Conjugate variables in quantum field theory and non-commutative coordinates

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Talk
Joint work with A. Much, S. Pottel

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Outline

Motivation

Structure of space-time: solution of a dynamical problem
Until now: only a dream
Trial: non-commutative space-time
Realization: e.g. via coordinate operators and Moyal product of fields

$$[Q_\mu, Q_\nu] = 2i\theta_{\mu\nu}$$

Typical interaction

$$S_{\text{int}} = g \int d^4x \, (\phi * \phi * \phi)(x)$$

$$(f * g)(x) = exp(\frac{i}{2}\theta_{\mu\nu}\frac{\partial}{\partial y_\mu}\frac{\partial}{\partial z_\nu})f(y)g(z)_{|y=z=x}$$

Q_μ enters via tensor product on state space. Why? How?

Conjugate pairs

Q_ν "conjugate" to P_μ

$$[P_\mu, Q_\nu] = i\eta_{\mu\nu}$$

Then

$$\hat{Q}_\nu = Q_\nu + \theta_{\nu\lambda}P^\lambda \to \left[\hat{Q}_\mu, \hat{Q}_\nu\right] = [Q_\mu, Q_\nu] + 2i\theta_{\mu\nu}$$

Q_ν not directly accessible, proceed via

$$[P_\mu, X_\nu] = iO_{\mu\nu}$$

X_ν "preconjugate" operator, "divide" by O to obtain Q
search for X: use geometrical and group theoretical notions realizable in (perturbative) QFT, ex.: scalar field
X: differential operators on (one-particle) wave fcts
charge-like operators on Fock space
differential operators on Green fcts
Q: $\exists$? $\in$ algebra of observables ?

X from geometry; $X^{(\nabla)}$

- $$X_\nu^{(\nabla)} f(p) = i\nabla_\nu f(p) \equiv i\left(\frac{\partial}{\partial p^\nu} - \frac{p_\nu}{m^2} p^\lambda \frac{\partial}{\partial p^\lambda}\right) f(p)$$

$X^{(\nabla)}$ "tangential derivative" [Coleman/Mandula] to $p^2 = m^2$
(covariant derivative on $p_0 = +\omega_p$ submanifold of $\mathbb{R}^4$; geometric notion!)
algebra:

$$\begin{aligned} P_\mu f(p) &= p_\mu f(p) \\ \left[P_\mu, X_\nu^{(\nabla)}\right] &= i(\eta_{\mu\nu} - \tfrac{1}{m^2} P_\mu P_\nu) \\ \left[X_\mu^{(\nabla)}, X_\nu^{(\nabla)}\right] &= \tfrac{i}{m^2} M_{\mu\nu} \qquad (M = \text{Lorentz, on-shell}) \end{aligned}$$

- On Fock space:

$$
\begin{aligned}
P_\mu(a,a^\dagger) &= \int \frac{d^3p}{2\omega_p} p_\mu a^\dagger(\mathbf{p}) a(\mathbf{p}) \\
P_\mu|\mathbf{p}> &= p_\mu|\mathbf{p}> \\
X_\nu^{(\nabla)}(a,a^\dagger) &= \frac{i}{2}\int \frac{d^3p}{2\omega_p}\left(a^\dagger(\mathbf{p})\nabla_\nu a(\mathbf{p}) - \nabla_\nu a^\dagger(\mathbf{p}) a(\mathbf{p})\right) \\
X_\nu^{(\nabla)}|\mathbf{p}> &= i(\nabla_\nu - \frac{3}{2}\frac{p_\nu}{m^2}|\mathbf{p}> \\
\left[P_\mu, X_\nu^{(\nabla)}\right] &= i\int \frac{d^3p}{2\omega_p}(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}) a^\dagger(\mathbf{p}) a(\mathbf{p}) \\
\left[X_\mu^{(\nabla)}, X_\nu^{(\nabla)}\right] &= \frac{i}{m^2} M_{\mu\nu} \qquad (M = \text{Lorentz, on Fock space})
\end{aligned}
$$

- On off-shell ("Green" fct.): $X^{(\mathrm{com})}$ differential operator

$$
\begin{aligned}
X^{(\mathrm{com})}_\mu &= M_{\mu\lambda}\frac{P^\lambda}{P^2} \qquad \text{Lorentz vector}\\
\left[P_\mu, X^{(\mathrm{com})}_\nu\right] &= i(\eta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2})\\
\left[X^{(\mathrm{com})}_\mu, X^{(\mathrm{com})}_\nu\right] &= iM_{\mu\nu}\frac{1}{P^2}
\end{aligned}
$$

$$
\phi = \phi(p) \qquad P_\mu\phi = p_\mu\phi(p)
$$

$$
X^{(\mathrm{com})}_\mu\phi(p) = i\left(\frac{\partial}{\partial p^\mu} - \frac{p_\mu}{p^2}p^\lambda\partial_\lambda\right)\phi(p)
$$

$X^{(\nabla)} \to Q^{(\nabla)}$

aim: construct $Q^{(\nabla)}$ such that

$$\left[P_\mu, Q_\nu^{(\nabla)}\right] |\mathbf{p}_1, ..., \mathbf{p}_n >= i\eta_{\mu\nu} n|\mathbf{p}_1, ..., \mathbf{p}_n >$$

$n = 0$ vacuum: r.h.s.=0, operator N projector ;
fits to charge-like P, X

$$\left[P_\mu, X_\nu^{(\nabla)}\right] |\mathbf{p} >= i\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right) |\mathbf{p} >$$

$n = 1$ contract with P^μ find r.h.s.=0: $(\text{matrix})_{\mu\nu}$ projector
replace $|\mathbf{p} >$ by $X(\nabla)^\nu |\mathbf{p} >$

$$\left[P_\mu, X_\nu^{(\nabla)}\right] X(\nabla)^\nu |\mathbf{p} >= i\eta_{\mu\nu} \nabla^\nu |\mathbf{p} >$$

$[P_\mu, X_\nu]$ proportional to $i\eta_{\mu\nu}$ on state $\epsilon(\mathbf{p})|\mathbf{p} >$, $\epsilon(\mathbf{p})$ non-trivial fct

alternatively: $n > 1$ r.h.s. invertible; invert

Inversion on “spin states”

$$\sum_{l=1}^{3} \epsilon_\mu^{(l)}(\mathbf{p})\epsilon_\nu^{(l)}(\mathbf{p}) = -\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)$$

spin sum of massive vector particle, introduce one-particle states

$$|\mathbf{p}, l, \mu> = \epsilon_\mu^{(l)}(\mathbf{p})|\mathbf{p}> \qquad \epsilon_\rho^{(l)}(\mathbf{p}) = \begin{pmatrix} \frac{p^l}{m} \\ -\delta_j^l + \frac{p^l p_j}{m(m+\omega_p)} \end{pmatrix}$$

$$\Longrightarrow \sum_{l=1}^{3} \epsilon_\mu^{(l)}(\mathbf{p})\epsilon_\nu^{(l)}(\mathbf{p})\nabla^\nu = -\nabla_\mu \qquad i[X^\nu, [i[P_\mu, X_\nu], a^\dagger]] = \nabla_\mu a^\dagger$$

$$\Longrightarrow [P_\mu, X_\nu]\eta^{\nu\rho}\epsilon_\rho^{(l)}|\mathbf{p}> = i\eta_{\mu\nu}\eta^{\nu\rho}\epsilon_\rho^{(l)}|\mathbf{p}>$$

$$\text{normalization}: <\mathbf{p}', l', \rho'|\mathbf{p}, l, \rho> = 2\omega_p\delta(\mathbf{p}' - \mathbf{p})\epsilon_{\rho'}^{(l')}(\mathbf{p})\epsilon_\rho^{(l)}(\mathbf{p}) \quad | -\eta^{\rho'\rho}$$

$$Q^{(l)}_{\rm eff}|\mathbf{p}>:=[X^\rho,\epsilon^{(l)}_\rho(\mathbf{p})a^\dagger(\mathbf{p})]|0> \;=\; -ie^{(l)}a^\dagger(\mathbf{p})|0>$$
$$e^{(l)} \;\equiv\; (-\delta^l_k+\frac{p^l p_k}{m(m+\omega_p)})\frac{\partial}{\partial p_k}$$

at $\mathbf{p}=0$ $\quad Q^{(l)}_{\rm eff}=i\partial/\partial p_l \quad l=1,2,3$

for finite $\mathbf{p}$: extend matrix of polarization vectors to boost $(L(p))_\sigma{}^\rho$

$$Q^{(\rm eff)}_\lambda|\mathbf{p}> \;=\; -i(L^{-1})_\lambda{}^\nu\nabla_\nu|\mathbf{p}>$$
$$Q^{(\rm eff)}_0|\mathbf{p}>=0 \qquad Q^{(\rm eff)}_j|\mathbf{p}>=ie_j|\mathbf{p}>$$

for commutators with P introduce $P^{(eff)}_j:=(L^{-1})_j{}^\nu P_\nu$

$$[P_\mu,Q^{(\rm eff)}_0] \;=\; \left[P^{(\rm eff)}_\mu,Q^{(\rm eff)}_0\right]=0$$
$$\left[P^{(\rm eff)}_\mu,Q^{(\rm eff)}_l\right]|\mathbf{p}> \;=\; i\eta_{\mu l}|\mathbf{p}>$$

analogy: quantized massive vector field; Lorentz covariant embedding of “3” into “4”; implementation of Darboux’s theorem!

Summary

• for $Q(\nabla) \Rightarrow Q_0^{(\mathrm{eff})} = Q_o = 0, Q_j^{(\mathrm{eff})} = ie_j \to i\partial/\partial p^j$ at $\mathbf{p} = 0$

refinement of Pauli's theorem:
if Q_0 conjugate to P_0 $\Rightarrow$ Q_0 not self-adjoint

reason: Fock states are asymptotic, time $=\pm\infty$ fixed

• $Q \leftrightarrow$ tensorial factor? yes: $\epsilon \times |... >$ needed for inversion of commutator $[P, X]$ and resolving the norm issue

• $Q(\nabla)$ is non-commutative:

$$\left[Q_j, Q_k\right] |\mathbf{p}> = -iM_{jk}|\mathbf{p}> \qquad \text{rotation}$$

origin: the polarization vectors

Further problems

- extend by tensoring from $n = 1$ to general n: explicit form?
- include spin ($\rightarrow$ gauge theories)
- construct a non-commutative theory with the help of Q

Thank you for your attention!

X from geometry; $X^{(<)}$

wedge variables

$$
\begin{aligned}
p_u &= \frac{1}{\sqrt{2}}(p_0 - p_1) & \qquad p_0 &= \frac{1}{\sqrt{2}}(p_v + p_u)\\
p_v &= \frac{1}{\sqrt{2}}(p_0 + p_1) & \qquad p_1 &= \frac{1}{\sqrt{2}}(p_v - p_u).
\end{aligned}
$$

mass-shell constraint: $2p_u p_v - p_a p_a = m^2$ summation over $a = 2, 3$
(both shells covered!)

$$
\begin{aligned}
\nabla^u &= \frac{\partial}{\partial p_u} - \frac{p^u}{m^2} p_\lambda \frac{\partial}{\partial p_\lambda} & \qquad \nabla^2 &= \frac{\partial}{\partial p_2} - \frac{p^2}{m^2} p_\lambda \frac{\partial}{\partial p_\lambda}\\
\nabla^v &= \frac{\partial}{\partial p_v} - \frac{p^v}{m^2} p_\lambda \frac{\partial}{\partial p_\lambda} & \qquad \nabla^3 &= \frac{\partial}{\partial p_3} - \frac{p^3}{m^2} p_\lambda \frac{\partial}{\partial p_\lambda}.
\end{aligned}
$$

tangential derivatives $\rightarrow X^{(<)} := i\nabla(p_u, p_v, p_a)$

note: in n-particle case def. of center of mass & relative variables possible

X from geometry; $X^{(<_0)}$

massless limit: $< \to <_0$ light wedge variables, non-trivial
spacetime: $(1,3) \to (1,1) \oplus (0,2)$ sym.: $SO(1,3) \to SO(1,1) \times SO(2)$
tangential derivatives

$$\nabla^u = \frac{1}{2}\left(\frac{\partial}{\partial p_u} - \frac{1}{p_u} p_v \frac{\partial}{\partial p_v}\right) \qquad \nabla^v = \frac{1}{2}\left(\frac{\partial}{\partial p_v} - \frac{1}{p_v} p_u \frac{\partial}{\partial p_u}\right)$$

$$\nabla^2 = \frac{\partial}{\partial p_2} - \frac{p^2}{p_a p^a} p_b \frac{\partial}{\partial p_b} \qquad \nabla^3 = \frac{\partial}{\partial p_3} - \frac{p^3}{p_a p^a} p_b \frac{\partial}{\partial p_b}$$

definition: $X^{(<_0)} = i\nabla(<_0)$ algebra:

$$\left[X_u^{(<_0)}, X_v^{(<_0)}\right] = i\frac{1}{P_a P^a} M_{uv} \qquad \left[X_2^{(<_0)}, X_3^{(<_0)}\right] = i\frac{1}{2P_u P_v} M_{23}$$

$$\left[P_\alpha, X_\beta^{(<_0)}\right] = \frac{-i}{2}\begin{pmatrix} \frac{-P_u}{P_v} & 1 \\ 1 & \frac{-P_v}{P_u} \end{pmatrix} \qquad \alpha, \beta = u, v$$

$$\left[P_a, X_b^{(<_0)}\right] = -i\begin{pmatrix} 1 + \frac{P_2 P_2}{P_b P^b} & \frac{-P_2 P_3}{P_b P^b} \\ \frac{-P_3 P_2}{P_b P^b} & 1 + \frac{P_3 P_3}{P_b P^b} \end{pmatrix} \qquad a, b = 2, 3$$

X from geometry; $X^{(\omega)}$

$$X_\nu^{(\omega)} = i(-\frac{p_\nu}{{\omega_p}^2} p^l \frac{\partial}{\partial p^l}) \qquad \omega_p^2 \equiv -p^l p_l$$

note:
$X^{(\omega)}$ even for $m = 0$ not a Lorentz vector, however relates to $X^{(\mathrm{SiSo})}$ algebra:

$$[P_\mu, X_\nu^{(\omega)}] = i\frac{p_\mu p_\nu}{\omega_p^2} \qquad [X_\mu^{(\omega)}, X_\nu^{(\omega)}] = 0$$

$1/\omega_p^2$: curvature of sphere $p_l p_l = p_0^2 = \text{fixed} \neq 0$
sphere *not* submanifold of double cone $p_0 = \pm\omega$.
$X^{(\omega)}$ causes some motion in p-space, this motion is not a translation
no $Q^{(\omega)}$ will result in $\mathbb{R}^4$

X from group theory; $X(\mathrm{x-conformal})$

$x^2 = 0$ invariant under $\delta^K_\mu x_\lambda = 2x_\mu x_\lambda - \eta_{\mu\lambda}x^2$
in Fourier space for function $f(x)$ of p-dim d

$$\begin{aligned} X_\mu(K(x - \mathrm{conf})) &:= K_\mu(x - \mathrm{conf}) = i\delta^K_\mu \\ \delta^K_\mu \tilde{f}(p) &= (-2(d-4)\tfrac{\partial}{\partial p^\mu} + 2p^\lambda \tfrac{\partial^2}{\partial p^\lambda \partial p^\mu} - p_\mu \tfrac{\partial^2}{\partial p^\rho \partial p_\rho})\tilde{f}(p) \end{aligned}$$

smooth transition off-shell $\rightarrow$ on-shell (s.b.)
on-shell: $p^2 = 0, \omega_p^2 = -p_l p^l, \partial/\partial p_0 = 0$
on Fock space:

$$\begin{aligned} K_0 &= \int \frac{d^3p}{2\omega_p}\omega_p \, a^\dagger(\mathbf{p})\partial^l\partial_l a(\mathbf{p}) \\ K_j &= \int \frac{d^3p}{2\omega_p} a^\dagger(\mathbf{p}) \left(p_j\partial^l\partial_l - 2p^l\partial_l\partial_j - 2\partial_j\right) a(\mathbf{p}) \end{aligned}$$

X from group theory; $X(\mathrm{p-conformal})$

$p^2 = 0$ invariant under under $\delta^K_\mu p_\lambda = 2p_\mu p_\lambda - \eta_{\mu\lambda} p^2$

find:

$$X_\mu^{(\mathrm{p-conf})}(a^\dagger, a) = 2(d - \frac{3}{2})\alpha P_\mu \qquad \alpha \in \Re$$

Hence no need to study further this case.

Realization of X in terms of field ϕ

find:
$X^{(\nabla)}$ non-local, depending explicitly on x
$X = X(\mathrm{x-conf}) = K$ local
reason: conserved current, $\exists$ charge
systematic search within Green fcts. of scalar field
demand for X_μ:
– Lorentz vector
– local in x-space
– conserved in time
– permits transition on/off-shell
– charge-like on Fock space
find: $X_\mu = K_\mu$ charge generating the special conformal transformations

Swieca/Völkel (CMP **29**(1973)319): essentially self-adjoint operator

Extension: (Anti)-deSitter

recapitulate: K preconjugate to P
i.e. Poincaré group $ISO(1,3)$ subgroup within rep$\{SO(2,4)\} \Rightarrow$
Minkowski space $\mathbb{R}^4 = ISO(1,3)/SO(1,3)$ means:
P singled out for finding (pre-)conjugate partners
extension to (anti-)deSitter space:
conformal algebra $so(2,4)$, generators $J_{ab}, a, b \in \{-1, 0, ..., 4\}$
compactify spaces, introduce contraction parameter ϵ, define

$$I_{4\mu} = P_\mu - \frac{\epsilon}{4} K_\mu \qquad I_{-1,\mu} = \frac{\epsilon}{4} P_\mu + K_\mu$$

$$\lim_{\epsilon \to 0} I_{4\mu} = P_\mu \qquad \lim_{\epsilon \to 0} I_{-1\mu} = K_\mu$$

$$\lim_{\epsilon \to 0} \left[I_{4\mu}, I_{-1\mu} \right] = 2i(\eta_{\mu\nu} D - M_{\mu\nu})$$

conclusion:
for spacetime with isometry group $SO(2,4)$: $J_{4,\mu}, J_{-1,\nu}$ candidates for (pre-)conjugate partners

Application: Deformation

A. Much: PhD-thesis; J.Math.Phys. 53(2012)081303

$\phi(x)$ quantum field, warped convolution with K yields non-constant, noncommutative spacetime

Inversion on standard states

$$n = 2 \qquad [P_\mu, X_\nu^{(\nabla)}]|\mathbf{p}_1, \mathbf{p}_2 >= 2i\left(\eta_{\mu\nu} - \frac{p_\mu^{(1)}p_\nu^{(1)}}{2m^2} - \frac{p_\mu^{(2)}p_\nu^{(2)}}{2m^2}\right)|\mathbf{p}_1, \mathbf{p}_2 >$$

inversion possible; go to cms; rotate to 0 y, z-components of $\mathbf{p}$
conjugation equs.

$$[P_\mu, Q_\nu]|\mathbf{p}, -\mathbf{p} >= \; 2i\eta_{\mu\nu}|\mathbf{p}, -\mathbf{p} >$$

$$Q_0 = -\frac{m^2}{p_x^2}X_0 \quad Q_1 = \frac{m^2}{2(m^2 + p_x^2)}X_1 \quad Q_2 = X_2 \qquad Q_3 = X_3$$

norm of $\mu = \nu = 0$ state opposite to norm of $\mu = \nu = j$;
impose Gupta-Bleuler condition

$$\left(-\frac{2m^2}{p_x^2}\alpha_0 + \frac{m^2}{m^2 + p_x^2}\alpha_1\right)|\mathbf{p}, -\mathbf{p} >= 0 \qquad \alpha \in \Re$$

solution only for $m^2 = 0$, $\alpha_1 = 2\alpha_0$;
inversion not consistent for $m^2 \neq 0$

result for $Q^{(\nabla)}$:

$m^2 \neq 0$	$Q_0 = 0$	3 spatial conjugate pairs standard Fock space, “Landau gauge“
$m^2 = 0$	$Q_0 = Q_1 = 0$	2 spatial conjugate pairs standard Fock space

check of results: off-shell

$$Q_\mu = X_\mu^{(com)} + (D - \hat{Y})\frac{P_\mu}{P^2}$$

$$\text{yields} \qquad Q_\mu \phi(p) = i\frac{\partial}{\partial p^\mu}\phi(p)$$

$$\text{satisfies} \qquad [P_\mu, Q_\nu] = i\eta_{\mu\nu} \quad \text{on} \quad \phi(p)$$

transition on-shell: $p_0 = \omega_p, \partial/\partial p_0 = 0, Q_0 = 0$

Results for $Q(K)$

case I: spacetime $(1,3)$ symmetry: $SO(2,4)$
$Q_0 = 0$ choice of Coulomb gauge
$Q_j|\mathbf{p}>= i\partial_j|\mathbf{p}>$
from: $Q_j|\mathbf{p}>=\frac{1}{2}\left(K_j + K^r\frac{P_rP_j}{P_0^2} + 2(D-i)^2\frac{P_j}{P_0^2}\right)D^{-1}|\mathbf{p}>$
with: $[P_0, Q_j]|\mathbf{p}>= -i\frac{p_j}{\omega_p}|\mathbf{p}>$ introduce Coulomb gauge polarization vectors $\epsilon^{(\lambda)}$
find within $[P_\mu, Q_\nu]\epsilon^{(\lambda)}|\mathbf{p}>= iC_{\mu\nu}\epsilon^{(\lambda)}|\mathbf{p}>$:
$\epsilon^{(0)} \to 0, \epsilon^{(1)} \to \epsilon^{(0)}$, transverse to transverse
choose quotient $\{\lambda = 0,1,2,3\}/\{\lambda = 0,1\}$ as state space
$\Rightarrow$ find 2 spatial conjugate pairs

check off-shell with

$$Q_\mu = \frac{1}{2}(P^\lambda M_{\mu\lambda} + M_{\mu\lambda}P^\lambda)\frac{1}{P^2} + \frac{1}{2}(D - Y - i)\frac{P_\mu}{P^2}$$

(Lagu/Laue Nuovo Cim. A20 217(1974))

case II: spacetime $(1,1)+(0,2)$

symmetry: conformal in these dimensions

split states: $|\mathbf{p}>=|p_1>|p_a>\equiv|p_1;p_a> \qquad a=2,3$

$$\text{find} \quad Q_a|p_1;p_a>=i\partial_a|p_1;p_a> \qquad a=2,3.$$

$$\text{from} \quad Q_a|p_1;p_a>=\left[K_a-(D-i)^2\frac{P_a}{P^bP_b}\right]D^{-1}|p_1;p_a>$$

2 spatial conjugate pairs in this subspace

in subspace $(1,1)$: 1 state with negative norm, 1 state with positive norm; Gupta-Bleuler condition; effectively no further conjugate pair

state space: quotient

state space symmetry: $SO(1,1)\times SO(2)$

Results for $Q(<_0)$

spacetime: (1,1)+(0,2) symmetry: $SO(1,1) \times SO(2)$
$SO(2)$-sector

$$\epsilon_a^{(2)} = \frac{1}{|\mathbf{p}|}\begin{pmatrix} p_2 \\ p_3 \end{pmatrix} \qquad \epsilon_a^{(3)} = \frac{1}{|\mathbf{p}|}\begin{pmatrix} -p_3 \\ p_2 \end{pmatrix} \qquad a = 2,3$$

$p_2 = |\mathbf{p}| \cos\alpha, p_3 = |\mathbf{p}| \sin\alpha$

$$\begin{aligned} Q^{(2)}_{\text{eff}}|\mathbf{p}> &= 0 \\ Q^{(3)}_{\text{eff}}|\mathbf{p}> &= \frac{i}{|\mathbf{p}|}(-p_3\frac{\partial}{\partial p_2} + p_2\frac{\partial}{\partial p_3})|\mathbf{p}> . \end{aligned}$$

$$\begin{aligned} \left[P_2, Q^{(3)}_{\text{eff}}\right]|\mathbf{p}> &= -i\sin\alpha|\mathbf{p}> \\ \left[P_3, Q^{(3)}_{\text{eff}}\right]|\mathbf{p}> &= i\cos\alpha|\mathbf{p}> \end{aligned}$$

1 spatial conjugate pair

SO(1, 1)-sector

$$p_u = \frac{c}{\sqrt{2}}(\cosh\phi - \sinh\phi) = \frac{c}{\sqrt{2}}e^{-\phi} \qquad c = \sqrt{2p_u p_v} = \sqrt{p_a p_a}$$
$$p_v = \frac{c}{\sqrt{2}}(\cosh\phi + \sinh\phi) = \frac{c}{\sqrt{2}}e^{+\phi} \qquad \phi = -\frac{1}{2}\ln\frac{p_u}{p_v} = \frac{1}{2}\ln\frac{p_v}{p_u}$$

$$\epsilon_\alpha^{(u)} = \frac{N}{\sqrt{2}}\begin{pmatrix} -\frac{p_u}{p_v} \\ 1 \end{pmatrix} \qquad \epsilon_\alpha^{(v)} = \frac{N}{\sqrt{2}}\begin{pmatrix} 1 \\ -\frac{p_v}{p_u} \end{pmatrix}$$

$$Q^u_{\text{eff}} = -i\frac{N}{\sqrt{2}}\nabla^v \simeq \frac{e^{-\phi}}{c}\frac{\partial}{\partial\phi} \qquad Q^v_{\text{eff}} = -i\frac{N}{\sqrt{2}}\nabla^u \simeq -\frac{e^{\phi}}{c}\frac{\partial}{\partial\phi}$$

vector under $SO(1,1)$
non-trivial commutator

$$[Q^\sigma_{\text{eff}}, Q^\tau_{\text{eff}}]|\cdots> = -\frac{1}{p_a p^a}(p^\sigma\frac{\partial}{\partial p_\tau} - p^\tau\frac{\partial}{\partial p_\sigma}) \equiv \frac{i}{P_a P^a}M^{\sigma\tau}|\cdots>$$

M generator of $SO(1,1)$; non-compact
2 conjugate pairs; motion on the two shells of the hyperboloid

Results for $Q(<)$

realize first $SO(1,1) \times SO(2)$ $\Rightarrow$ 2+1 conjugate pairs

standard Fock space (in wedge variables)

$m^2 \neq 0$: rotations M_{12}, M_{13}, boosts M_{02}, M_{03} exist

$\Rightarrow$ full $SO(1,3)$ realized

Lorentz covariance: re-identify physical states together with polarization vectors

Summary

• for $Q(\nabla), Q(K) \Rightarrow Q_0 = 0, Q_j = i\partial/\partial p^j$ (universality)

refinement of Pauli's theorem:
if Q_0 conjugate to P_0 $\Rightarrow Q_0$ not self-adjoint

reason: Fock states are asymptotic, time $= \pm\infty$ fixed

• for $Q(<)$ operator ordering differently defined, Q_u, Q_v involve "time",
2 pairs in non-compact sector; 1 spatial pair

• $Q \leftrightarrow$ tensorial factor? yes: $\epsilon \times |... >$ needed for inversion of commutator $[P, X]$ and resolving the norm issue

Further problems

extend by tensoring from $n = 1$ to general n: explicit form?

• include spin ($\rightarrow$ gauge theories)
e.g. via susy and supercurrent (contains K)

• understand effect of conformal anomaly for Q