# Prospectives in Index Theory

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## Contents



**Abstract**

In this lecture the following topics will be discussed in chronological order.

1. Direct connections
2. Localise periodic cyclic homology, algebraic $K$-theory.
3. Algebraic $T$-theory. $K$-theory vs. $T$-theory.
   Define a new completion procedure of semigroups.
4. Define the analytic and topological indices associated with *any* short exact sequence of associtive algebras.

5. Extend the problem of the index formula for exact sequences of associative algebras.
6. Define *non-commutative topology.*

# 1 Atiyah - Singer Index theorem.

[1], [2] deal with *smooth* manifolds.

Let $(D)$ be an elliptic operator. Then

$$AnalyticIndex(D) = TopologicalIndex(D) \tag{1}$$

$$Index(D) = Ch(D) \cup Todd(D) \cap [M]. \tag{2}$$

Un this lecture we make reference to [1].

## 2 Teleman Index Formula.

N. Teleman [3], [4] extended the Atiyah - Singer formula to *topological* manifolds. This formula tells that the index formula is a *topological statement*.

The Teleman formula expresses the indices of an elliptic operator as co-homological *classes*. We intend to represent them as co-homologic *chains*.

## 3 Connes - Moscovici Local Index Formula.

Holds for *differential and pseudo-differential* elliptic operators on *smooth manifolds.* One introduces the following formulas.

Suppose $A$ is a differential or pseudo-differential elliptic operator. Let $B$ be a *parametrix* for the operaotor $A$. Define $S_0 = 1 - BA$ and $S_1 = 1 - AB$. Let $f_0, f_1 \dots f_n$ be smooth functions on $M$. One defines

$$Index(D) = Tr(f_0 \ R(D) \ f_1 \ R(D) \dots f_n \ R(D)) \tag{3}$$

where

$$L(D) \ = \begin{pmatrix} S_0 & -(1 + S_0 B) \\ A & S_1 \end{pmatrix} \tag{4}$$

$$P(D) = L(D) \ e_1 \ L(D)^{-1} \tag{5}$$

and

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6}$$

$$e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

$$R(D) = P - e_2 = \begin{pmatrix} S_0^2 & S_0(1 + S_0)B \\ S_1 A & -S_1^2 \end{pmatrix}. \tag{8}$$

The operator $R(D)$ has the folllowing properties

1. it is a compact operator
2. the operator $P$ is an *idempotent*
3. the *index* of the operator $P$ is the *analytical index* of $D$.
   Clearly

$$Index(D) = Trace R(D) = Trace(S_0^2) - Trace(S_1^2). \tag{9}$$

$R(D)$ is the image of the *simbol* $[\sigma(D)]$ through the *connecting homomorphism* associated to the exact sequence of pseudo-differential operators $\partial : K^1(C^\infty(M)) \longrightarrow K^0(C^\infty(M))$.

The Connes - Moscovici formula [6] is

$$Tr(f_0 R(D) f_1 R(D) \dots f_n R(D)) \ = \tag{10}$$

$$Cont.Ch(\sigma D) Td(M)[f_0 f_1 \dots f_n]. \tag{11}$$

The Alexander - Spanier $f_0 f_1 \dots f_n$ cocycle has *three functions* in this formula.

1. The cycle has to be *totally skew symmetric* for the formula to work.
2. It is used to *localise* the procedure.
3. This formula gives the components of the Chern character of the manifold $M$, once at a time, for each $n$ in the expression (3).

**Remark 1** *The classical* $K^1$*-group* $K^1(C^\infty(M)) = \mathbb{Z}$ *is* too small *to allow obtaining* all components of the Chern character *of the manifold.*

# 4 Connes - Sullivan - Teleman Index Formula.

This work produces a *local expressions* for the index of an elliptic operator on *topological manifolds*; it solves the problem formulated in section §2.

This method still produces the *components of the Chern character* once of a time. To produce *all components* of the Chern character (the analogue of the Thom - Hirzebruch formula on topological manifolds), we have to introduce *new notions*: a *new connection*, called *direct connection* and a new $K$-theory, called $T$-*theory*. This will be shown in the remaining part of this lecture.

## 5 Modified Hochschild homology.

For any associative ring $\mathcal{A}$ introduce the operator

$$\tilde{b}_n \ = \sum_{k=1}^{k=n} (-1)^k C_{2n}^k (db)^{k-1} \tag{12}$$

$$\tilde{b} : C_n(\mathcal{A}) \longrightarrow C_{n-1}(\mathcal{A}). \tag{13}$$

The *modified Hochschild homology* $\tilde{b}$ has the following properties

1. it is a boundary
2. the corresponding homology is at least as big as the homology of the Hochschild complex
3. The modified Hochschild boundary *commutes with the Alexander - Spanier boundary*
$$\tilde{b}\, d \ + \ d\, \tilde{b} \ = 0. \tag{14}$$
4. it can be *localised*.

We recall that the *periodic cyclic homology* is the homology of the bi-complex $(b,\ B)$. We may speak about *periodic local cyclic homology* as the homology of the *bi-complex* $(\tilde{b}, d)$.

Recall that the *periodic cyclic homology* of the associative algebra $\mathcal{A}$ is the homology the bi-complex $(b, B)$.

The periodic local cyclic homology is going to be the homology in the *non - commutative topology* introduced in section §8.

# 6 Algebraic $T$-theory.

The classical algebraic $K^i$-groups, $i = 0, 1$, are defined as follows,

$$K^0(\mathcal{A}) := \text{conjugated classes of idempotents of the matrix algebra of } \mathcal{A} \tag{15}$$

$$K^1(\mathcal{A}) := GL_*(\mathcal{A})/[GL_*(\mathcal{A}), GL_*(\mathcal{A})]; \tag{16}$$

here we have assumed that $\mathcal{A}$ is an unital associative algebra and $GL_*(\mathcal{A})$ are the invertible elements of the matrix algebra over $\mathcal{A}$.

The new groups are denoted $T^i$. They are defined as follows, see see Teleman [9]

In both groups $T^i$ the addition is given by the direct sum of equivalence classes of *conjugated matrices.* The elements of $u_1 \sim u_2$ are *conjugated* provided there exists an invertible element $u$ such that

$$u_1 = u\, u_2\, u^{-1}. \tag{17}$$

**Definition 2**

$$T^0(\mathcal{A}) := \textit{Grothendieck completion of conjugated classes of} \tag{18}$$

$$\textit{idempotentsof the matrix algebra of } \mathcal{A}. \tag{19}$$

*To define $T^1$ we consider*

$$T^1(\mathcal{A}) := \frac{1}{2}\, GL_*(\mathcal{A})/\sim . \tag{20}$$

*Two conjugated invertible elements $u_1, u_2$ are $\sim$ equivalent provided there exist invertible elements $\tilde{\xi}_i$*

$$\xi_i = \begin{pmatrix} \tilde{\xi}_i & 0 \\ 0 & \tilde{\xi}_i^{-1} \end{pmatrix} \tag{21}$$

such that

$$u_1 + \xi_1 = u_2 + \xi_2. \tag{22}$$

The factor $\frac{1}{2}$ is necessary to insure that $\sim$ passes to the quotient for an exact sequence of associative algebras

$$0 \longrightarrow \Lambda' \longrightarrow \Lambda_1 \oplus \Lambda_2 \longrightarrow \Lambda \longrightarrow 0 \tag{23}$$

there correponds an exact sequence of $T^i$ groups

$$T_1^{loc}(\Lambda') \overset{(i_{1*}, i_{2*})}{\longrightarrow} T_1^{loc}(\Lambda_1) \oplus T_1^{loc}(\Lambda_2) \overset{j_{1*} - j_{2*}}{\longrightarrow} T_1^{loc}(\Lambda') \overset{\partial}{\longrightarrow} \tag{24}$$

$$T_0^{loc}(\Lambda') \otimes \mathbb{Z}[\frac{1}{2}] \overset{(i_{1*}, i_{2*})}{\longrightarrow} T_0^{loc}(\Lambda_1) \otimes \mathbb{Z}[\frac{1}{2}] \oplus T_0^{loc}(\Lambda_2) \otimes \mathbb{Z}[\frac{1}{2}] \overset{j_{1*} - j_{2*}}{\longrightarrow} T_0^{loc}(\Lambda') \otimes \mathbb{Z}[\frac{1}{2}] \tag{25}$$

# 7 T-completion

The group $T_1$ was obtained by completing the semigroup of conjugacy classes of invertible elements not by using the Grothendieck, but by a new procedure. We explain it.

The data is: a semigroup $K$ and a subgroup $U$. The involution $I$ is assumed to be an involution on $K$. In our case $K = \mathit{invertible\ elements}$, $I = \mathit{inversion}$ and $U = \{A \oplus A^{-1}\}$.

Two elements $u_1, u_2$ of the subgroup $K$ are called emphequivalent $\sim$ provided there exists two elements $\xi_1, \xi_2 \in U$ such that

$$u_1 \ + \ \xi_1 \ = \ u_2 \ + \ \xi_2. \tag{26}$$

## 8 Non-commutative Topology.

For any associative ring $\mathcal{A}$ one associates the *non - commutative topology* over $\mathcal{A}$.

In the *non - commutative topology*, the *homology* is the $T_i(\mathcal{A})$-theory and the periodic local cyclic homology.

In the next section §9 we will define the Chern character on $T_i(\mathcal{A})$.

# 9 Chern character of idempotents

**Theorem 3** *Let $e \in \mathbb{M}_*$ be an idempotent over the ring $\mathcal{A}$.*
*Let*

$$\Psi_{2n} \ = \ e(de)^n. \tag{27}$$

*Then*

$$\tilde{b}\, \Psi_{2n} \ = \ n\, d\Psi_{2n}. \tag{28}$$

*Therefore, $\{\Psi_{2n}\}_{n\in\mathbb{Z}}$ defines a cycle in the homology of the bi - complex $(\tilde{b},\, d)$. This by definition the* Chern character *of the idempotent $e$.*

**Remark 4** *The components of the idempotent $e$ may by discontinuous.*

Analogously one defines the Chern character of elements in $T_1(\mathcal{A})$.

# 10 The index formula in Non - commutative Topology.

Let

$$0 \longrightarrow \mathcal{J}^{'} \longrightarrow \mathcal{J} \longrightarrow \mathcal{J}^{"} \longrightarrow 0 \tag{29}$$

be an exact sequence of localised associative rings. We assume the last two rings to be unital. The first ring is going to be unitarised.

Let $\sigma$ be an invertible element belonging to $\mathbb{M}_{ast}(\mathcal{J}^{"})$. We associate the exact sequence in algebraic $T_*$-theory.

Let $\sigma \in T_1^{loc}(\mathcal{J}^{"})$ be an invertible element.

STEP I: With this element we associate

1. the *topological index* $TopInd^T(\sigma) := \partial[\sigma] \in T_0^{loc}(\mathcal{J}^{'})$
2. the *analytical index* $AnaInd(\sigma) \in T_0^{loc}(\mathcal{J}^{'})$
3. by the same formulas defined in section §3. This means that the analytic index is the Chern character of the element $R(\sigma)$, where $R(\sigma)$ is defined by the formula (8).

   These are the *basic* definitions of the topological and analytical indices.

It is important to realise that these definitions apply to *any* exact sequence of localised associative rings.

The reader will remark that the *Todd class* does not appear in the formula for the topological index.

STEP II: Apply the Chern character to both topological and analytic indices. One obtains the *topological and analytical indices with values in the local cyclic homology.*

At this point the problem is how much regularity is available. This involves the problem of multiplying distributions.

The problem consists of seeing whether the local cyclic homology classes could be pushed up to the *diagonal*. If the manifold is *smooth and the exact sequence consists of pseudo-differential operators*, the corresponding index theorem is the Atiyah - Singer formula. For topological manifolds one obtains the Connes - Sullivan - Teleman formula.

## References


[1] Atiyah M. F., Singer I. Seminar on the Atiyah - Singer index theorem. Princeton Lecture Notes Nr. 57, 1965

[2] Atiyah M. F., Singer I. M. The index of elliptic operators I. Annals of Mathematics Vol. 87, 484 - 530, 1968

[3] The index of signature operators on Lipschitz manifolds. Publ. Math. IHES Nr.58, Pg. 251 - 290, 1983

[4] The Index theorem on topolpogical manifolds. Acta Mathematica 153, pg. 117 - 152, 1984

[5] Non - commutative differential geometry. IHES Publications, Vol. 62, 257 - 360, 1985

[6] Cyclic cohomology, the Novikov conjecture and hyperbolic groups. Topology Vol. 29, Pg. 345 - 388, 1990 The index of elliptic operators II

[7] Quasiconformal mappings, Operators on Hilbert spaces and local formulae for characteristic classes. Topology 33, pg 663-681, 1984.

[8] Kubarski J., Teleman N.: Linear Direct Connections, in Proc. 7-th Int. Conf. "The Mathematical Legacy of C. Ehresmann,, Betlewo (Polonia), Maggio 2005, pp. 425 - 436, Banach Center Publications, Polish Academy of Science, Warszawa 2007.

[9] Teleman N.: Local Algebraic K-Theory, arXiv:1307.7014, 26 July 2013.