QUANTUM GEOMETRY OF NON-GEOMETRIC FLUXES

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Quantum Structure of Spacetime

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Quantum spacetime structures: Dualities and new geometries

Bayrischzell 2016

Open strings and noncommutative gauge theory

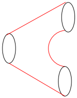
(Chu & Ho '98; Schomerus '99; Seiberg & Witten '99; . . .)

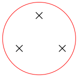
- D-branes in B-fields provide realisations of noncommutative/nonassociative spaces
- 2-point function on boundary of disk: ordering 
- 2-form (B-field) deforms to noncommutative 2-bracket
- Quantization produces star-products of fields, encoded in scattering amplitudes
- When 3-form $H = \mathrm{d}B \neq 0$, cyclicity and associativity restored on-shell in amplitudes $S_{\mathrm{DBI}} = \int \sqrt{g + \mathcal{F}}$
  (Cornalba & Schiappa '02; Herbst, Kling & Kreuzer '01)
- Massless bosonic modes: A_μ, X^i gauge and scalar fields: Low-energy dynamics described by noncommutative gauge theory

Closed strings and nonassociative gravity(?)

(Blumenhagen & Plauschinn '10; Lüst '10; Blumenhagen *et al.* '11)

- Closed strings winding and propagating in non-geometric flux compactifications provide realisations of noncommutative/nonassociative spaces





- 3-point function on sphere: orientation
- 3-vector (R-flux) deforms to nonassociative 3-bracket
- Encoded in off-shell correlators through triproducts of fields, violates strong constraint of DFT
- Quantization of phase space produces nonassociative star-product (Mylonas, Schupp & RS '12; Bakas & Lüst '13)
- Massless bosonic modes: $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ background geometry, gravity: Low-energy dynamics described by noncommutative/nonassociative gravity?

Non-geometric flux compactification

$$M \xrightarrow{T^d} W \ , \qquad [H] \in H^3(M,\mathbb{Z})$$

$$M = T^3 \xrightarrow{T^d} T^{3-d} \ , \qquad [H] \ = \ k \in \mathbb{Z}$$

T^3 with H-flux gives geometric and non-geometric fluxes via T-duality

(Hull '05; Shelton, Taylor & Wecht '05)

$$H_{abc} \xrightarrow{T_a} f^a{}_{bc} \xrightarrow{T_b} Q^{ab}{}_c \xrightarrow{T_c} R^{abc}$$

- H-flux $(d=0)$: $H = \mathrm{d}B$, gerbes on M
- Metric flux $(d=1)$: $\mathrm{d}e^a = -\frac{1}{2}\, f^a{}_{bc}\, e^b \wedge e^c$, twisted torus
  Globally $M \xrightarrow{S^1} T^2$ degree $[H]$

Non-geometric flux compactification

- Q-flux $(d = 2)$: T-folds, stringy transition functions, locally $M \xrightarrow{T^2} S^1$
  Canonical structure yields closed string noncommutativity
  (Lüst '10; Blumenhagen *et al.* '11; Condeescu, Florakis & Lüst '12; Andriot *et al.* '12):

$$[x^i, x^j] = \tfrac{\mathrm{i}\, \ell_s^4}{3\hbar}\, Q^{ij}{}_k\, w^k \ , \qquad [x^i, w^j] = 0 = [w^i, w^j]$$

- R-flux $(d = 3)$: g, B not even locally defined
  T-duality sends $Q^{ij}{}_k \longmapsto R^{ijk}$, $w^k \longmapsto p_k$:

$$[x^i, x^j] = \tfrac{\mathrm{i}\, \ell_s^4}{3\hbar}\, R^{ijk}\, p_k \ , \qquad [x^i, p_j] = \mathrm{i}\, \hbar\, \delta^i{}_j \ , \qquad [p_i, p_j] = 0$$

  Twisted Poisson structure on $\mathcal{M} = T^* M$:
  Closed string nonassociativity (Jacobiator) $[x^i, x^j, x^k] = \ell_s^4\, R^{ijk}$

Topological nonassociative tori

(Mathai & Rosenberg '04; Bouwknegt, Hannabuss & Mathai '06; Ellwood & Hashimoto '06; Grange & Schäfer-Nameki '07)

- H-flux $(d=0)$: $M = \mathrm{Spec}(\mathcal{A})$; T-dual is $\widehat{\mathcal{A}} = \mathcal{A} \rtimes_\alpha \widehat{\mathbb{R}}^d$, $\alpha : \mathbb{R}^d \longrightarrow \mathrm{Aut}(\mathcal{A})$
- Metric flux $(d=1)$: $\mathrm{Spec}(\widehat{\mathcal{A}}) = H_{\mathbb{R}}/H_{\mathbb{Z}} =$ Heisenberg nilmanifold
- Q-flux $(d=2)$: $\widehat{\mathcal{A}} = C^*(H_{\mathbb{Z}}) \otimes \mathcal{K} =$ field of noncommutative 2-tori T^2_θ , $\theta = k\,x$, $x \in S^1$ T-fold
- R-flux $(d=3)$: $\widehat{\mathcal{A}} = \mathcal{K}\big(L^2(\widehat{T}^3)\big) \rtimes_{u_\phi} \widehat{T}^3 =$ nonassociative 3-torus T^3_ϕ , $\phi \in Z^3\big(\widehat{T}^3, U(1)\big)$ associated to H non-geometric R-flux

Geometrical meaning of n-brackets

- Nambu–Poisson structures: Smooth manifold M with n-Lie algebra structure $\{-,\dots,-\} : C^\infty(M)^{\wedge n} \longrightarrow C^\infty(M)$ satisfying:

  1. Fundamental identity:

  $$\begin{aligned}\big\{f_1,\dots,f_{n-1},\{g_1,\dots,g_n\}\big\} &= \big\{\{f_1,\dots,f_{n-1},g_1\},\dots,g_n\big\} \\ &\quad + \cdots + \big\{g_1,\dots,\{f_1,\dots,f_{n-1},g_n\}\big\}\end{aligned}$$

  2. Generalized Leibniz rule:

  $$\{f\,g,h_1,\dots,h_{n-1}\} = f\,\{g,h_1,\dots,h_{n-1}\} + \{f,h_1,\dots,h_{n-1}\}\,g$$

- Example: $M=\mathbb{R}^3$ or T^3, Nambu–Poisson 3-bracket: (Nambu '73)

  $$\{x^i,x^j,x^k\} = R^{ijk}$$

  Extend by linearity and generalized Leibniz rule; quantization gives **Nambu–Heisenberg algebra**

n-plectic manifolds

- Multisymplectic manifolds: Manifold M with closed $n+1$-form ω such that $\omega(X,-)=0 \Longleftrightarrow X=0$
- 1-plectic $\equiv$ symplectic; 2-plectic $\equiv$ 3-form ω
- If $\dim(M)=n+1$, ω^{-1} gives Nambu–Poisson structure
- Multiphase spaces in Nambu mechanics (generalizing Poisson phase spaces in Hamiltonian dynamics)
- Starting point for higher quantization

Quantization of 2-plectic manifolds

- Symplectic manifold (M, ω) with $[\omega] \in H^2(M, \mathbb{Z})$ encodes prequantum line bundle with connection (L, ∇), $F_\nabla = \omega$ (first Chern class)
- 2-plectic manifold (M, H) with $[H] \in H^3(M, \mathbb{Z})$ encodes prequantum abelian gerbe with 2-connection $(\mathcal{G}, A, B)$, $H = \mathrm{d}B$ (Dixmier–Douady class)
- Quantize bracket on Hamiltonian 1-forms (Baez, Hoffnung & Rogers '10):

$$\{\alpha, \beta\} = H(X_\alpha, X_\beta, -)$$

$\alpha, \beta \in \Omega^1(M)$, $\mathrm{d}\alpha = H(X_\alpha, -)$

- Jacobiator: $\{\alpha, \beta, \gamma\} = \mathrm{d}H(X_\alpha, X_\beta, X_\gamma) \neq 0$ (Lie 2-algebra of Hamiltonian 1-forms)

Quantization of 2-plectic manifolds: $\mathbb{R}^3$

Trick: Map 2-plectic forms to symplectic forms by transgressing $\mathcal{G}$ to prequantum line bundle over loop space of M (Sämann & RS '12)

- Transgression:

$$
\begin{array}{ccc}
 & \mathcal{L}M \times S^1 & \\
 \overset{ev}{\swarrow} & & \overset{\oint}{\searrow} \\
M & & \mathcal{L}M
\end{array}
$$

$$\mathcal{T} = \big(\oint\big)_! \circ ev^* : \Omega^{n+1}(M) \longrightarrow \Omega^n(\mathcal{L}M)$$

$$(\mathcal{T}\alpha)_x\big(v_1(\tau)\,,\,\dots\,,\,v_n(\tau)\big) = \oint \mathrm{d}\tau\ \alpha\big(v_1(\tau)\,,\,\dots\,,\,v_n(\tau), \dot{x}(\tau)\big)$$

- Symplectic form on loop space $\mathcal{L}M$:

$$\mathcal{T}H = \oint \mathrm{d}\tau\ H_{ijk}\,\dot{x}^k(\tau)\,\delta x^i(\tau) \wedge \delta x^j(\tau)$$

Quantization of 2-plectic manifolds: $\mathbb{R}^3$

- Invert to get Poisson bracket:

$$\{f,g\} := \oint \mathrm{d}\tau \; Q^{ij}{}_k \, \frac{\dot{x}^k(\tau)}{|\dot{x}(\tau)|^2} \Big(\frac{\delta}{\delta x^i(\tau)} f \Big) \Big(\frac{\delta}{\delta x^j(\tau)} g \Big)$$

- Quantization gives:

$$\big[x^i(\tau)\,,\, x^j(\rho)\big] \;=\; \mathrm{i}\,\hbar\, Q^{ij}{}_k \, \dot{x}^k(\tau)\, \delta(\tau-\rho) + \mathcal{O}(Q^2)$$

- Agrees with 1-form quantization (Baez, Hoffnung & Rogers '10):

$$\mathcal{T}\{\alpha,\beta\} = \{\mathcal{T}\alpha, \mathcal{T}\beta\}$$

  Quantized 1-forms on M $\Longrightarrow$ Quantized functions on $\mathcal{L}M$

  Since $\mathcal{T}\mathrm{d}H(X_\alpha, X_\beta, X_\gamma) = \delta\mathcal{T}H(X_\alpha, X_\beta, X_\gamma) = 0$, Jacobi identity recovered

- Gives $[x^i, x^j] = \mathrm{i}\,\hbar\, Q^{ij}{}_k \, w^k$ after integration over τ, ρ where $x^i = \oint \mathrm{d}\tau \; x^i(\tau)$, $\; w^k = \oint \mathrm{d}\tau \; \dot{x}^k(\tau)$

Higher geometric quantization

(Bunk, Sämann & RS '??)

- Sections $\Gamma(M,L)$: Module over $C^\infty(M)$, with structure of vector space over $\mathbb{C}$ and pairing valued in $C^\infty(M)$

  Defines prequantum Hilbert space and prequantization is representation of Poisson Lie algebra $(C^\infty(M),\{\cdot,\cdot\})$

- Bundle gerbe modules $\Gamma(M,\mathcal{G})$: Module category over $\mathsf{Vec}(M)$, with structure of module category over Hilb and bifunctor valued in $\mathsf{Vec}(M)$

  Defines prequantum 2-Hilbert space

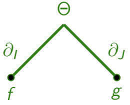
- Example: 2-Hilbert space $\Gamma(\mathbb{R}^3,\mathcal{G})$ carries action of 2-group String(3)

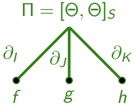
Deformation quantization

- Formality maps U_n : multivector fields $\longrightarrow$ differential operators, define quasi-isomorphisms between d.g. L_∞-algebras relating Schouten brackets to Gerstenhaber brackets (Kontsevich '03):

$$U_n(\mathcal{X}_1,\dots,\mathcal{X}_n) \;=\; \sum_{\Gamma\in G_n} w_\Gamma\, D_\Gamma(\mathcal{X}_1,\dots,\mathcal{X}_n)$$

- Star-product and 3-bracket:





$$f\star g \;=\; \sum_{n=0}^{\infty}\frac{(\mathrm{i}\,\hbar)^n}{n!}\,U_n(\Theta,\dots,\Theta)(f,g) \;=:\; \Phi(\Theta)(f,g)$$

$$[f,g,h]_\star \;=\; \sum_{n=0}^{\infty}\frac{(\mathrm{i}\,\hbar)^n}{n!}\,U_{n+1}(\Pi,\Theta,\dots,\Theta)(f,g,h) \;=:\; \Phi(\Pi)(f,g,h)$$

Formality conditions

(Mylonas, Schupp & RS '12)

- $[\Phi(\Theta), \star]_G = \mathrm{i}\,\hbar\,\Phi([\Theta,\Theta]_S)$ quantifies nonassociativity:

$$(f\star g)\star h - f\star(g\star h) = \frac{\hbar}{2\,\mathrm{i}}\,\Phi(\Pi)(f,g,h) = \frac{\hbar}{2\,\mathrm{i}}\,[f,g,h]_\star$$

- For constant R-flux:

$$f\star g \;=\; \cdot\Big(\mathrm{e}^{\frac{\mathrm{i}\,\ell_s^4}{12\hbar}\,R^{ijk}\,p_k\,\partial_i\otimes\partial_j}\;\mathrm{e}^{\frac{\mathrm{i}\,\hbar}{2}\,\left(\partial_i\otimes\tilde\partial^i-\tilde\partial^i\otimes\partial_i\right)}(f\otimes g)\Big)$$

$$(f\star g)\star h \;=\; \varphi\big(f\star(g\star h)\big) \;:=\; \star\Big(\mathrm{e}^{\frac{\ell_s^4}{6}\,R^{ijk}\,\partial_i\otimes\partial_j\otimes\partial_k}\big(f\otimes(g\otimes h)\big)\Big)$$

- 2-cyclicity: $\int f\star g = \int g\star f = \int f\,g$

  3-cyclicity: $\int (f\star g)\star h = \int f\star(g\star h)$

Quantization of Nambu–Poisson brackets

- $\Pi = \frac{1}{6}\, R^{ijk}\, \partial_i \wedge \partial_j \wedge \partial_k$ defines Nambu–Poisson trivector with brackets:

$$\{f,g,h\} := \Pi(\mathrm{d}f, \mathrm{d}g, \mathrm{d}h) = R^{ijk}\, \partial_i f\, \partial_j g\, \partial_k h$$

for $f,g,h \in C^\infty(M)$.

- Candidate for quantized Nambu–Poisson 3-bracket:
$\{f,g,h\} \ \longmapsto\ [f,g,h]_\star$

- Evidence:
  1. $[x^i, x^j, x^k]_\star = \ell_s^4\, R^{ijk}$
  2. $[f,g,h]_\star = \ell_s^4\, \{f,g,h\} + \cdots$
  3. Formality condition $[\Phi(\Pi), \star]_G = \mathrm{i}\, \hbar\, \Phi([\Pi, \Theta]_S)$ encodes quantum Leibniz rule:

$$[f \star g, h, k]_\star - [f, g \star h, k]_\star + [f, g, h \star k]_\star = f \star [g,h,k]_\star + [f,g,h]_\star \star k$$

  4. Quantum fundamental identity??

n-triproducts

(Aschieri & RS '15)

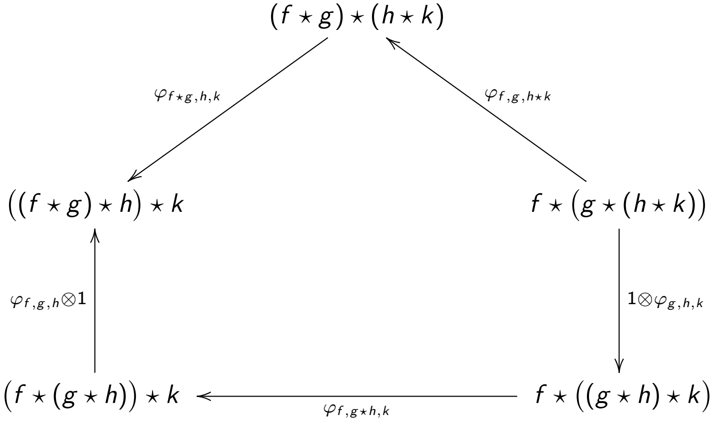
$$
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\ \mathscr{Q}\ } & \mathcal{M} \\
{\scriptstyle \pi^*}\uparrow & & \downarrow{\scriptstyle s^*_{\bar p}} \\
M & \overset{\cdots\cdots\cdots}{\underset{\mathscr{Q}_{\bar p}}{\longrightarrow}} & M
\end{array}
\qquad , \qquad \pi(x,p) = x \quad , \quad s_{\bar p}(x) = (x, \bar p)
$$

- $$\mu^{(n)}_{\bar p}(f_1,\dots,f_n)(x) \ := \ s^*_{\bar p}\big[\pi^* f_1 \star \big(\pi^* f_2 \star (\pi^* f_3 \star (\cdots \star \pi^* f_n)\cdots)\big)\big](x,p)$$
  $$= \ \bar\star\Big[\exp\Big(\frac{\ell_s^4}{12}\sum_{1\le a<b<c\le n} R^{ijk}\,\partial^a_i \otimes \partial^b_j \otimes \partial^c_k\Big)(f_1\otimes\cdots\otimes f_n)\Big]$$
- $$\int \mu^{(n)}_{\bar p}(f_1,\dots,f_n) \ = \ \int f_1 \,\bar\star\, \cdots \,\bar\star\, f_n$$
- For $\bar p = 0$ gives triproducts induced by closed string vertex operators

  (Blumenhagen *et al.* '11)

Association relations

$$f \star (g \star h) \xrightarrow{\varphi_{f,g,h}} (f \star g) \star h$$



Hopf cocycle twist quantization

(Majid '95)

- Drinfel'd twist for a Hopf algebra $H(\Delta, S, \varepsilon, \cdot)$:

$$F = F_{(1)} \otimes F_{(2)} \;\in\; H \otimes H$$

$$(F \otimes 1)\,\Delta_1 F = (1 \otimes F)\,\Delta_2 F \qquad \text{2-cocycle condition}$$

- Maps H to new Hopf algebra $H_F(\Delta_F, S_F, \varepsilon, \cdot)$ with

$$\Delta_F = F\,\Delta\,F^{-1}$$

- Deforms (quantizes) any H-module algebra A to "braided-commutative" algebra:

$$f \star g = \cdot\big(F^{-1}(f \otimes g)\big) = F^{-1}_{(1)} f \cdot F^{-1}_{(2)} g$$

Quasi-Hopf cochain twist quantization

- For an arbitrary 2-cochain twist F, H_F is a quasi-Hopf algebra:

$$\Delta_2\,\Delta = \varphi\,\Delta_1\,\Delta\,\varphi^{-1}$$

- Associator $\varphi = \varphi_{(1)} \otimes \varphi_{(2)} \otimes \varphi_{(3)} \in H^{\otimes 3}$ is a 3-cocycle:

$$\varphi = \partial^* F := F_{23}\,\Delta_2\,F\,\Delta_1\,F^{-1}\,F_{12}^{-1}$$

- Quantizes A to "quasi-associative" algebra:

$$(f \star g) \star h = \varphi_{(1)} f \star (\varphi_{(2)} g \star \varphi_{(3)} h)$$

- Example: $\mathfrak{g}$ = Lie algebra of symmetries of manifold $\mathcal{M}$,
  $H = U(\mathfrak{g})$, $A = C^\infty(\mathcal{M})$ quantized to A_F;
  Similarly, $\Omega^\bullet(\mathcal{M})$ quantized to $\Omega^\bullet_F(\mathcal{M})$, etc.

Twist quantization functor

- Simultaneously deforms all H-covariant constructions as functorial isomorphism of closed braided monoidal categories of left modules:

$$\mathcal{Q}_F \, : \, {}^{H}\mathcal{M} \longrightarrow {}^{H_F}\mathcal{M}$$

- Associator $\equiv$ associativity isomorphisms $\Phi_{V,W,Z} \, : \, (V \otimes W) \otimes Z \longrightarrow V \otimes (W \otimes Z)$:

$$\Phi_{V,W,Z}\big((v \otimes w) \otimes z\big) = \varphi_{(1)} v \otimes \big(\varphi_{(2)} w \otimes \varphi_{(3)} z\big)$$

- Braiding $\equiv$ commutativity isomorphism $\Psi_{V,W} \, : \, V \otimes W \longrightarrow W \otimes V$:

$$\Psi_{V,W}(v \otimes w) = F^{-2}_{(1)} w \otimes F^{-2}_{(2)} v$$

- A_F is commutative and associative in the category ${}^{H_F}\mathcal{M}$;
Framework for differential geometry internal to quasi-Hopf representation categories (Barnes, Schenkel & RS '14)

Cochain twist quantization of R-space

(Mylonas, Schupp & RS '13)

- Nonabelian Lie algebra $\mathfrak{g}$ of Bopp shifts generated by vector fields:

$$P_i = \partial_i \quad , \quad \tilde{P}^i = \tilde{\partial}^i \quad , \quad M_{ij} = p_i \, \partial_j - p_j \, \partial_i$$

$\sigma^{ij} \, M_{ij} \ : \ x^i \longmapsto x^i + \sigma^{ij} \, p_j$

- Quasi-Hopf deformation of $U(\mathfrak{g})$ by cochain twist:

$$\mathcal{F} = \exp \Big[\tfrac{\mathrm{i} \, \ell_s^4}{12 \hbar} \, R^{ijk} \, \big(P_i \otimes M_{jk} + M_{jk} \otimes P_i \big) + \tfrac{\mathrm{i} \, \hbar}{2} \, \big(P_i \otimes \tilde{P}^i - \tilde{P}^i \otimes P_i \big) \Big]$$

- Quantization functor on category of quasi-Hopf module algebras generates nonassociative algebras through associator

$$\varphi = \partial^* \mathcal{F} = \exp \big(\tfrac{\ell_s^4}{6} \, R^{ijk} \, P_i \otimes P_j \otimes P_k \big)$$

Nonassociative differential calculus

- $(\Omega^\bullet, \wedge, \mathrm{d})$ with d equivariant under covariant action of $H = U(\mathfrak{g})$
- Action of H on $\Omega^\bullet$ given by Lie derivatives $\mathcal{L}_h$:

$$M_{ij}\,\mathrm{d}x^k := \mathcal{L}_{M_{ij}}(\mathrm{d}x^k) = \delta_j{}^k\,\mathrm{d}p_i - \delta_i{}^k\,\mathrm{d}p_j$$

- Deformed exterior product $\omega \wedge_\star \eta := \wedge\big(\mathcal{F}^{-1}(\omega \otimes \eta)\big)$ noncommutative and nonassociative:

$$\mathrm{d}x^I \wedge_\star \mathrm{d}x^J = -\mathrm{d}x^J \wedge_\star \mathrm{d}x^I = \mathrm{d}x^I \wedge \mathrm{d}x^J$$

$$(\mathrm{d}x^I \wedge_\star \mathrm{d}x^J) \wedge_\star \mathrm{d}x^K = \mathrm{d}x^I \wedge_\star (\mathrm{d}x^J \wedge_\star \mathrm{d}x^K)$$

- Deformed $A_{\mathcal{F}}$-bimodule structure:

$$x^i \star \mathrm{d}x^j = \mathrm{d}x^j \star x^i + \tfrac{\mathrm{i}\,\ell_s^4}{12\hbar}\,R^{ijk}\,\mathrm{d}p_k$$

- Graded 2-cyclicity: $\int \omega \wedge_\star \eta = (-1)^{|\omega|\,|\eta|} \int \eta \wedge_\star \omega = \int \omega \wedge \eta$
  Graded 3-cyclicity: $\int (\omega \wedge_\star \eta) \wedge_\star \lambda = \int \omega \wedge_\star (\eta \wedge_\star \lambda)$

Summary of open issues

- Relation between topological nonassociative tori and phase space formulation

- Higher geometric quantization:
  - Working properly with loop space variables
  - Extension to non-torsion $[H] \in H^3(M, \mathbb{Z})$
  - Representation of Lie 2-algebra of Hamiltonian 1-forms
  - Polarisation

- Quantization of Nambu–Poisson structures: Viability of phase space model

Summary of open issues

- Deformations of configuration space geometry: Remove momentum dependence up to $O(d,d)$-symmetry of $\gamma = \mathrm{d}x^i \otimes \mathrm{d}p_i + \mathrm{d}p_i \otimes \mathrm{d}x^i$ via foliated tensor fields of $T\mathcal{M} \cong L \oplus L^*$

$$\imath_Z T = 0 = \mathcal{L}_Z T \ , \qquad Z \in \Gamma(\mathcal{M}, L^*)$$

  e.g. Foliation $s_{\bar{p}}$, $L = TM$, $Z = \tilde{\partial}^i$

- Nonassociative gravity?: Gravity on phase space $\Longrightarrow$ gravity on configuration space

  (Aschieri & RS '15; Barnes, Schenkel & RS '15; Blumenhagen & Fuchs '16)

- Generalisation to curved backgrounds, flux formulation of DFT

  (Blumenhagen *et al.* '13)