Chern-Simons theory, string dualities, and quantum geometry

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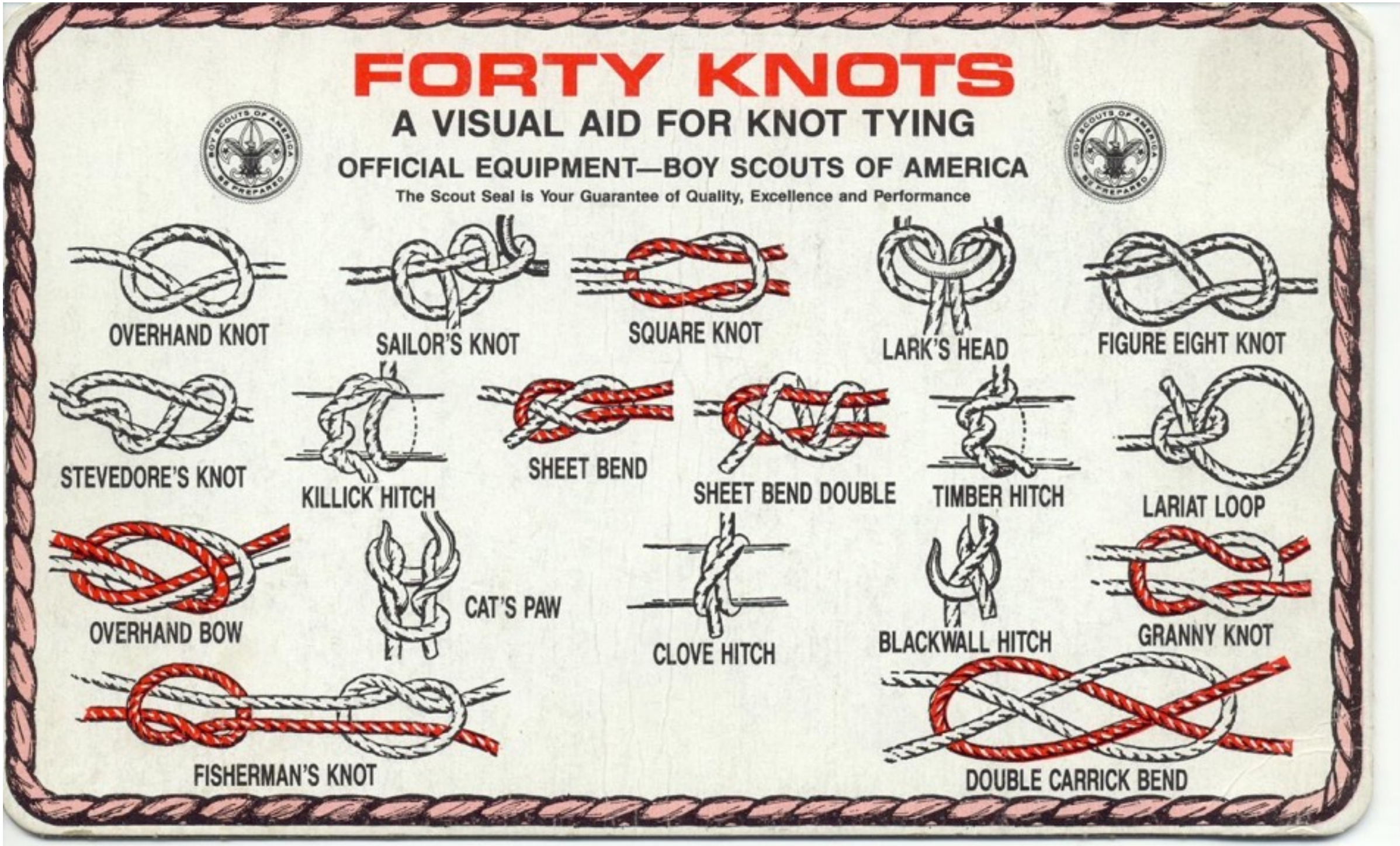
Bayrischzell, May 2016

Why knots?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A VISUAL AID FOR KNOT TYING
OFFICIAL EQUIPMENT—BOY SCOUTS OF AMERICA
The Scout Seal is Your Guarantee of Quality, Excellence and Performance
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SAILOR'S KNOT
SQUARE KNOT
LARK'S HEAD
FIGURE EIGHT KNOT
STEVEDORE'S KNOT
KILLICK HITCH
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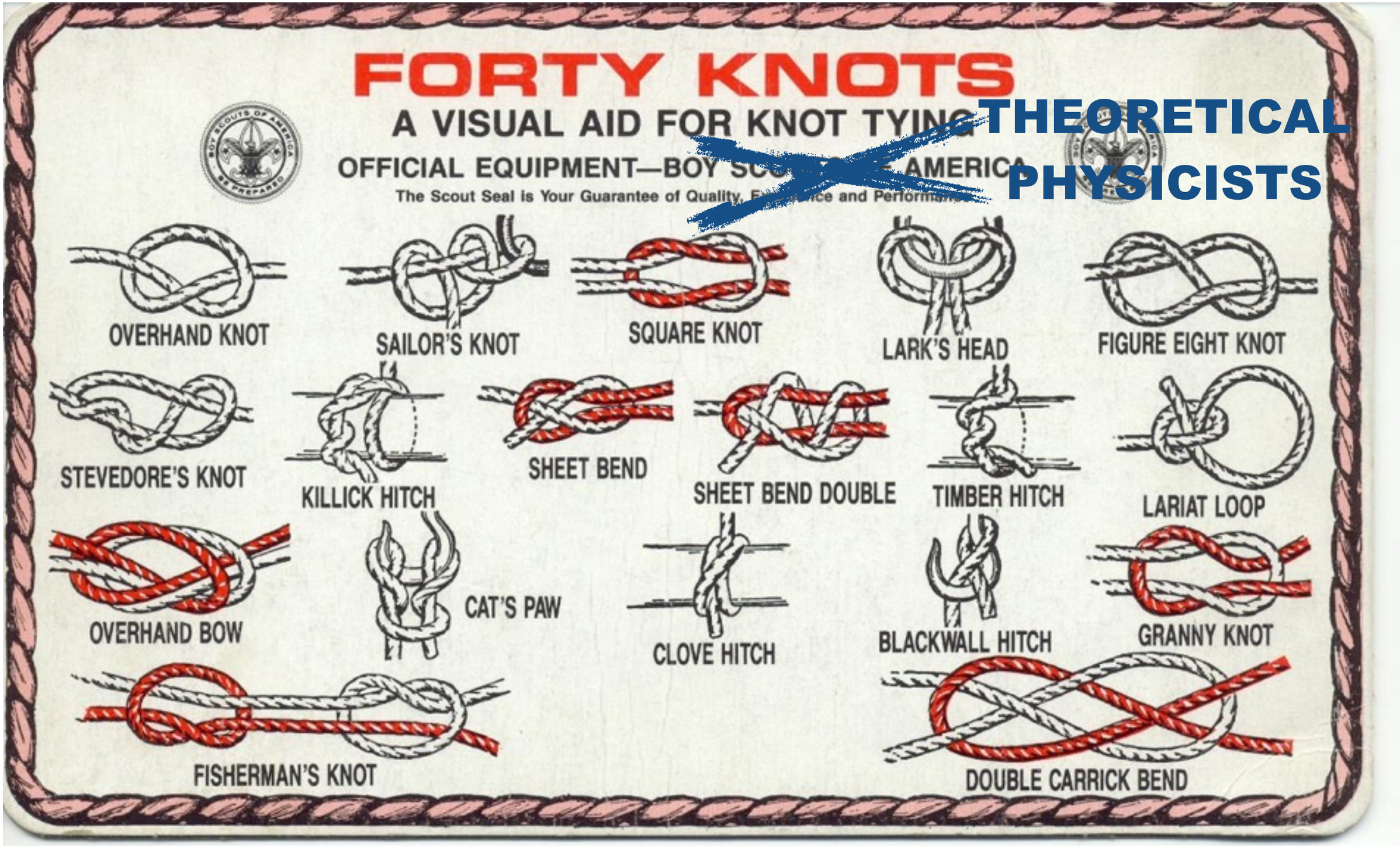
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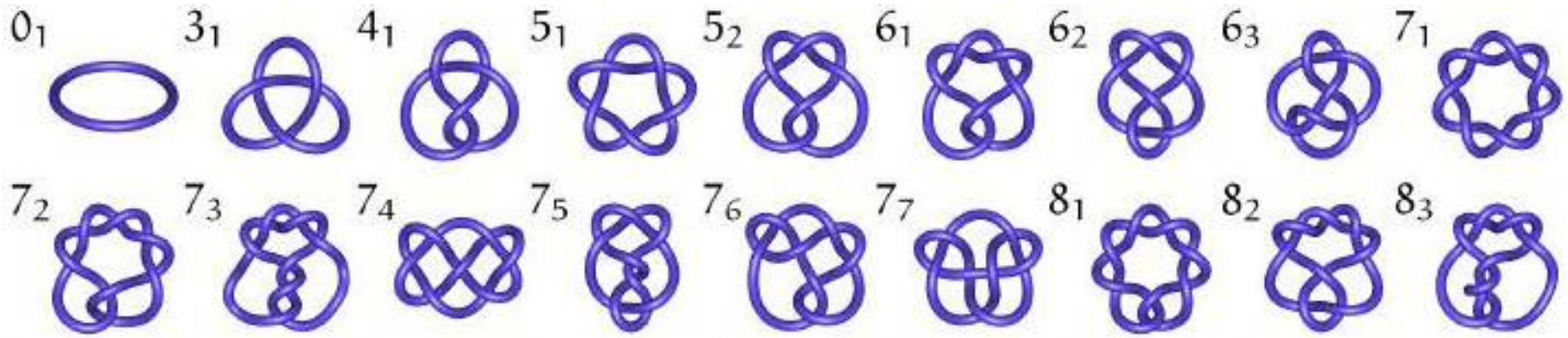

Mathematics: find (perfect) knot invariants

Why knots?

1. Observables in Topological Field Theories (TQFT)

- Wilson loops in Chern-Simons theory

2. Counting of BPS states

- Witten index, indices in SUSY theories
- Gopakumar-Vafa invariants for Calabi-Yau manifolds
- Ooguri-Vafa (LMOV): new knot invariants

3. Tests of dualities in string theory

- geometric transitions
- mirror symmetry

4. Relation to M5-branes and SUSY gauge theories

- 3d-3d correspondence

Why knots: TQFT observables

Consider Chern-Simons theory on M_3 *[Witten, 1989]*:

$$S = \frac{k}{4\pi} \int \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

Why knots: TQFT observables

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Polynomial knot invariants from Wilson loop observables:

$$P_R(a,q) = \left\langle \mathrm{Tr}_R\, e^{\oint A} \right\rangle = \int \mathcal{D}A \left(\mathrm{Tr}_R\, e^{\oint A} \right) e^{\frac{ik}{4\pi} S}$$

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HOMFLY polynomial for SU(N) gauge group:

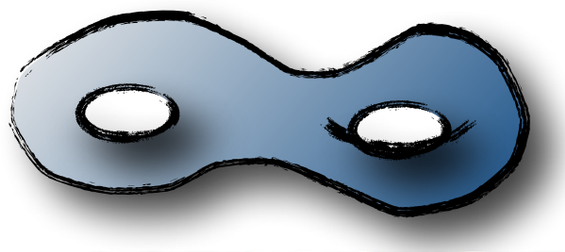
$$q = e^{\frac{2\pi}{k+N}}, \qquad a = q^N$$

Jones polynomial for SU(2), Alexander polynomial for a=1

Why knots: 3d-3d duality

Recall AGT conjecture *[Alday-Gaiotto-Tachikawa, 2009]:*

6d (2,0) theory of M5 branes



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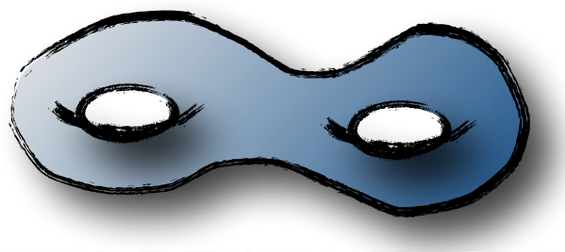
4d, N=2 theories



$$Z^{Nekrasov} \simeq \text{conformal blocks in Liouville theory}$$

Lift this to 3+3 dimensions:

3d, N=2 theories

Knot complement

$$Z^{3d,N=2} \simeq \text{knot invariants, } P_R(a,q)$$

Gromov-Witten theory, BPS states

Engineer string theory background *[Witten, 1993]*:

total 10-dim:	$\mathrm{R}^4 \times T^*S^3$
N D4-branes:	$\mathrm{R}^2 \times S^3$
D4-brane:	$\mathrm{R}^2 \times$ *Lagrangian brane*

and consider **conifold transition** *[Ooguri-Vafa, 1999]*:

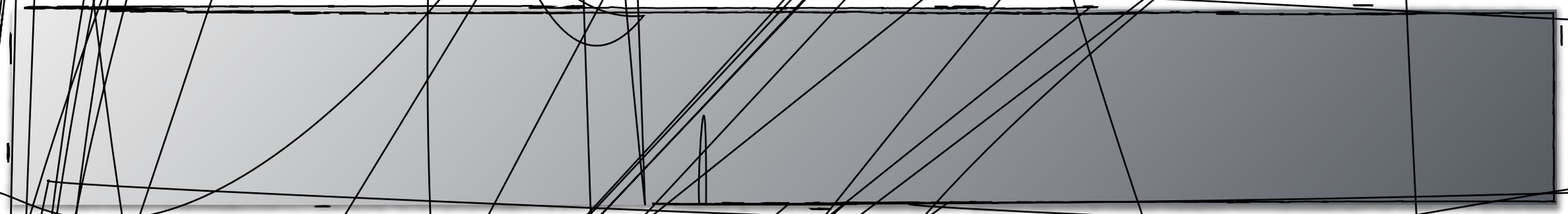
Chern-Simons

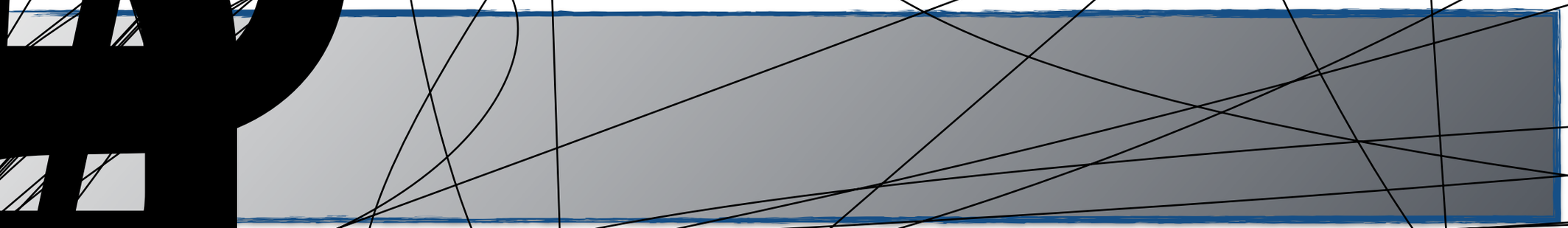
Gromow-Witten

$$T^* S^3 \qquad\qquad \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$$

Why knots: counting of BPS states

Certain Chern-Simons operator takes form *[Ooguri-Vafa, 1999]:*

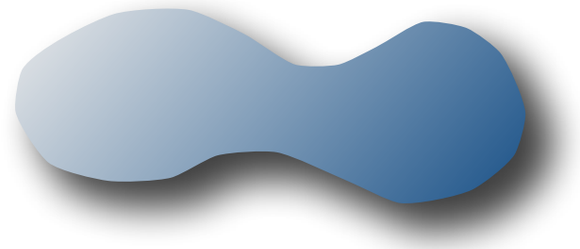




B-model viewpoint

Mirror symmetry:

A-model on toric manifold = B-model on mirror manifold



A-polynomial, *A(x,y)=0*

1. Encodes asymptotics of colored Jones polynomial:

$$J_r(q) \underset{q=e^{\hbar}\to 1, x=q^r}{\simeq} \exp\left(\frac{1}{\hbar}S_0(x) + S_1(x) + \hbar S_2(x) + \dots\right)$$

B-model picture: $S_k(x)$ from topological recursion

2. Recursions for colored Jones:

$$\widehat{A}(\hat{M}, \hat{L}; q) J_r = 0$$

$$\hat{M} J_r = q^r J_r, \qquad \hat{L} J_r = J_{r+1}, \qquad \hat{L}\hat{M} = q\hat{M}\hat{L}$$

Volume (AJ) conjecture:

$$\lim_{q\to 1} \widehat{A}(\hat{M}, \hat{L}; q) \;=\; A(M, L)$$

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???

3. Moduli space of flat connections for SL(2,C) Chern-Simons

4. Moduli space of vacua of a dual 3d N=2 theory (3d-3d duality)

Examples: trefoil & figure-8 knot

(Classical) A-polynomial for **trefoil knot:**

$$A(M,L) = (L-1)(L-M^3)$$

Quantum curve - recursion relations for colored Jones polynomials:

$$\frac{q^{2r}(q^r-q)}{q^{2r}-q}\,J_{r-1} + q\left(1+q^{-r}-q^r+\frac{q-q^r}{q^{2r}-q}-\frac{q^r-1}{q^{2r+1}-1}\right)J_r + \frac{q-q^{-r}}{1-q^{2r+1}}\,J_{r+1} = 0$$

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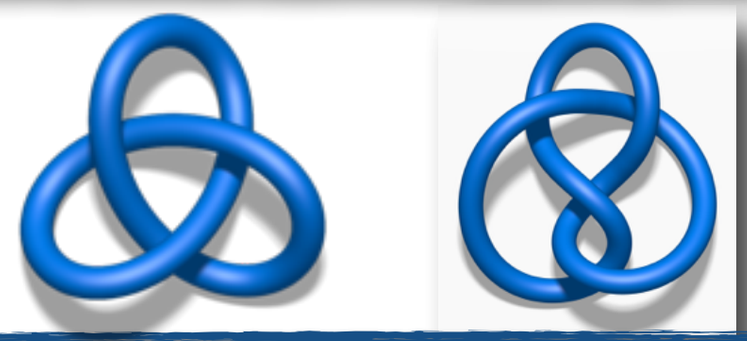

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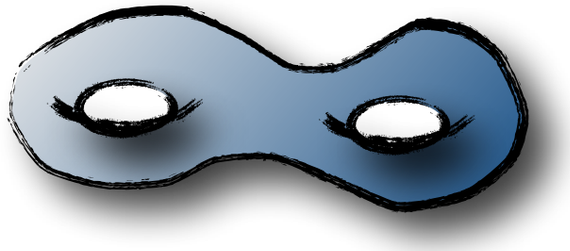
Super-A-polynomial for **figure-8 knot:**

$$\begin{aligned} A^{\text{super}}(M,L,a,t) \quad = \quad & a^4t^5(M^2-1)^2M^4 + a^2t^2M^4(1+a^2t^3M^2)^2L^3 + \\ & +a^2t(M^2-1)(1+t(1-t)M^2+2a^2t^3(t+1)M^4-2a^2t^4(t+1)M^6+ \\ & +a^4t^6(1-t)M^8-a^4t^8M^{10})L-(1+a^2t^3M^2)(1+a^2t(1-t)M^2+ \\ & +2a^2t^2(t+1)M^4+2a^4t^4(t+1)M^6+a^4t^5(t-1)M^8+a^6t^7M^{10})L^2. \end{aligned}$$

Knots, BPS states, and algebraic curves

$N_{R,i,j}$



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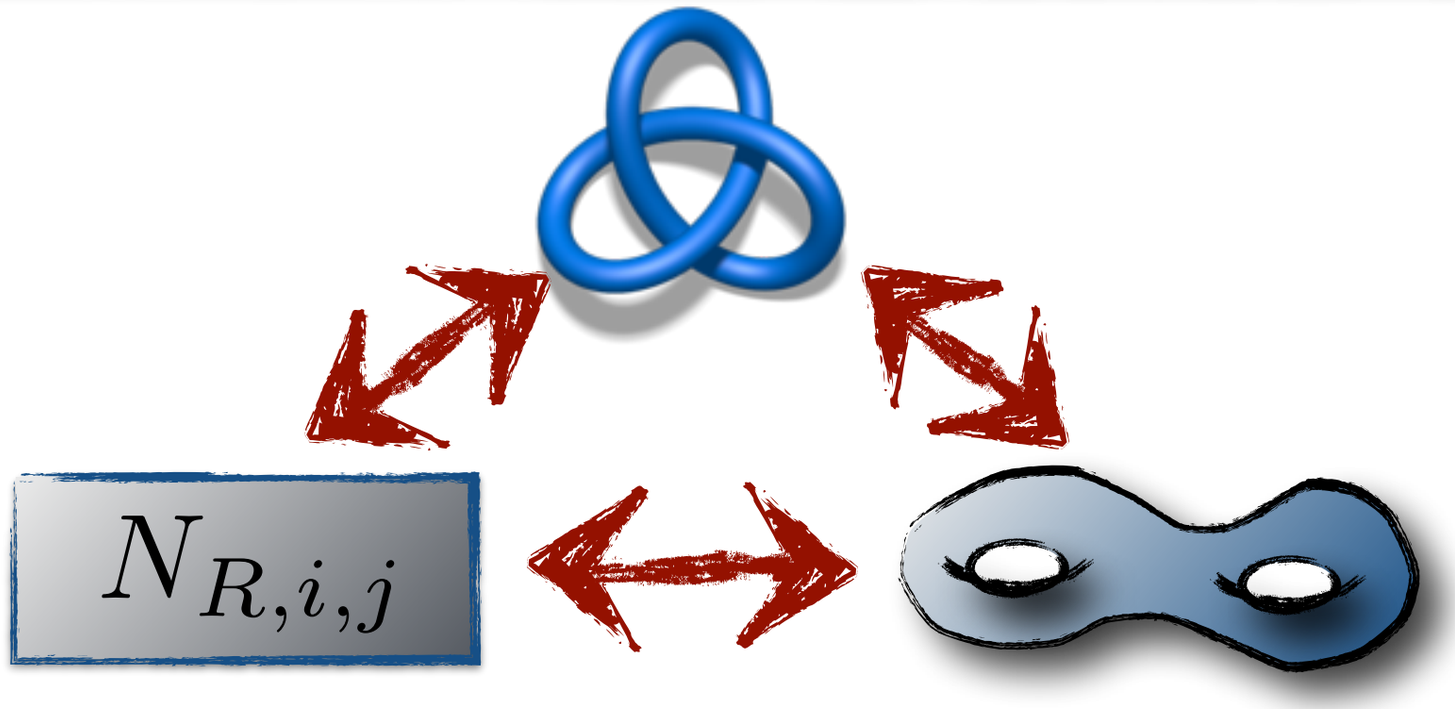

Note:

1. Quantum **A-polynomials** annihilate HOMFLY polynomials

2. **BPS degeneracies** defined via colored HOMFLY polynomials

Dual A-polynomials

1. HOMFLY polynomial - SU(*N*) Chern-Simons theory: $a = q^N$

$$\widehat{A}(\hat{M}, \hat{L}; a, q) P_r(a, q) = 0$$

Consider the generating function: $P(x) = \sum_{r=0}^{\infty} P_r \, x^r$

$$\begin{aligned} \hat{M}P(x) &= \sum_r P_r (qx)^r = P(qx) \\ \hat{L}P(x) &= \sum_r P_{r+1} x^r = \frac{1}{x} P(r) \end{aligned}$$

$$\begin{aligned} \hat{x}P(x) &= xP(x) \\ \hat{y}P(x) &= P(qx) \\ \hat{y}\hat{x} &= q\hat{x}\hat{y} \end{aligned}$$

It follows:

$$\widehat{\mathcal{A}}(\hat{x}, \hat{y}; a, q) P(x) = 0$$

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2. Now restrict Ooguri-Vafa functional to $R=S^r$ (i.e. consider $V=[x]$)

$$\sum_{R=S^r} P_R(a, q) \mathrm{Tr}_R V = \sum_r P_r(a, q) x^r = P(x)$$

Extremal invariants

Focus on the minimal/maximal power of *a* in colored HOMFLY:

$$P_{\square}^{4_1}(a,q) = a^{-3}\frac{q}{(1-q^2)} + a^{-1}\frac{q^4+1}{q\,(q^2-1)} + a\frac{\left(q^4+1\right)}{q\,(1-q^2)} + a^3\frac{q}{q^2-1}$$

Introduce minimal/maximal colored HOMFLY:

$$P_r^{\pm}(q) = \Big\{\text{coefficient of min/max power of } a \text{ in } P_r(a,q)\Big\}$$

Can consistently consider S^r and **extermal BPS invariants…**

$$f_{S^3}(a,q) = P_{S^3}(a,q) - P_{\square}(a,q)P_{S^2}(a,q) + \frac{1}{3}P_{\square}(a,q)^3 - \frac{1}{3}P_{\square}(a^3,q^3)$$

…and the corresponding **extremal A-polynomials**

$$\mathcal{A}^{\pm}(x,y) \simeq \mathcal{A}(xa^{\mp c_{\pm}}, y, a^{\pm})|_{a=0}$$

BPS invariants from curves

Solving $A(x,y) = 0$ for $y=y(x)$ encodes combinations of $N_{r,i,j}$

$$y(x) = \lim_{q\to 1} \frac{\hat{y}P(x)}{P(x)} = \lim_{q\to 1} \frac{P(qx)}{P(x)} = \prod_{r\geq 1;i} (1-x^r a^i)^{-rb_{r,i}/2}$$

$$b_{r,i} = \sum_j N_{r,i,j}$$

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$$b_{r,i} = \sum_j N_{r,i,j}$$

In consequence:

$$x\frac{y'(x)}{y(x)} = \frac{1}{2}\sum_{r\geq 1} r^2 b_r \frac{x^r}{1-x^r} \equiv \sum_{n\geq 0} a_n x^n$$

$$b_r = \frac{2}{r^2}\sum_{d|r} \mu(d) a_{r/d}$$

$$\mu(p) = \begin{cases} 1 & \text{all } p_i \text{ are different and the number of prime factors } k \text{ is even} \\ -1 & \text{all } p_i \text{ are different and the number of prime factors } k \text{ is odd} \\ 0 & \text{there is (at least one) repeated factor } p_i^2 \text{ in the product } p = p_1 \cdots p_k \end{cases}$$

Some explicit results…

[S. Garoufalidis, P. Kucharski, P.S. - CMP (2016), arXiv: 1504.06327]

5_2 knot and Catalan numbers

Minimal A-polynomial:

$$\mathcal{A}^-(x,y) = 1 - y^2 - xy^4$$

The solution is the generating function for Catalan numbers:

$$y(x)^2 = \frac{-1+\sqrt{1+4x}}{2x} = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}(-x)^n = 1 - x + 2x^2 - 5x^3 + 14x^4 - 42x^5 + \dots$$

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We get quite non-trivial integrality prediction:

$$b_r^- = \frac{1}{r^2}\sum_{d|r} \mu\left(\frac{r}{d}\right)(-1)^{d+1}\binom{2d-1}{d-1} \in \mathbb{Z}$$

Note: general answer for any *r* !

5_2 knot and improved integrality

r	b_r^-	b_r^+	$6\frac{b_r^-}{r}$	$2\frac{b_r^+}{r}$
1	-1	-1	-6	-2
2	1	3	3	3
3	-1	-15	-2	-10
4	2	110	3	55
5	-5	-950	-6	-380
6	13	9021	13	3007
7	-35	$-91\,763$	-30	$-26\,218$
8	100	982 652	75	245 663
9	-300	$-10\,942\,254$	-200	$-2\,431\,612$
10	925	125 656 950	555	25 131 390
11	-2915	$-1\,479\,452\,887$	-1590	$-268\,991\,434$
12	9386	17 781 576 786	4693	2 963 596 131
13	$-30\,771$	$-217\,451\,355\,316$	$-14\,202$	$-33\,454\,054\,664$
14	102 347	2 698 753 797 201	43 86	38 536 256 743
15	$-344\,705$	$-33\,922\,721\,455\,050$	$-137\,88$	$-4\,523\,029\,527\,340$

Much stronger than Ooguri-Vafa integrality of BPS numbers!

5_2 knot, full dependence on a

$$
\begin{aligned}
\mathcal{A}(x,y;a) \;=\; & a^{10}x^2y^{16} - 2a^9x^3y^{16} + a^8x^2y^{12}\left(x^2y^4 - 3y^2 - 4\right) + a^7xy^{12}\left(x^2\left(4y^2+3\right) - 1\right) + \\
& + a^6x^2y^8\left(-x^2y^6 + 5y^4 + 3y^2 + 6\right) + a^5xy^{10}\left(2 - x^2\left(3y^2+2\right)\right) + \\
& - a^4x^2\left(3y^6 + 4y^4 - 3y^2 + 4\right)y^4 - a^3x\left(y^2-1\right)y^4\left(x^2\left(y^2-1\right) + y^2 + 3\right) + \\
& + a^2x^2\left(-3y^6 + 5y^4 - 3y^2 + 1\right) + ax\left(-3y^4 + 4y^2 - 2\right) - y^2 + 1
\end{aligned}
$$

Integrality of $b_{r,i}$ invariants:

$$
\begin{pmatrix}
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & -4 & 11 & -10 & 3 & 0 & 0 \\
-1 & 4 & 0 & -12 & 23 & -71 & 154 & -162 & 80 \\
2 & -12 & 14 & 8 & 40 & -226 & 594 & -1542 & 2944 \\
-5 & 36 & -66 & -30 & 132 & -184 & 1550 & -6108 & 16525 \\
13 & -114 & 302 & -94 & -419 & -660 & 3387 & -12042 & 56209 \\
-35 & 372 & -1296 & 1168 & 1843 & -1400 & -1372 & -27398 & 135350 \\
100 & -1244 & 5382 & -8014 & -4222 & 12390 & 23462 & -42626 & 163928 \\
-300 & 4240 & -21932 & 45383 & -6044 & -79628 & -31246 & 116368 & 589812 \\
925 & -14676 & 88457 & -234124 & 160251 & 359514 & -192014 & -1022096 & -86854
\end{pmatrix}
$$

4_1 knot

Minimal A-polynomial:

$$\mathcal{A}^-(x,y) = x - y^4 + y^6$$

We get quite non-trivial integrality prediction:

$$b_r^- = -\frac{1}{r^2}\sum_{d|r}\mu\left(\frac{r}{d}\right)\binom{3d-1}{d-1} \in \mathbb{Z}$$

r	$b_r^- = -b_r^+$	$2\frac{b_r^-}{r}$
1	-1	-2
2	-1	-1
3	-3	-2
4	-10	-5
5	-40	-16
6	-171	-57
7	-791	-226

Growth of BPS numbers

From Stirling formula:

$$\lim_{r\to\infty} \frac{b_r^-}{b_{r+1}^-} = \frac{4}{27}$$

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Minimal A-polynomial:

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Zero of *y*-discriminant of minimal A-polynomial:

$$\mathrm{Disc}_y\mathcal{A}^-(x,y)=-64x^3(-4+27x)^2$$

Twist knots K_p

Extremal A-polynomials:

$$\mathcal{A}^-_{K_p}(x,y) = x - y^4 + y^6, \qquad \mathcal{A}^+_{K_p}(x,y) = 1 - y^2 + xy^{4|p|+2}, \qquad p \le -1,$$
$$\mathcal{A}^-_{K_p}(x,y) = 1 - y^2 - xy^4, \qquad \mathcal{A}^+_{K_p}(x,y) = 1 - y^2 + xy^{4p+4}, \qquad p \ge 2.$$

Extremal BPS degeneracies for $p<0$:

$$b^-_{K_p,r} = -\frac{1}{r^2}\sum_{d|r}\mu(\frac{r}{d})\binom{3d-1}{d-1}, \qquad b^+_{K_p,r} = \frac{1}{r^2}\sum_{d|r}\mu(\frac{r}{d})\binom{(2|p|+1)d-1}{d-1}$$

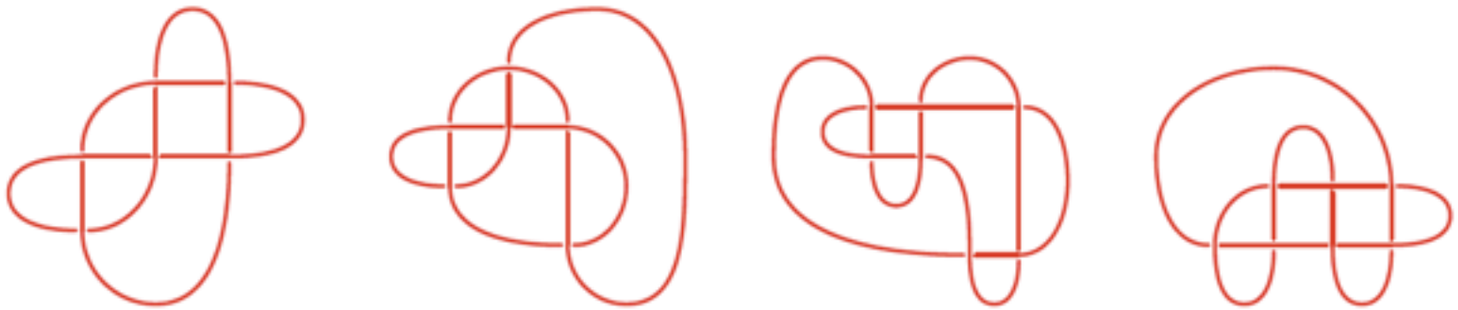
Extremal BPS degeneracies for $p>1$:

$$b^-_{K_p,r} = \frac{1}{r^2}\sum_{d|r}\mu(\frac{r}{d})(-1)^{d+1}\binom{2d-1}{d-1}, \qquad b^+_{K_p,r} = \frac{1}{r^2}\sum_{d|r}\mu(\frac{r}{d})(-1)^{d}\binom{(2p+2)d-1}{d-1}$$

Torus knots, $Y=y^2$

Knot	$\mathcal{A}^-(x,y)$	$\mathcal{A}^+(x,y)$
$\mathbf{3_1}$	$-1+x+Y-xY+2xY^2+x^2Y^3$	$-1+Y+xY^4$
$\mathbf{5_1}$	$-1+x+Y-xY+2xY^2-2xY^3+$ $+3xY^4+2x^2Y^5-x^2Y^6+3x^2Y^7+x^3Y^{10}$	$-1+Y+xY^4-xY^5+2xY^6+x^2Y^{11}$
$\mathbf{7_1}$	$-1+x+Y-xY+2xY^2-2xY^3+$ $+3xY^4-3xY^5+4xY^6+3x^2Y^7-2x^2Y^8$ $+6x^2Y^9-3x^2Y^{10}+6x^2Y^{11}+3x^3Y^{14}+$ $-x^3Y^{15}+4x^3Y^{16}+x^4Y^{21}$	$-1+Y+xY^4-xY^5+2xY^6-2xY^7+$ $+3xY^8+2x^2Y^{13}-x^2Y^{14}+3x^2Y^{15}+x^3Y^{22}$
$\mathbf{9_1}$	$-1+x+Y-xY+2xY^2-2xY^3+$ $+3xY^4-3xY^5+4xY^6-4xY^7+5xY^8+$ $+4x^2Y^9-3x^2Y^{10}+9x^2Y^{11}-6x^2Y^{12}+$ $+12x^2Y^{13}-6x^2Y^{14}+10x^2Y^{15}+6x^3Y^{18}+$ $-3x^3Y^{19}+12x^3Y^{20}-4x^3Y^{21}+10x^3Y^{22}+$ $+4x^4Y^{27}-x^4Y^{28}+5x^4Y^{29}+x^5Y^{36}$	$-1+Y+xY^4-xY^5+2xY^6-2xY^7$ $+3xY^8-3xY^9+4xY^{10}+3x^2Y^{15}+$ $-2x^2Y^{16}+6x^2Y^{17}-3x^2Y^{18}+6x^2Y^{19}+$ $+3x^3Y^{26}-x^3Y^{27}+4x^3Y^{28}+x^4Y^{37}$
$\mathbf{11_1}$	$-1+x+Y-xY+2xY^2-2xY^3+$ $+3xY^4-3xY^5+4xY^6-4xY^7+$ $+5xY^8-5xY^9+6xY^{10}+5x^2Y^{11}+$ $-4x^2Y^{12}+12x^2Y^{13}-9x^2Y^{14}+18x^2Y^{15}+$ $-12x^2Y^{16}+20x^2Y^{17}-10x^2Y^{18}+15x^2Y^{19}+$ $+10x^3Y^{22}-6x^3Y^{23}+24x^3Y^{24}-12x^3Y^{25}+$ $+30x^3Y^{26}-10x^3Y^{27}+20x^3Y^{28}+10x^4Y^{33}+$ $-4x^4Y^{34}+20x^4Y^{35}-5x^4Y^{36}+15x^4Y^{37}+$ $+5x^5Y^{44}-x^5Y^{45}+6x^5Y^{46}+x^6Y^{55}$	$-1+Y+xY^4-xY^5+2xY^6-2xY^7+$ $+3xY^8-3xY^9+4xY^{10}-4xY^{11}+$ $+5xY^{12}+4x^2Y^{17}-3x^2Y^{18}+9x^2Y^{19}+$ $-6x^2Y^{20}+12x^2Y^{21}-6x^2Y^{22}+10x^2Y^{23}+$ $+6x^3Y^{30}-3x^3Y^{31}+12x^3Y^{32}-4x^3Y^{33}+$ $+10x^3Y^{34}+4x^4Y^{43}-x^4Y^{44}+5x^4Y^{45}+x^5Y^{56}$

Some other knots…



6_2, 6_3, 7_3, and 7_5 knots

8_{19}, 8_{20}, and 8_{21} knots

10_{124}, 10_{132}, and 10_{139} knots

Extremal A-polynomials, $Y=y^2$

Knot	$\mathcal{A}^-(x,y)$	$\mathcal{A}^+(x,y)$
6_2	$x-Y^4+Y^5$	$1-Y+2xY-xY^2+x^2Y^2+xY^3$
6_3	$x+Y-Y^2$	$-1+Y+xY^2$
7_3	$-1+Y+xY^4$	$-1+Y+xY^6-xY^7+2xY^8+x^2Y^{15}$
7_5	$x+Y^7-Y^8$	$x^2+2xY^3-xY^4+xY^5+Y^6-Y^7$
8_{19}	$-1+x+Y-xY+3xY^3+xY^4-4xY^5+$ $+5xY^6+x^2Y^6-x^2Y^7+5x^2Y^8+3x^2Y^9+$ $-6x^2Y^{10}+10x^2Y^{11}-x^3Y^{12}+x^3Y^{13}+$ $+3x^3Y^{14}-4x^3Y^{15}+10x^3Y^16-x^4Y^{18}+$ $+x^4Y^{19}-x^4Y^{20}+5x^4Y^{21}+x^5Y^{26}$	$-1+Y-xY^9$
8_{20}	$x^2-xY^2+xY^3-2xY^4-Y^7+Y^8$	$-x+x^2-Y+2xY-5x^2Y+4x^3Y+$ $+Y^2-xY^2+11x^2Y^2-9x^3Y^2+6x^4Y^2+$ $+xY^3-7x^2Y^3+17x^3Y^3-7x^4Y^3+4x^5Y^3+$ $+4x^2Y^4-12x^3Y^4+9x^4Y^4-2x^5Y^4+x^6Y^4+$ $+5x^3Y^5-6x^4Y^5+x^5Y^5-x^3Y^6+2x^4Y^6-x^5Y^6$
8_{21}	$x-Y^4+Y^5$	$x^4+3x^3Y+3x^2Y^2-7x^3Y^2+xY^3-10x^2Y^3+$ $+2x^3Y^3-3xY^4+17x^2Y^4-x^3Y^4-Y^5+2xY^5+$ $-11x^2Y^5+Y^6-3xY^6+6x^2Y^6-x^2Y^7$
10_{124}	$x^2-2xY^{10}+xY^{11}-xY^{12}+Y^{20}-Y^{21}$	$x^7+7x^6Y^7-x^6Y^8+x^6Y^9-2x^6Y^{10}+$ $+21x^5Y^{14}-6x^5Y^{15}+5x^5Y^{16}-3x^5Y^{17}-x^5Y^{19}+$ $+x^5Y^{20}+35x^4Y^{21}-15x^4Y^{22}+10x^4Y^{23}+8x^4Y^{24}+$ $-3x^4Y^{25}-3x^4Y^{26}+35x^3Y^{28}-20x^3Y^{29}+10x^3Y^{30}+$ $+22x^3Y^{31}-9x^3Y^{32}+2x^3Y^{34}+21x^2Y^{35}-2x^3Y^{35}+$ $-15x^2Y^{36}+5x^2Y^{37}+18x^2Y^{38}-9x^2Y^{39}+5x^2Y^{40}+$ $+3x^2Y^{41}+7xY^{42}-6xY^{43}+xY^{44}+5xY^{45}+$ $-3xY^{46}+3xY^{47}+Y^{49}-xY^{49}-Y^{50}+xY^{50}$

Refined open BPS invariants

Generating function of colored superpolynomials:

$$P(x) = \sum_r P_r(a,q,t)x^r = \prod_{r\geq 1;i,j,k;n\geq 0} \left(1 - x^r a^i t^j q^{k+2n+1}\right)^{N_{r,i,j,k}}$$

Semi-classical BPS degeneracies: $b_{r,i,j} = \sum_k N_{r,i,j,k}$

…can be determined from the relation:

$$x\frac{\partial_x y(x,a,t)}{y(x,a,t)} = \frac{1}{2}\sum_{r,i,j} r^2 b_{r,i,j} \frac{x^r a^i t^j}{1 - x^r a^i t^j}$$

where *y=y(x)* arises from the super-A-polynomial equation:

$$\mathcal{A}^{\rm super}(x,y;a,t) = 0$$

Refined invariants from super-A-polynomials

Verify integrality of refined BPS degeneracies ($-t=T^2$):

Knot	$\sum_{r,i,j} \tilde{b}_{r,i,j} x^r a^i T^j$
$\mathbf{4_1}$	$(-1+aT-aT^2+2aT^3-2a^2T^4+a^2T^5)x+(-1+2aT+(-a-a^2)T^2+(3a+a^2)T^3+$ $-4a^2T^4+(a^2+a^3)T^5)x^2+(-3+7aT+(-3a-5a^2)T^2+(10a+5a^2+a^3)T^3+$ $+(-21a^2-2a^3)T^4+(6a^2+13a^3)T^5)x^3+(-10+30aT+(-12a-32a^2)T^2+$ $+(42a+28a^2+14a^3)T^3+(-117a^2-21a^3-2a^4)T^4+(35a^2+114a^3+5a^4)T^5)x^4+$ $+(-40+143aT+(-55a-198a^2)T^2+(198a+165a^2+132a^3)T^3+$ $+(-690a^2-180a^3-42a^4)T^4+(210a^2+912a^3+84a^4+5a^5)T^5)x^5+\dots$
$\mathbf{6_1}$	$(-1+aT-2aT^2)x+(-1+2aT+(-3a-a^2)T^2)x^2+(-3+7aT+(-10a-5a^2)T^2)x^3+$ $+(-10+30aT+(-42a-32a^2)T^2)x^4+(-40+143aT+(-198a-198a^2)T^2)x^5+\dots$
$\mathbf{5_2}$	$(-1+T+(-1-a)T^2+2aT^3-2aT^4+(a+2a^2)T^5)x+(T^2+(-1-2a)T^3+$ $+(1+5a+a^2)T^4+(-8a-4a^2)T^5)x^2+(T^3+(-3-4a)T^4+(2+16a+4a^2)T^5)x^3+$ $+(2T^4+(-6-10a)T^5)x^4+4T^5x^5+\dots$
$\mathbf{3_1}$	$(-1-T^2+2aT^3+aT^5)x+(T^2-aT^3+T^4-5aT^5)x^2+$ $+(-2T^4+7aT^5)x^3+(T^4-3aT^5)x^4+\dots$
$\mathbf{5_1}$	$(-1-T^2+2aT^3-T^4)x+(T^2-aT^3+4T^4)x^2-5T^4x^3+2T^4x^4+\dots$

Summary

Various links between knots, Chern-Simons theory, and string theory / M-theory

BPS degeneracies for knots encoded in algebraic curves

Extremal A-polynomials

Extremal BPS degeneracies:

- explicit formulas: statements in number theory
- ``improved integrality'': beyond M-theory prediction
- refined BPS invariants

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Open questions

- **Explain ``improved integrality'', construct explicitly relevant physical systems**
- **Derive and prove new statements in number theory**
- **Relate BPS counting to knot homologies**
- **Understand relations to matrix models, e.g.:**
  - efficient computations with topological recursion
  - a plethora of null-vector like quantum curves
  - homological deformation

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