $(Q_u - Q)(Q_v - Q)$, where $u, v \in G$ and we use the notation $Q_u = uQu^{-1}$.

In fact, one checks that the matrices $E - E_0$ has coefficients within the ideal generated by products of the form $(Q_u - Q)(Q_v - Q)$, $u, v \in G$.

Before we start, a useful analytic result.

Lemma

Let $p \in [1, \infty)$ and suppose that $T : H \to H$ is a bounded operator with $T^2 - T \in \mathcal{L}^p(H)$. Then there exists a bounded idempotent $E : H \to H$ with $T - E \in \mathcal{L}^p(H)$. In particular, suppose that $E, F : H \to H$ are two idempotents with $[E, F] \in \mathcal{L}^2(H)$. Then there exists an idempotent $E' : EH \to EH$ such that $EFE - E' \in \mathcal{L}^1(EH)$.

We fix two objects of our category, say invertible elements $u, v \in G$.

Suppose first that we have a pair of elements of $Mor_1(u, v)$ of the form

$$\{u \xrightarrow{g} v\} \text{ and } \{u \xrightarrow{k} v\}$$

Step 1.

Set

$$K = \left\{ \begin{array}{cc} P_g P_k & P_g(1 - P_k P_g) \\ P_k(1 - P_g P_k) & (P_k P_g)^2 - 2P_k P_g \end{array} \right\}.$$

and

$$L = \left\{ \begin{array}{cc} 2P_k P_g - (P_k P_g)^2 & P_k(1 - P_g P_k) \\ P_g(1 - P_k P_g) & -P_g P_k \end{array} \right\},$$

where we use the notation

$$P_g = diag\ (gPg^{-1}, gPg^{-1}).$$

By the above lemma we can choose, for each pair $u, g \in G$, a bounded idempotent E_g^u such that

1. $[E_g^u, P_g] = 0$ and
2. $E_g^u - P_g Q_u P_g \in \mathcal{L}^1(H)$

$$T_{(g,k)}^{(u,v)} :== (E_g \oplus E_k^0 \oplus E_g^0 \oplus E_k^0)(K \oplus L)(E_k \oplus E_g^0 \oplus E_k^0 \oplus E_g^0)$$
$$: \mathrm{Rg}\,(E_k \oplus E_g^0 \oplus E_k^0 \oplus E_g^0) \to \mathrm{Rg}\,(E_g \oplus E_k^0 \oplus E_g^0 \oplus E_k^0)$$

$T_{(g,k)}^{(u,v)}$ is a Fredholm operator

Definition

$$Mor_2\left(\{u \xrightarrow{g} v\}\ ,\ \{u \xrightarrow{k} v\} \right) = Det\left(T_{(g,k)}^{(u,v)}\right).$$

The main point of the above is a "fancy" way of writing the following. Suppose that P's and Q's commute. The pairing of the "Fredholm module" $\left\{ \begin{matrix} 0 & P_g P_k \\ P_k P_g & 0 \end{matrix} \right\}$ to a "K_0-class" $[Q_u] - [Q_v]$ produces a Fredholm operator

$$P_g \oplus P_k : P_k Q_u H \oplus P_g Q_v H \to P_g Q_v H \oplus P_k Q_u H$$

and the two-morphisms are given by its determinant line. The concrete choices made above allow us to compose two-morphisms.

To be more concrete, let us sketch the construction of the composition

$$Mor_2\left(\{u \xrightarrow{g} v\}, \{u \xrightarrow{k} v\}\right) \times Mor_2\left(\{u \xrightarrow{k} v\}, \{u \xrightarrow{l} v\}\right) \longrightarrow Mor_2\left(\{u \xrightarrow{g} v\}, \{u \xrightarrow{l} v\}\right)$$

In the construction of the line of two morphisms it is anly the K-part that plays non-trivial role, so what we have is the following picture.

$$E_k \oplus E_g^0 \xrightarrow{K_{g,k}} E_g \oplus E_k^0$$

$$E_l \oplus E_k^0 \xrightarrow{K_{k,l}} E_k \oplus E_l^0$$

We replace $K_{g,k}$ by $K_{g,k} \oplus 1_{E_l H}$ and $K_{k,l}$ by $K_{k,l} \oplus 1_{E_g H}$. The composition

$$\left(K_{k,l} \oplus 1_{E_g H}\right) \circ \left(K_{g,k} \oplus 1_{E_l H}\right)$$

differs from $K_{k,l} \oplus 1_{E_k H}$ by an operator in $\mathcal{L}^1(H)$, hence there is a canonical perturbation isomorphism producing the composition of corresponding two-morphisms.

Given two strings

$$\{u \xrightarrow{g_1} w_1 \xrightarrow{g_2} w_2 \longrightarrow \ldots\ldots \xrightarrow{g_k} w_k \xrightarrow{g_{k+1}} v\}$$

and

$$\{u \xrightarrow{k_1} v_1 \xrightarrow{k_2} v_2 \longrightarrow \ldots\ldots \xrightarrow{k_n} v_n \xrightarrow{k_{n+1}} v\}$$

we refine them to the form

$$\{u \xrightarrow{g_1} v_1 \xrightarrow{g_1} \ldots \xrightarrow{g_1} v_n \xrightarrow{g_1} w_1 \xrightarrow{g_2} \ldots \xrightarrow{g_k} w_k \xrightarrow{g_{k+1}} v\}$$

and

$$\{u \xrightarrow{k_1} v_1 \xrightarrow{k_2} \ldots \xrightarrow{k_n} v_n \xrightarrow{k_{n+1}} w_1 \xrightarrow{k_{n+1}} \ldots \xrightarrow{k_{n+1}} w_k \xrightarrow{k_{n+1}} v\}$$

and the line of two-morphisms is given by the tensor product of the lines of two-morphisms beween two length one strings constructed above.

Theorem

The construction above does define a two-category.

The above construction produces non-trivial cocycles!

Set $H = L^2(\mathbb{T}^2)$, Q and P the corresponding projections onto functions holomorphic in the first and second variable. Then we can choose $G = C^\infty(\mathbb{T}^2)^{inv}$. The computation of the corresponding group 3-cocycle gives, for a constant function λ,

$$c(z_1, z_2, \lambda) = \lambda.$$

In particular, c gives an extension of the Tate symbol from $K_3^{alg}(\mathbb{C}((z_1))((z_2)))$ to $K_3^{alg}(C^\infty(\mathbb{T}^2))$.