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On Noncommutativity and physics:
Hopf algebras in Noncommutative Geometry

Noncommuative principal bundles on projective bases

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Given the relevance of principal bundles in geometry it is natural to study NC principal bundles. These are not understood as well as noncommutative Vector Bundles are.

The underlying geometrical structure of a gauge field theory is that of a principal bundle with its gauge group of transformations. NC gauge theories are useful in describing low energy regimes of string theories and allow to consider T-duality transformations (Morita equivalence) fully within gauge field theory context (on NC space).

Main examples of NC principal bundles, like the NC instanton bundle, are algebraically understood as Hopf-Galois extensions. This notion is particularly useful when the noncommutative base space is affine (the function algebra being given in terms of generators and relations). I present a relaxed notion, based on a sheaf theoretic approach, that allows to consider cases where the base is a projective variety (e.g. projective space).

Principal Bundle

$E \longrightarrow M$ is a P-bundle (P is the structure group):

- $E \longrightarrow M$ is a bundle
- E is a P-space (P is the structure group)
- The action of P on E preserves the fibers of E.

$E \longrightarrow M$ is a principal P-bundle:

$E/P \simeq M$ is homeomorphism

$$\begin{aligned} E \times P &\longrightarrow E \times_M E \\ (e, p) &\longmapsto (e, ep) \end{aligned} \quad \text{is homeomorphism}$$

(injectivity: freeness of the P action; surjectivity: trasitivity on the fibers)

Consider E, M, P affine varieties and let $\mathcal{O}(E)$, $\mathcal{O}(M)$, $\mathcal{O}(P)$ be the algebra of coordinate functions on E, M, P respectively. Then $\mathcal{O}(P)$ is a Hopf algebra, and $\mathcal{O}(E)$ is an $\mathcal{O}(P)$-comodule algebra with coaction

$$\begin{aligned} \mathcal{O}(E) &\to \mathcal{O}(E) \otimes \mathcal{O}(P) \\ f &\mapsto \delta(f) = f_0 \otimes f_1 \end{aligned} \qquad\qquad (f_0 \otimes f_1)(e,p) = f(ep)$$

$$\mathcal{O}(M) \simeq \mathcal{O}(E/P) = \mathcal{O}(E)^{co\,\mathcal{O}(P)} = \{f \in \mathcal{O}(E),\, \delta(f) = f \otimes 1\}$$

Bijectivity of $E \times P \longrightarrow E \times_M E$ then becomes bijectivity of

$$\chi : \mathcal{O}(E) \otimes_{\mathcal{O}(E)^{co\,\mathcal{O}(P)}} \mathcal{O}(E) \longrightarrow \mathcal{O}(E) \otimes \mathcal{O}(P)$$

$$f \otimes_{\mathcal{O}(E)^{co\,\mathcal{O}(P)}} f' \longmapsto f f'_0 \otimes f'_1$$

Def. $\mathcal{O}(E)^{co\,\mathcal{O}(P)} \subset \mathcal{O}(E)$ is Hopf-Galois extension if χ is a bijection.

If P is semisimple surjectivity is enough to recover principality of the bundle

The definition applies also in the noncommutative case and many examples of Hopf-Galois extensions are obtained with noncommutative algebras and quantum groups, e.g. via quanization $\mathcal{O}(E) \to \mathcal{O}_q(E)$, $\mathcal{O}(H) \to \mathcal{O}_q(H)$.

We study the case where M is a projective variety. Specifically:

$E = G$ complex semisimple algebraic group, P parabolic subgroup.

Example:P upper Borel subgroup (upper triangular matrices) in SL_2. Then

$$\mathsf{SL}_2 \to \mathsf{SL}_2/P \cong \mathbf{P}^1(\mathbb{C})$$

is a principal bundle.

$P \subset \mathsf{SL}_2$ gives

$$\mathcal{O}(\mathsf{SL}_2) \xrightarrow{\pi} \mathcal{O}(P)$$

with

$$\mathcal{O}(\mathsf{SL}_2) = \mathbb{C}[a, b, c, d]/(ad - bc - 1)$$

$$\mathcal{O}(P) = \mathcal{O}(\mathsf{SL}_2)/(c) = \mathbb{C}[t, p, s]/(ts - 1)$$

$\mathcal{O}(\mathsf{SL}_2)$ is $\mathcal{O}(P)$ comodule algebra with

$$\delta = (\mathbf{I} \otimes \pi)\Delta : \mathcal{O}(\mathsf{SL}_2) \longrightarrow \mathcal{O}(\mathsf{SL}_2) \otimes \mathcal{O}(P)$$

$$\mathcal{O}(\mathsf{SL}_2)^{co\,\mathcal{O}(P)} = \mathbb{C}$$

Holomorphic fuctions on the Riemann sphere are just the constants (Liuville)

Notice $\mathcal{O}(\mathsf{SL}_2)^{co\,\mathcal{O}(P)} \subset \mathcal{O}(\mathsf{SL}_2)$ is not Hopf-Galois.

Solution:
$\mathbf{P}^1$ as a ringed space with the sheaf of regular functions. (Obtained form its homogeneous coordinate ring $\mathbb{C}[a, c]/(ac - ca)$).

The algebraic description of the principal bundle $\mathsf{SL}_2 \to \mathsf{SL}_2/P$ given in terms of **local trivializations**.

Then quantize to noncommutative algebras and quantum groups.

Consider the classical notion of principal bundle $(E \to M, P)$ as:
-$E \to M$ surjection,
-P that acts freely
-P transitive on the fibers
-E is locally trivial i.e. there is an open covering $\{U_i\}$ of M with $E|_{U_i} \simeq U_i \times P$

Inspired and generalizing work by [Pflaum], [Cirio, Pagani] we define:

Definition . 1 . A **(quantum) ringed space** $(M, \mathcal{O}_M)$ is a pair consisting of a classical topological space M and a sheaf over M of (non commutative) algebras.

Definition . 2 . Let $(M, \mathcal{O}_M)$ be a quantum ringed space and H a Hopf algebra. $\mathcal{F}$ is a **quantum principal bundle** over $(M, \mathcal{O}_M)$ if:

- $\mathcal{F}$ is a sheaf of H comodule algebras;

- There exists an open covering $\{U_i\}$ of M such that:

1. $\mathcal{F}(U_i)^{\mathrm{co}H} = \mathcal{O}_M(U_i)$,

2. $\mathcal{F}$ is *locally cleft*, that is $\mathcal{F}(U_i)$ is a cleft extension of $\mathcal{F}(U_i)^{\mathrm{co}H}$, i.e. $\mathcal{F}(U_i) \cong \mathcal{F}(U_i)^{\mathrm{co}H} \otimes H$, as $\mathcal{F}(U_i)$ modules and H comodules, for all i.

Remark . The condition of being locally cleft implies immediately that each $\mathcal{F}(U_i)$ is a Hopf-Galois extension of $\mathcal{F}(U_i)^{\mathrm{co}H}$.

In the example we used (and we will use):
If $\mathcal{B}$ is a basis for a topology on M. Then a sheaf $\mathcal{F}$ defined for the open sets in $\mathcal{B}$ (with gluing conditions) extends to a unique sheaf on M.
If $\{U_i\}$ is an open cover of M, then $U_{i_1} \cap \cdots \cap U_{i_r}$ form a basis for a topology on M.

Non commutative example:

$$\mathcal{O}_q(\mathrm{SL}_2) = \mathbb{C}_q\langle a, b, c, d\rangle / I_M + (ad - q^{-1}bc - 1)$$

where I_M is the ideal of the Manin relations.

As in previous example. Important point: The Ore condition is satisfied for the localization with a^{-1} and c^{-1}.

Program:

-Develop the approach in [Ciccoli, Fioresi, Gavarini] to quantum projective varieties to adapt it to quantum principal bundles.

G is a semisimple complex algebraic group, P a parabolic subgroup, G/P is a projective variety.

Given a representation ρ of P on some vector space V, we can construct a vector bundle associated to it, namely

$$\mathcal{V} := G \times_P V = G \times V / \simeq \, , \quad (gh, v) \simeq (g, \rho(h)v) \, , \qquad \forall h \in P, g \in G, v \in V.$$

The space of global sections of this bundle is identified with the induced module

$$H^0\Big(G\big/P, \mathcal{V}\Big) = \Big\{ f \colon G \to V \,\Big|\, f \text{ is regular, } f(gh) = \rho(h)^{-1} f(g) \Big\} \, .$$

Let $\chi \colon P \longrightarrow \mathbb{k}^*$ character of P, (1- dim rep. of P on $L \cong \mathbb{k}$). Then $L^{\otimes n}$ is again a one dimensional representation of P with character χ^n. Let $\mathcal{L}^n := G \times_P L^{\otimes n}$.

Define

$$\mathcal{O}(G/P)_n \;:=\; H^0\Big(G\Big/P, \mathcal{L}^n\Big)$$

$$\mathcal{O}(G/P) \;:=\; \oplus_{n\geq 0}\, \mathcal{O}(G/P)_n \;\subseteq\; \mathcal{O}(G)$$

Assume $\mathcal{L}$ is very ample, i.e. is generated by global sections f_0, f_1, $\dots$, $f_N \in \mathcal{O}(G/P)_1$; so that the algebra $\mathcal{O}(G/P)$ *is graded and generated in degree 1* (by the f_i's). Then $\mathcal{O}(G/P)$ is the homogeneous coordinate ring of the projective variety $G\Big/P$, with respect to the embedding given via the global sections of $\mathcal{L}$.

Reformulation in Hopf Algebraic terms

The character χ is just a group like element $\chi \in \mathcal{O}(P)$.

It can be lifted to a $t \in \mathcal{O}(G)$ in such a way that:

Proposition . *Let G/P be embedded into some projective space via some very ample line bundle. Then there exists a $t \in \mathcal{O}(G)$ (properly chosen section of the line bundle) such that*

$$\Delta_\pi(t) \;:=\; \Big((\mathit{id} \otimes \pi) \circ \Delta\Big)(t) \;=\; t \otimes \pi(t) \tag{2.2}$$

$$\pi\Big(t^m\Big) \neq \pi\Big(t^n\Big) \quad \forall\;\, m \neq n \in \mathbb{N} \tag{2.3}$$

$$\mathcal{O}(G/P)_n \;=\; \left\{ f \in \mathcal{O}(G) \;\middle|\; (id \otimes \pi)\Delta(f) = f \otimes \pi\big(t^n\big) \right\} \tag{2.4}$$

$$\mathcal{O}(G/P) \;=\; \bigoplus_{n\in\mathbb{N}} \mathcal{O}(G/P)_n \tag{2.5}$$

and $\mathcal{O}(G/P)$ is generated in degree 1, i.e. by $\mathcal{O}(G/P)_1$

Vice-versa, if such t exists, it gives a projective embedding of G/P.

Quantum version

Let $\mathcal{O}_q(G)$ be a quantum group and $\mathcal{O}_q(P)$ a quantum subgroup, quantizations respectively of G and P as above. We want to give a quantization of a classical section t, that is, an element $d \in \mathcal{O}_q(G)$, reducing to t when $q = 1$, that produces a quantum deformation of the projective embedding of G/P.

Data:

$$\mathcal{O}_q(G) \xrightarrow{\pi} \mathcal{O}_q(P) := \mathcal{O}_q(G)/I_q(P)$$

$I_q(P)$ being a quantization of the ideal $I_q(P)$ defining P.

A **Quantum section** of the line bundle $\mathcal{L}$ on $G\big/P$ is a quantization $d \in \mathcal{O}_q(G)$ of the classical section $t \in \mathcal{O}(G)$ such that

$$(id \otimes \pi)\Delta(d) = d \otimes \pi(d)$$

Define now:

$$\mathcal{O}_q(G/P) := \sum \mathcal{O}_q(G/P)_n$$

$$\mathcal{O}_q(G/P)_n := \{f \in \mathcal{O}_q(G) \,|\, (\mathit{id} \otimes \pi)\Delta(f) = f \otimes \pi\Big(d^n\Big)\}.$$

From [CFG] we have

$\mathcal{O}_q(G/P)$ is a *graded subalgebra* of $\mathcal{O}_q(G)$

$$\mathcal{O}_q(G/P)_r \cdot \mathcal{O}_q(G/P)_s \subseteq \mathcal{O}_q(G/P)_{r+s}, \quad \mathcal{O}_q(G/P) = \bigoplus_{n\in\mathbb{N}} \mathcal{O}_q(G/P)_n$$

$\mathcal{O}_q(G/P)$ is a *graded* (left) $\mathcal{O}_q(G)$–comodule algebra, via the restriction of the comultiplication Δ in $\mathcal{O}_q(G)$,

$$\Delta|_{\mathcal{O}_q(G/P)} : \mathcal{O}_q(G/P) \longrightarrow \mathcal{O}_q(G) \otimes \mathcal{O}_q(G/P)$$

Hence $\mathcal{O}_q(G/P)$ is a quantum homogeneous projective variety.

Remark . In the commutative case $\mathcal{O}(G/P)_r$ is the module of the algebraic sections of the line bundle $\mathcal{L}^r \to G/P$. We then interpret $\mathcal{O}_q(G/P)_r$ as the module of sections of a corresponding quantum line bundle, since it is a quantization of $\mathcal{O}(G/P)_r$.

From now on we assume that $\mathcal{O}_q(G/P)$ is generated in degree one, namely by $\mathcal{O}_q(G/P)_1$.

Quantum Grassmannian and flag are examples of this construction, they are both generated in degree one.

Example . Let us consider the case $G = \mathrm{SL}_n(\mathbb{C})$ and P the maximal parabolic subgroup of G:

$$P = \left\{ \begin{pmatrix} t_{r\times r} & p_{r\times n-r} \\ 0_{n-r\times r} & s_{n-r\times n-r} \end{pmatrix} \right\} \subset \mathrm{SL}_n(\mathbb{C})$$

The quotient G/P is the Grassmannian Gr of r spaces into the n dimensional vector space $\mathbb{C}^n$. It is a projective variety and it can be embedded, via the Plücker embedding, into the projective space $\mathbf{P}^N$ where $N = \binom{n}{r}$. This embedding corresponds to the character:

$$P \ni \begin{pmatrix} t & p \\ 0 & s \end{pmatrix} \mapsto \det(t) \in \mathbb{C}^\times$$

The coordinate ring $\mathcal{O}(\mathsf{Gr})$ of Gr, with respect to the Plücker embedding, is realized as the graded subring of $\mathcal{O}(\mathsf{SL}_n)$ generated by the determinants d_I of the minors obtained by taking (distinct) rows $I = (i_1, \dots, i_r)$ and columns $1, \dots, r$. In fact one can readily check that

$$(\mathit{id} \otimes \pi)\Delta(d_I) = d_I \otimes \pi\Big(d\Big), \qquad d = \det(a_{ij})_{1 \leq i,j \leq r}$$

Similarly quantum Grassmannian $\mathcal{O}_q(\mathsf{Gr})$ is defined as the graded subring of $\mathcal{O}_q(\mathsf{SL}_n)$ generated by all of the quantum determinants D_I of the minors obtained by taking (distinct) rows $I = (i_1, \dots, i_r)$ and columns $1, \dots, r$. It is a quantum deformation of $\mathcal{O}(\mathsf{Gr})$ and a quantum homogeneous space for the quantum group $\mathcal{O}_q(\mathsf{SL}_n)$. Again one can readily check that $d = D_{1 \dots r}$ is a quantum section and that

$$(\mathit{id} \otimes \pi)\Delta(D_I) = D_I \otimes \pi\Big(d\Big),$$

where $\mathcal{O}_q(P)$ is the quantum subgroup of $\mathcal{O}_q(G)$ defined by the ideal $I_P = (a_{ij})$ for $r + 1 \leq i \leq n$ and $1 \leq j \leq r$ and $\pi : \mathcal{O}_q(G) \longrightarrow \mathcal{O}_q(P)$ as before.

From Quantum Projective Varieties to Quantum Principal Bundles

The quantum section d determines (via the coproduct) the other quantum sections d_i spanning $\mathcal{O}_q(G/P)_1$.

Define V_i to be the open in G where the classical sections corresponding to the d_i do not vanish. Let U_i be the projection of V_i on the base space G/P.

Provided that $S_i := \{d_i^r\}$ is Ore in $\mathcal{O}_q(G)$ consider the $\mathcal{O}_q(P)$-comodule algebras

$$\mathcal{F} : U_i \to \mathcal{O}_q(V_i) := \mathcal{O}_q(G) S_i^{-1} \ .$$

These generate a sheaf of $\mathcal{O}_q(P)$-comodule algebras over the quantum ringed space $(G/P, \mathcal{O}_{G/P})$ where

$$\mathcal{O}_{G/P}(U_i) := \mathcal{O}_q(V_i)^{\mathrm{co}\,\mathcal{O}_\mathrm{q}(\mathrm{P})}$$

is the subalgebra of coinvariants.

Examples of Quantum Principal Bundles

Consider projective space $\mathbf{P}^N = G/P$ where $G = \mathsf{SL}_n(\mathbb{C})$, and the quantum projective space $\mathcal{O}_q(\mathbf{P}^N)$.

The sheaf $\mathcal{F}$ generated by $\mathcal{F} : U_i \to \mathcal{O}_q(V_i) := \mathcal{O}_q(\mathsf{SL}_n(\mathbb{C}))S_i^{-1}$ defines a quantum principal bundle on quantum projective space according to Definition 2, where the quantum ringed space $(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N})$ is generated by the coinvariants in $\mathcal{O}_q(V_i)$.

Quantum principal bundles according to Definition 2 on nontrivial opens of quantum Grassmannians $Gr(r, n)$ are similarly constructed.