From quantum circle bundles to quantum sphere bundles,

going beyond Pimsner's construction

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Let $V \to X$ a Hemitian vector bundle, $S(V)$ its sphere bundle.

Gysin sequence in K-theory: six-term exact sequence

$$\begin{tikzcd} K^0(X) \arrow[r, "\alpha"] & K^0(X) \arrow[r, "\pi^*"] & K^0(S(V)) \arrow[d, "\delta_{0,1}"] \\ K^1(S(V)) \arrow[u, "\delta_{1,0}"] & K^1(X) \arrow[l, "\pi^*"] & K^1(X) \arrow[l, "\alpha"] \end{tikzcd} , \quad (1)$$

where α is the multiplication by the Euler class of the vector bundle.

Example

For a line bundle L, $S(V)$ is a circle bundle and

$$\chi(L) = 1 - [L].$$

In view of the Serre–Swan theorem, noncommutative vector bundles are finitely generated projective modules.
Hermitian vector bundles are finitely generated projective Hilbert modules.

What about sphere bundles?

- Circle bundles –> complete answer
- General sphere bundles –> work in progress

1 Motivation

2 Pimsner algebras, circle bundles, and Gysin sequences

3 Towards noncommutative sphere bundles: subproduct systems

Definition

A *self-Morita equivalence bimodule (SMEB)* over B consist of a full right Hilbert C*-module E over B and an isomorphism

$$\phi : B \to \mathcal{K}(E).$$

Example: $B = C(X)$ and $E = \Gamma(\mathcal{L})$ the module of sections of a Hermitian $\mathcal{L} \to X$.

Self Morita equivalences over B form a group: the *Picard group* of B, denoted $\mathrm{Pic}(B)$.

Out of internal tensor products, construct

$$\mathcal{F}_E := B \oplus \bigoplus_{n \geq 1} E^{\otimes n}$$

On $\mathcal{F}_E$ define the *shift operators* by

$$T_\eta(\xi_1 \otimes \cdots \xi_n) = \eta \otimes \xi_1 \otimes \cdots \xi_n, \quad T_\eta a = \eta a.$$

The *Toeplitz algebra* $\mathcal{T}_E$ as the C*-subalgebra of $\mathcal{L}(\mathcal{F}_E)$ generated by the shifts.

The Cuntz–Pimsner algebra of E is the quotient

$$0 \longrightarrow \mathcal{K}(\mathcal{F}_E) \xrightarrow{j} \mathcal{T}_E \xrightarrow{\pi} \mathcal{O}_E \longrightarrow 0. \tag{2}$$

Examples: Cuntz and Cuntz–Krieger algebras, graph algebras, crossed products by $\mathbb{Z}$...

For any f.g.p. module: realisation in terms of generators and relations.

Let $\{\eta_i\}_{i=1}^n$ be a finite frame for E, i.e.

$$\xi = \sum_{j=1}^{n} \eta_j \langle \eta_j, \xi \rangle_B, \qquad \forall \xi \in E.$$

Then $\mathcal{O}_E$ is the *universal C*-algebra* generated by B together with n operators $S_1, \dots, S_n$, satisfying

$$S_i^* S_j = \langle \eta_i, \eta_j \rangle_B, \quad \sum\nolimits_j S_j S_j^* = 1, \quad \text{and} \quad bS_j = \sum\nolimits_i S_i \langle \eta_i, \phi(b)\eta_j \rangle_B,$$

for $b \in B$, and $j = 1, \dots, n$.

We have a circle action γ on $\mathcal{O}_E$ called the *gauge action*.
This is defined on generators by

$$\gamma_z S_i = z S_i, \quad \forall i = 1, \dots, n.$$

We denote by $\mathcal{O}_E^{\gamma}$ the fixed point for this action.

Proposition

E is a self-Morita equivalence bimodule if and only if $\mathcal{O}_E^{\gamma} \simeq B$.

Let A be a C^*-algebra with an action $\{\sigma_z\}_{z\in S^1}$.
For each $n \in \mathbb{Z}$, one can define the spectral subspaces

$$A_{(n)} := \left\{\xi \in A \mid \sigma_z(\xi) = z^{-n}\,\xi \quad \text{for all } z \in S^1\right\}.$$

Then $A_{(0)} = A^\gamma$, $A^*_{(n)} = A_{(-n)}$ and that $A_{(n)}A_{(m)} \subseteq A_{(n+m)}$.

Theorem (A.–Kaad–Landi)

Suppose that the circle action $\{\sigma_z\}$ satisfies

$$A^*_{(1)}A_{(1)} = A_{(0)} = A_{(1)}A^*_{(1)}.$$

Then the Pimsner algebra $\mathcal{O}_{A_{(1)}}$ is isomorphic to A.

Proposition (Gabriel–Grensing)

Let A be a unital, commutative C-algebra. Suppose that the first spectral subspace $E = A_{(1)}$ generates A as a C*-algebra, and that it is finitely generated projective over $B = A_{(0)}$.*

Then the following facts hold

1. $B = C(X)$ *for some compact space* X*;*
2. $E = \Gamma(\mathcal{L})$ *for some line bundle* $\mathcal{L} \to X$*;*
3. $A = C(P)$*, where* $P \to X$ *is the principal circle bundle over* X *associated to the line bundle* $\mathcal{L}$*.*

Pimsner 1997: The defining extension is semi-split. Hence it induces six term exact sequences in KK-theory.

These simplify by using:

- The class of the correspondence $E \in KK(B, B)$;
- The class of the Morita equivalence $[\mathcal{F}_E] \in KK(\mathcal{K}_B(\mathcal{F}_E), B)$;
- The class of the inclusion $j : \mathcal{K}(\mathcal{F}_E) \to \mathcal{T}_E$.
- The class of the KK-equivalence $[\alpha]^{-1} \in KK(\mathcal{T}_E, B)$, which is the inverse to the class of the inclusion $\alpha : B \hookrightarrow \mathcal{T}_E$.
- The classes satisfy:

$$[\mathcal{F}_E] \otimes_B (1 - [E]) = [j] \otimes_{\mathcal{T}_E} [\alpha]^{-1}$$

.

Let [ext] be the class of the defining extension and
$[\partial] := [\text{ext}] \otimes_{\mathcal{K}(\mathcal{F}_E)} [\mathcal{F}_E] \in KK_1(\mathcal{O}_E, B)$ the class of the product.
For $C = \mathbb{C}$ we get exact sequences in KK-theory. In K-theory:

$$\begin{tikzcd} K_0(B) \arrow[r, "1-[E]"] & K_0(B) \arrow[r, "j_*"] & K_0(\mathcal{O}_E) \arrow[d, "{[\partial]}"] \\ K_1(\mathcal{O}_E) \arrow[u, "{[\partial]}"] & K_1(B) \arrow[l, "j_*"] & K_1(B) \arrow[l, "1-[E]"] \end{tikzcd} , \tag{3}$$

The C*-algebra of the odd-dimensional quantum sphere $C(S_q^{2n+1})$ is the universal C*-algebra generated by $n+1$ elements $\{z_i\}_{i=0,\ldots,n}$ and relations:

$$
\begin{aligned}
z_i z_j &= q^{-1} z_j z_i && 0 \le i < j \le n \,, \\
z_i^* z_j &= q z_j z_i^* && i \neq j \,, \\
[z_n^*, z_n] = 0, \qquad [z_i^*, z_i] &= (1-q^2) \sum_{j=i+1}^{n} z_j z_j^* && i = 0, \ldots, n-1 \,,
\end{aligned}
$$

and a sphere relation:

$$z_0 z_0^* + z_1 z_1^* + \ldots + z_n z_n^* = 1 \,.$$

This C*-algebra can be realised as a graph C*-algebra.

Let $\mathrm{m} = (m_0, \dots, m_n)$ any weight vector.
Weighted circle action on $C(S_q^{2n+1})$, whose fixed point algebra is the algebra of functions on the weighted projective spaces $C(\mathbb{WP}^n(\mathrm{m}))$:

$$\sigma_\xi^{\mathrm{m}}(z_i) = \xi^{m_i} z_1 \qquad \xi \in \mathbb{T}^1. \tag{4}$$

Brzeziński-Szymański (BS16): let m be a weight vector such that there exists $0 \le j \le n-1$ with m_j coprime with m_n. Denote by m_n the weight vector $(m_0, \dots, m_{n-1})$. Then there exists an exact sequence of C*-algebras

$$0 \longrightarrow \mathcal{K}^{\oplus m_n} \longrightarrow C(\mathbb{WP}^n(\mathrm{m})) \longrightarrow C(\mathbb{WP}_q^{n-1}(\mathrm{m}_n)) \longrightarrow 0, \tag{5}$$

Proposition (Brzeziński–Szymański 2016)

Let m *be a weight vector with the property that for each* $j \geq 1$ *there exists* $i < j$ *such that* $\gcd(m_i, m_j) = 1$. *Let* $M = \sum_{i=1}^{n} m_i$. *Then the K-theory groups of the quantum weighted projective spaces are given by*

$$K_0(C(\mathbb{WP}_q^n(\mathrm{m})) = \mathbb{Z}^{1+M}, \quad K_1(C(\mathbb{WP}_q^n(\mathrm{m})) = 0.$$

Proposition (A. 2016)

Let m *be a weight vector satisfying the previous assumptions. Then the* C^**-algebra* $C(\mathbb{WP}^n(\mathrm{m}))$ *is KK-equivalent to* $\mathbb{C}^{1+M}$.

Let E denote the Hilbert C*-module given by the first spectral subspace for the weighted circle action on $C(S_q^{2n+1})$.
The Pimsner algebra over $B = C(\mathbb{WP}^n)$ for the module $E^{\otimes d}$ is the C*-algebra of a quantum lens space, i.e.

$$\mathcal{O}_{E^{\otimes d}} \simeq C(L_q^{2n+1}(d \cdot N_{\mathrm{m}}; \mathrm{m})) \simeq C(S^{2n+1})^{\mathbb{Z}_r}.$$

Free action (principal circle bundle) for $N_{\mathrm{m}} = \prod_{i=0}^{n} m_i$.

Idea: use the KK-equivalence and Pimsner's exact sequences.

The exact sequence in K-theory becomes of the form

$$0 \longrightarrow K_0\big(C(L_q(d))\big) \longrightarrow \mathbb{Z}^{M+1} \xrightarrow{1-\mathrm{A}^d} \mathbb{Z}^{M+1} \longrightarrow K_0\big(C(L_q(d))\big) \longrightarrow 0 \ ,$$

where A is a matrix of pairings.

Proposition

Let m *be a weight vector as above. Then for any* $d \in \mathbb{N}$ *we have*

$$K_0\big(C(L_q(d))\big) \simeq \mathrm{Coker}(1-\mathrm{A}^d), \qquad K_1\big(C(L_q(d))\big) \simeq \mathrm{Ker}(1-\mathrm{A}^d)$$

- In the case of SMEBs, Pimsner's construction can be thought of as a noncommutative associated circle bundle construction.
- The corresponding six-term exact can be interpreted as a *Gysin sequence* in K-theory and K-homology for the 'line bundle' E over the 'noncommutative base space' B.
- Multiplication by the Euler class is replaced with the Kasparov product with $1 - [E]$.
- Applications include computations of K-theory groups (e.g. of weighted lens spaces).

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3 Towards noncommutative sphere bundles: subproduct systems

Arveson (1989) considered collections $\{H_t\}_{t\in\mathbb{R}}$ of Hilbert spaces such that

$$H_t \otimes H_s \simeq H_{t+s}.$$

Definition (Fowler 2002)

A product system over a semigroup P is a collection $\{X_p\}_{p\in\mathbb{P}}$ of C*-correspondences over B with multiplication

$$X_p \otimes_B X_q = X_{p+q}, \quad \forall p, q \in P.$$

Given a single C*-correspondence (E, ϕ) over B, the collection

$$\{E^{\otimes_\phi n}\}_{n\in\mathbb{N}}$$

is a product system over the semigroup $\mathbb{N}$.

One can construct two universal C*-algebras $\mathcal{T}_X$ and $\mathcal{O}_X$, called the *Nica–Toeplitz* and *Cuntz–Nica–Pimsner* algebra of the product system X.

$\mathcal{T}_X$ and $\mathcal{O}_X$ generalise $\mathcal{T}_E$ and $\mathcal{O}_E$ and are *universal* for Toeplitz and Cuntz–Pimsner representations.

Theorem (Fletcher, 2016)

Cuntz–Nica–Pimsner algebras of compactly aligned product systems over $\mathbb{N}^k$ *are* k*-fold iterated Cuntz–Pimsner algebras.*

Torus bundles can be modelled using Cuntz–Nica–Pimsner algebras for product systems over $\mathbb{N}^k$.

Definition (Shalit–Solel 2009)

A *standard subproduct system* over $\mathbb{N}$ is a collection $\{X_n\}_{n\in\mathbb{N}}$ of C*-correspondences over B such that

1. $X_0 = B$;
2. X_{n+m} is a complemented submodule of $X_n \otimes_B X_m$ for all $n, m \in \mathbb{N}$.

Given a single C*-correspondence (E, ϕ) over B, the collection

$$\{E^{\otimes_\phi n}\}_{n\in\mathbb{N}}$$

is a product system over the semigroup $\mathbb{N}$ and trivially a standard subproduct system.

Let $\{X_n\}_{n\in\mathbb{N}}$ be a standard subproduct system and take $E = X(1)$.
Then $X(n)$ is a complemented submodule of $E^{\otimes n}$, with orthogonal projection

$$p_n : E^{\otimes n} \to X(n).$$

The module

$$\mathcal{F}_X := \bigoplus_{n\geq 0} X_n$$

is a submodule of the full Fock module

$$\mathcal{F}_E := \bigoplus_{n\geq 0} E^{\otimes n}$$

Example

Let $B = \mathbb{C}$, $E = \mathbb{C}^d$ and consider the symmetric tensor product $X_n = (\mathbb{C}^d)^{\odot n}$.
Then $\mathcal{F}_E$ is the full Fock space and $\mathcal{F}_X$ is the symmetric Fock space.

Let $\xi \in X_n$. The X-shift operator associated to ξ is given by

$$T_n(\xi)(\eta) = p_{n+m}(\xi \otimes \eta),$$

for all $\eta \in X_m$.

Definition (Viselter 2011)

The C*-subalgebra of $\mathcal{L}(\mathcal{F}_X)$ generated by the X-shifts is the *Toeplitz algebra* of the subproduct system X.

It agrees with the usual notion of Toeplitz algebra for a single correspondence.

Canonical $U(1)$-gauge action.

Inside $\mathcal{T}_X$ one can find a gauge invariant ideal $\mathcal{I}_X$, which agress with $\mathcal{K}(\mathcal{F}_E)$ for the case of a single correspondence.

Definition

The Cuntz–Pimsner algebra of the subproduct system X is the quotient

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{T}_X \xrightarrow{\pi} \mathcal{O}_X \longrightarrow 0. \tag{6}$$

Example

For the symmetric subproduct system we get back the odd spheres:

$$0 \longrightarrow \mathcal{K}(H^2(S^{2d-1})) \longrightarrow \mathcal{T}_d \xrightarrow{\pi} C(S^{2d-1}) \longrightarrow 0.$$

Let again $E = \mathbb{C}^d$ with orthonormal basis e_i.

Consider the q-deformed symmetric subproduct system, i.e.

$$X_1 = E, \quad X_2 = E \otimes E/\mathrm{span}\{e_i \otimes e_j - q e_j \otimes e_i\}, \quad \cdots$$

Then $\mathcal{T}_X$ is the C*-algebra $C(B^{2n}_{q^2})$, a.k.a. the *quantum ball* of Hong & Szymański.

Extension involving the Vaksman–Soibelman spheres:

$$0 \longrightarrow \mathcal{K} \longrightarrow C(B^{2n}_{q^2}) \xrightarrow{\pi} C(S^{2d-1}_q) \longrightarrow 0.$$

- Cuntz–Pimsner algebras are a model for circle bundles.
- Cuntz–Nica–Pimsner algebras of product systems can model torus bundles.
- There is evidence that Cuntz–Pimsner algebas of subproduct systems are suitable to encode spherical symmetries.
- Open questions:
  - How to go from spheres to sphere bundles (work in progress).
  - KK-equivalences and exact sequences