q-Virasoro constraints, matrix models and exact partition functions

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Outline

1. Motivation, toy model
2. Virasoro constraints for Hermitian matrix model
3. q-calculus and matrix models
4. 3D partition functions
5. Summary

Motivation, toy model

toy model for path integral

$$\int_{-\infty}^{+\infty} f(x) dx$$

and the Ward identities are

$$\int_{-\infty}^{+\infty} \frac{d}{dx} f(x) dx = 0$$

Not much use!!!

Motivation, toy model

the generating function:

$$\int_{-\infty}^{+\infty} f(x)dx \quad \Rightarrow \quad Z(g,\{t\}) = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2g}} \ e^{\sum\limits_{s=0}^{\infty} \frac{t_s}{s!} x^s} \ dx$$

the Ward identities:

$$\int_{-\infty}^{+\infty} \frac{d}{dx}\left(x^{n+1} e^{-\frac{x^2}{2g}} \ e^{\sum\limits_{s=0}^{\infty} \frac{t_s}{s!} x^s}\right) dx = 0 \ , \quad n \geq -1$$

formal Diff on $\mathbb{R}$ and operators $l_n = -\frac{d}{dx}(x^{n+1}...)$:

$$[l_n, l_m] = (n-m) l_{n+m}$$

Motivation, toy model

Virasoro constraints:

$$L_n Z(g, \{t\}) = 0\ , \quad n \geq -1$$

with

$$L_n = (n+1)! \frac{\partial}{\partial t_n} - \frac{1}{g}(n+2)! \frac{\partial}{\partial t_{n+2}} + \sum_{s=1}^{\infty} \frac{(n+s)!}{(s-1)!} t_s \frac{\partial}{\partial t_{s+n}}\ , \quad n \geq 1$$

$$L_0 = 1 - \frac{2}{g}\frac{\partial}{\partial t_2} + \sum_{s=1}^{\infty} s t_s \frac{\partial}{\partial t_s}$$

$$L_{-1} = t_1 - \frac{1}{g}\frac{\partial}{\partial t_1} + \sum_{s=2}^{\infty} t_s \frac{\partial}{\partial t_{s-1}}$$

Infinity many PDE's!!!

Motivation, toy model

Answer:

$$Z(g,\{t\}) = e^{t_0}\sqrt{2\pi g}\sum_{p=0}^{\infty}\frac{1}{2^p p!}B_{2p}(t_1,\dots,t_{2p})g^p$$

here B_n are Bell's polynomials.

We can get this answer either by direct calculation of the integrals or by using the Virasoro constrains.

Idea is to do the same for more complicated integrals and find more complicated symmetries

Virasoro constraints for Hermitian matrix model

the Hermitian matrix model:

$$Z(\{t\}) = \int\limits_{u(N)} dM \; e^{\sum\limits_{s=0}^{\infty} \frac{t_s}{s!} \mathrm{Tr}(M^s)} ,$$

where $M^\dagger = M$ and the measure is invariant under $M \to U^\dagger M U$ with $U \in U(N)$.

In terms of eigenvalues of M:

$$Z(\{t\}) = \int\limits_{\mathbb{R}^N} \prod_{i=1}^{N} d\lambda_i \; \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{\sum\limits_{s=0}^{\infty} \frac{t_s}{s!} \sum\limits_{i=1}^{N} \lambda_i^s}$$

Virasoro constraints for Hermitian matrix model

Ward identities:

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} d\lambda_i \sum_{l=1}^{N} \frac{\partial}{\partial \lambda_l} \left(\lambda_l^{n+1} \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{\sum\limits_{s=0}^{\infty} \frac{t_s}{s!} \sum\limits_{i=1}^{N} \lambda_i^s} \right) = 0 \ ,$$

where

$$l_n = - \sum_{l=1}^{N} \frac{\partial}{\partial \lambda_l} \left(\lambda_l^{n+1} \cdots \right)$$

satisfy

$$[l_n, l_m] = (n-m) l_{n+m}$$

Virasoro constraints for Hermitian matrix model

After some rewriting we get the Virasoro constraints:

$$L_n Z(\{t\}) = 0 \ , \quad n \geq -1 \ ,$$

where

$$L_{-1} = \sum_{k=0}^{\infty} t_k \frac{\partial}{\partial t_{k-1}} \ ,$$

$$L_0 = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k} + N^2 \ ,$$

$$L_n = \sum_{k=0}^{n} (n-k)!k! \frac{\partial^2}{\partial t_k \partial t_{n-k}} + \sum_{k=0}^{\infty} \frac{k(k+n)!}{k!} t_k \frac{\partial}{\partial t_{k+n}} \ , \quad n \geq 1$$

Virasoro constraints for Hermitian matrix model

Let us think for the moment, we naturally have the representation of Heisenberg algebra:

$$\text{creation operator:} \quad \alpha_{-n} = \frac{\sqrt{2}}{(n-1)!} t_n \ ,$$

$$\text{annihilation operator:} \quad \alpha_n = \frac{n!}{\sqrt{2}} \frac{\partial}{\partial t_n} \ ,$$

and we can check that

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \alpha_{n-m} \alpha_m : \ , \quad n \geq -1$$

but it can be extended to all n's and we get the full Virasoro algebra with the central charge $c = 1$.

Thus we deal with the free boson $\phi(x) = \sum_n a_n x^{-n}$

Looking for an operator $S(x)$ such that

$$[L_n, S(x)] = \frac{d}{dx} \mathrm{O}(x) \ ,$$

we can get easily the solution of Virasoro constraints

$$Z(\{t\}) = \mathrm{Q}^N \ , \quad \mathrm{Q} = \int dx \ S(x) \ ,$$

we immediately get

$$L_n Q^N |0\rangle = L_n(\{t_k\}) \mathrm{Z}(\{t_k\}) = 0$$

This is indeed the Hermitian matrix model, in this argument only contour of integration is not specified.
explain the details on the blackboard

One can keep playing this game. The symmetries of integrals are not only differential operators, but also the difference operators, e.g. q-derivative

$$D_q f(x) = \frac{f(qx) - f(x)}{x(q-1)}$$

such that

$$\lim_{q \to 1} D_q f(x) = \frac{df}{dx}(x) \; .$$

We have

$$\int_{-\infty}^{\infty} D_q f(x) dx = \frac{1}{q-1} \int_{-\infty}^{\infty} f(qx) \frac{dx}{x} - \frac{1}{q-1} \int_{-\infty}^{\infty} f(x) \frac{dx}{x} = 0 \; .$$

Thus we can start to insert the following operators

$$T_n^q = -D_q(x^{n+1}...)$$

which satisfy the following algebra

$$q^n T_n^q T_m^q - q^m T_m^q T_n^q = ([n]_q - [m]_q) T_{n+m}^q \ ,$$

or

$$[T_n^q, T_m^q] = q^{-n-m}([n]_q - [m]_q) \left([2]_q T_{n+m}^{q^2} - T_{n+m}^q \right)$$

This is the deformation of the Virasoro algebra.

Comment on the symmetries of the integral:

$$Z(\{t\}) = \int d^N x \ f(x_1, ..., x_N) \ e^{\sum\limits_{k=0}^{\infty} \frac{t_k}{k!} \sum\limits_{i=1}^{N} x_i^k} \ ,$$

$$DZ(\{t\}) = \int d^N x \ f(x_1, ..., x_N) \ \sigma_D(x_1, ..., x_N) \ e^{\sum\limits_{k=0}^{\infty} \frac{t_k}{k!} \sum\limits_{i=1}^{N} x_i^k} = 0$$

there can be ideal generated by the operators

$$(D - \tilde{D}) \ e^{\sum\limits_{k=0}^{\infty} \frac{t_k}{k!} \sum\limits_{i=1}^{N} x_i^k} = 0$$

It gets more complicated when we look at higher order differential operators (e.g., $\partial^k x^n$) etc.

the symmetry problem is complicated even for finite dimensional integral, we do not know how to solve

We take a different path, we will use the representation of the Heisenberg algebra or its deformations

Deformations of Heisenberg ($p = qt^{-1}$, $t = q^{\beta}$):

$$[a_n, a_m] = \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}})\delta_{n+m,0}\ , \quad n, m \in \mathbb{Z}\backslash\{0\}\ ,$$

$$[P, Q] = 2\ ,$$

the deformed Virasoro

$$[T_n, T_m] = -\sum_{\ell} f_\ell (T_{n-\ell} T_{m+\ell} - T_{m-\ell} T_{n+\ell})$$

$$-\frac{(1-q)(1-t^{-1})}{(1-p)}(p^n - p^{-n})\delta_{n+m,0}$$

$q = e^{\hbar}$, we have the small $\hbar$ expansion

$$T_n = 2\delta_{n,0} + \hbar^2 \beta \left(L_n + \frac{Q_\beta^2}{4}\delta_{n,0} \right) + O(\hbar^4)$$

3D partition functions

the representation of deformed Heisenberg

$$a_{-n} = (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) t_n \ , \quad a_n = \frac{1}{n}(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}}) \frac{\partial}{\partial t_n} \ , \quad n \in \mathbb{Z}_{>0} \ ,$$

$$\sqrt{\beta} Q = t_0 \ , \quad P = 2\sqrt{\beta} \frac{\partial}{\partial t_0} \ , \quad |0 >= 1$$

So we do the similar thing, construct the operators S such that

$$[T_n, \int dx \ S(x)] = 0$$

3D partition functions

$$Z(\{t\}) = \oint \prod_{i=1}^{N} \frac{dw_i}{2\pi i w_i} \prod_{i \neq j} \frac{(w_i w_j^{-1}; q)_\infty}{(t w_i w_j^{-1}; q)_\infty} \, e^{\sum_{k=0}^{\infty} t_k \sum_j w_j^k}$$

such that

$$T_n Z(\{t\}) = 0 \; , \quad n > 0$$

3D gauge theory on $D^2 \times S^1$,

$N = 2$ $U(N)$ vector with adjoint chiral

3D partition functions

q-Virasoror modulo double (a'la Faddeev):

$$q_1 = e^{2\pi i \tau} \ , \qquad t_1 = e^{2\pi i \tau \beta}$$
$$q_2 = e^{2\pi i \frac{\tau}{\tau - 1}} \ , \qquad t_2 = e^{2\pi i \beta \frac{\tau}{\tau - 1}}$$

the gauge theory interpretation

$$\tau = \frac{\omega}{\omega_1} \ , \qquad \omega = \omega_1 + \omega_2$$

$$S^3 \ , \qquad \omega_1 |z_1|^2 + \omega_2 |z_2| = 1$$

3D $U(N)$ vector with adjoint scalar on S^3

$$Z(\{t\}, \{\tilde{t}\})$$

3D partition functions

two commuting q-Virasoro constraints:

$$T_n Z(\{t\}, \{\tilde{t}\}) = 0 \ , \qquad n > 0$$

$$\tilde{T}_n Z(\{t\}, \{\tilde{t}\}) = 0 \ , \qquad n > 0$$

Summary

• there are many generalizations (including elliptic deformations)

• BPS/CFT correspondence (magic)

• it would be nice to understand the symmetries of the integrals from the first principles