# Crossed modules of Hopf algebras: an approach via monoids

**Gabriella Böhm**

Wigner Research Centre for Physics, Budapest

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# Crossed module

of **groups** [Whitehead 1941]

- generalization of a normal subgroup $N \triangleleft G$ to non-injective $N \to G$
- diverse applications
- equivalent to:
  - strict 2-group (= category object in the category of groups)
  - simplicial group whose Moore complex has length 1

  concise categorical proof [G Janelidze 2003]

of **groupoids** [Brown, İçen 2003]

of **Hopf algebras** [Fernández Vilaboa, López López, Villanueva Novoa 2006]

- working definitions
- bits of the equivalent forms
- no abstract categorical treatment

of **monoids** in monoidal categories **?**
a simplified review of [GB arXiv:1803.03418 1803.04124 1803.04622]

# Monoids in monoidal categories

a monoidal category consists of

- a category $\mathsf{C}$
- functors $\mathsf{C} \times \mathsf{C} \xrightarrow{\text{juxtaposition}} \mathsf{C} \xleftarrow{I} 1$
- coherent natural isomorphisms $(- -) - \rightarrow -(- -)$

  $I - \rightarrow - \leftarrow - I$ (omitted throughout)

**examples:** $(\mathsf{set}, \times)$, $(\mathsf{span}, \square)$, $(vec, \otimes)$, $(\mathsf{clg}, \otimes)$.

a monoid in a monoidal category consists of

- an object $A$
- morphisms $AA \xrightarrow{m} A \xleftarrow{u} I$ s.t.

$$\begin{tikzcd} AAA \arrow[r, "m1"] \arrow[d, "1m"'] & AA \arrow[d, "m"] \\ A \arrow[r, "m"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "u1"] \arrow[d, "1u"'] \arrow[dr, equal] & AA \arrow[d, "m"] \\ A \arrow[r, "m"'] & A \end{tikzcd}$$

commute

**examples:** ordinary monoids, small categories, algebras, bialgebras.

a monoid morphism is $A \xrightarrow{f} A'$ s.t. $f.m = m'.ff$ and $f.u = u'$

# Factorization of monoids

for monoid morphisms $A \xrightarrow{f} C \xleftarrow{g} B$ s.t. $q := AB \xrightarrow{fg} CC \xrightarrow{m} C$ is invertible

monoid morphism ⇔ monoid morphisms s.t.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, bend right, "a"'] & C \arrow[d, "c"] & B \arrow[l, "g"'] \arrow[dl, bend left, "b"] \\
 & D &
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, bend right, "a"'] & C \arrow[d, "c"] & B \arrow[l, "g"'] \arrow[dl, bend left, "b"] \\
 & D &
\end{tikzcd}$$

$$\begin{tikzcd}
BA \arrow[r, "ba"] \arrow[d, "gf"'] & DD \arrow[d, "m"] \\
CC \arrow[d, "m"'] & D \\
C \arrow[d, "q^{-1}"'] \arrow[r, phantom, "(*)"] & \\
AB \arrow[r, "ab"'] & DD \arrow[uu, "m"']
\end{tikzcd}$$

commutes

$C \xrightarrow{c} D$ ↦ $A \xrightarrow{f} C \xrightarrow{c} D \xleftarrow{c} C \xleftarrow{g} B$

$C \xrightarrow{q^{-1}} AB \xrightarrow{ab} DD \xrightarrow{m} D$ ↤ $A \xrightarrow{a} D \xleftarrow{b} B$

multiplicative iff $(*)$ commutes

# Idea

| | | | |
|---|---|---|---|
| **view** | **groups** | **groupoids** | **Hopf algebras** |
| as distinguished | monoids | categories | bialgebras |
| i.e. distinguished<br>**monoids** in the category of | sets | spans | coalgebras |

and **apply** the **factorization theory** of monoids to relate
category objects and crossed modules

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and **apply** the **factorization theory** of monoids to relate

category objects and crossed modules

# Split epimorphisms versus actions

category object in the category of ordinary monoids:

$$\begin{tikzcd} B \arrow[r, "i" description] & A \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & A\underset{B}{\square}A = \{(a,a')\in A\times A \mid s(a)=t(a')\} \arrow[l, "c"'] \end{tikzcd}$$

pullback monoid

for $A\underset{B}{\square}I = \{a\in A \mid s(a)=1\} \overset{j}{\rightarrowtail} A \xleftarrow{i} B$,

$$q = (A\underset{B}{\square}I)\times B \xrightarrow{j\times i} A\times A \xrightarrow{m} A\,, \qquad (y,b)\mapsto yi(b)$$

has the inverse $a\mapsto (ai(s(a)^{-1}), s(a))$ whenever $B$ is a **group**

$$\begin{tikzcd} B \arrow[r, "i" description] & A \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & A\underset{B}{\square}A \arrow[l, "c"'] \end{tikzcd} \quad \Leftrightarrow \quad B\times Y \xrightarrow{\triangleright} Y \text{ s.t.}$$

$$\begin{array}{ll} b\triangleright(yy') = (b\triangleright y)(b\triangleright y') & b\triangleright 1 = 1 \\ (bb')\triangleright y = b\triangleright(b'\triangleright y) & 1\triangleright y = 1 \end{array}$$

$$\begin{tikzcd} B \arrow[r, shift right=1, "i"'] & A \arrow[l, shift right=1, "s"'] \end{tikzcd} \quad \mapsto \quad B\times(A\underset{B}{\square}I) \xrightarrow{i\times j} A\times A \xrightarrow{m} A \xrightarrow{q^{-1}} (A\underset{B}{\square}I)\times B \xrightarrow{p_1} A\underset{B}{\square}I$$

$$(b,y)\mapsto i(b)yi(b^{-1})$$

$$\begin{tikzcd} B \arrow[r, shift right=1, "(1{,}-)"'] & Y\times B \arrow[l, shift right=1, "p_2"'] \end{tikzcd} \quad \leftarrow\!\shortmid \quad B\times Y \xrightarrow{\triangleright} Y$$

# Reflexive graphs versus pre-crossed modules

$$\begin{tikzcd}
B \arrow[r, "i" description] & A \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & A\underset{B}{\square}A \arrow[l, "c"']
\end{tikzcd}$$

monoid morphism ⇔ monoid morphisms s.t.

$$\begin{tikzcd}
A\underset{B}{\square}I \arrow[r, "j"] \arrow[dr, bend right, "k"'] & A \arrow[d, "t"] & B \arrow[l, "i"'] \arrow[dl, bend left, equal] \\
 & B &
\end{tikzcd}$$

$$\begin{tikzcd}
A\underset{B}{\square}I \arrow[r, "j"] \arrow[dr, bend right, "k"'] & A \arrow[d, "t"] & B \arrow[l, "i"'] \arrow[dl, bend left, equal] \\
 & B &
\end{tikzcd}$$

$$\begin{tikzcd}
B \times (A\underset{B}{\square}I) \arrow[r, "1\times k"] \arrow[d, "i\times j"'] & B \times B \arrow[d, "m"] \\
A \times A \arrow[d, "m"'] & B \\
A \arrow[d, "q^{-1}"'] & \\
(A\underset{B}{\square}I) \times B \arrow[r, "k\times 1"'] & B \times B \arrow[uu, "m"']
\end{tikzcd}$$

commutes; i.e.

$$k(b \triangleright y)b = bk(y)$$

$1^{\text{st}}$ Peiffer condition

# Category objects versus crossed modules

$$\begin{tikzcd} B \arrow[r, "i" description] & A \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & A\underset{B}{\square}A \arrow[l, "c"'] \end{tikzcd}$$

for $A\underset{B}{\square}I \xrightarrow{j} A \xrightarrow{1\square i} A\underset{B}{\square}A \xleftarrow{i\square 1} A$

$q_2 = (A\underset{B}{\square}I)\times A \xrightarrow{j\times 1} A\times A \xrightarrow{(1\square i)\times(i\square 1)} (A\underset{B}{\square}A)\times(A\underset{B}{\square}A) \xrightarrow{m} A\underset{B}{\square}A,\ (y,a)\mapsto(yit(a),a)$

has the inverse $(a',a)\mapsto(a'i(t(a)^{-1}),a)$ whenever $B$ is a **group**

$$\begin{tikzcd}
A\underset{B}{\square}I \arrow[r, "(1\square i).j"] \arrow[dr, bend right, "j"'] & A\underset{B}{\square}A \arrow[d, "c"] & A \arrow[l, "i\square 1"'] \arrow[dl, bend left, equal] & \Leftrightarrow & A\underset{B}{\square}I \arrow[r, "(1\square i).j"] \arrow[dr, bend right, "j"'] & A\underset{B}{\square}A \arrow[d, "c"] & A \arrow[l, "i\square 1"'] \arrow[dl, bend left, equal] \\
& A & & & & A &
\end{tikzcd}$$

s.t.

$$\begin{tikzcd}
A\times(A\underset{B}{\square}I) \arrow[r, "1\times j"] \arrow[d, "1\times j"'] & A\times A \arrow[dd, "m"] \\
A\times A \arrow[d, "(i\square 1)\times(1\square i)"'] & \\
(A\underset{B}{\square}A)\times(A\underset{B}{\square}A) \arrow[d, "m"'] & A \\
A\underset{B}{\square}A \arrow[d, "q_2^{-1}"'] & \\
(A\underset{B}{\square}I)\times A \arrow[r, "j\times 1"'] & A\times A \arrow[uu, "m"']
\end{tikzcd}$$

i.e. $y'y = (k(y')\triangleright y)y'$

2nd Peiffer condition

**yields an equivalence of the categories of**
**strict 2-groups and crossed modules**

**generalizes to any monoidal category with pullbacks** (e.g. set $\rightsquigarrow$ span)

# **Problem:** in some examples — e.g. in clg — there are no pullbacks

for coalgebra maps $A \xrightarrow{s} B \xleftarrow{t} C$

cotensor product

- $A\underset{B}{\square}C := \{\sum_i a^i \otimes c^i \in A \otimes C \mid \sum_i a^i_1 \otimes s(a^i_2) \otimes c^i = \sum_i a^i \otimes t(c^i_1) \otimes c^i_2\}$
  is a subcoalgebra iff $a \mapsto a_1 \otimes s(a_2)$ and $c \mapsto t(c_1) \otimes c_2$ are coalgebra maps
  (then the counits $\varepsilon$ induce coalgebra maps $A \xleftarrow{1\otimes\varepsilon} A \otimes C \xrightarrow{\varepsilon\otimes 1} C$ )

- a factorization

$$\begin{tikzcd}
 & & D \arrow[dll, bend right, "a"'] \arrow[d, red] \arrow[drr, bend left, "c"] & & \\
A & A \otimes C \arrow[l, "1\otimes\varepsilon"'] & A\underset{B}{\square}C \arrow[l, tail, "j"'] \arrow[r, tail, "j"'] & A \otimes C \arrow[r, "\varepsilon\otimes 1"'] & C
\end{tikzcd}$$

exists

iff $d \mapsto a(d_1) \otimes c(d_2)$ is a coalgebra map

**Idea:** only **relative** pullbacks wrt a suitable class of spans

# Admissible class of spans

**definition** a class $\mathcal{S}$ of spans in any category is admissible if

$$X \xleftarrow{f} A \xrightarrow{g} Y \ \in \mathcal{S} \Rightarrow \ X' \xleftarrow{f'} X \xleftarrow{f} A \xrightarrow{g} Y \xrightarrow{g'} Y' \ \in \mathcal{S} \ \forall f', g' \text{ and}$$

$$\Rightarrow \ X \xleftarrow{f} A \xleftarrow{h} B \xrightarrow{h} A \xrightarrow{g} Y \ \in \mathcal{S} \ \forall h.$$

**examples**

- the class of all spans in any category is admissible
- in clg the class $\mathcal{S}_{\text{clg}} := \{ \ X \xleftarrow{f} A \xrightarrow{g} Y \ \mid a \mapsto f(a_1) \otimes g(a_2) \text{ is a coalgebra map}\}$ is admissible

# Relative pullback

**definition** the $\mathcal{S}$-relative pullback of any $A \xrightarrow{s} B \xleftarrow{t} C$ is $A \xleftarrow{p_A} A\underset{B}{\Box}C \xrightarrow{p_C} C \in \mathcal{S}$

s.t. in

$$\begin{tikzcd}
D \arrow[dr, "h"] \arrow[drr, bend left, "c"] \arrow[ddr, bend right, "a"'] & & \\
& A\underset{B}{\Box}C \arrow[r, "p_C"] \arrow[d, "p_A"'] & C \arrow[d, "t"] \\
& A \arrow[r, "s"'] & B
\end{tikzcd}$$

- the blue square commutes
- if $A \xleftarrow{a} D \xrightarrow{c} C \in \mathcal{S}$ and the exterior commutes then $\exists !\ h$
- $A \xleftarrow{p_A} A\underset{B}{\Box}C \xleftarrow{f} D \xrightarrow{g} E$ & $C \xleftarrow{p_C} A\underset{B}{\Box}C \xleftarrow{f} D \xrightarrow{g} E \in \mathcal{S} \Rightarrow A\underset{B}{\Box}C \xleftarrow{f} D \xrightarrow{g} E \in \mathcal{S}$

  $E \xleftarrow{g} D \xrightarrow{f} A\underset{B}{\Box}C \xrightarrow{p_A} A$ & $E \xleftarrow{g} D \xrightarrow{f} A\underset{B}{\Box}C \xrightarrow{p_C} C \in \mathcal{S} \Rightarrow E \xleftarrow{g} D \xrightarrow{f} A\underset{B}{\Box}C \in \mathcal{S}$

**example** if $A \xrightarrow{s} B \xleftarrow{t} C$ are coalgebra maps s.t. $a \mapsto a_1 \otimes s(a_2)$, $c \mapsto t(c_1) \otimes c_2$ are coalgebra maps then

$$\begin{tikzcd}
A\underset{B}{\Box}C \arrow[r, "(\varepsilon\otimes 1).j"] \arrow[d, "(1\otimes\varepsilon).j"'] & C \arrow[d, "t"] \\
A \arrow[r, "s"'] & B
\end{tikzcd}$$

is an $\mathcal{S}_{\mathrm{clg}}$-relative pullback

# Relative category

**theorem** Let $\mathcal{S}$ be an admissible class of spans in a category s.t. for $A \xrightarrow{s} B \xleftarrow{t} C$ for which $A = A \xrightarrow{s} B, \quad B \xleftarrow{t} C = C \ \in \mathcal{S}$, there exists the $\mathcal{S}$-relative pullback $A\underset{B}{\square}C$. Then for any $B$ for which $B = B = B \ \in \mathcal{S}$, there is a monoidal category:

- objects are the spans $B \xleftarrow{t} A \xrightarrow{s} B$ s.t. $A = A \xrightarrow{s} B, \ B \xleftarrow{t} C = C \ \in \mathcal{S}$
- morphisms are the span morphisms
- monoidal product is the $\mathcal{S}$-relative pullback with the unit $B = B = B$

**example** for a cocommutative coalgebra $B$ there is a monoidal category:

- objects are the spans $B \xleftarrow{t} A \xrightarrow{s} B$ of coalgebras s.t. $a \mapsto a_1 \otimes s(a_2)$ and $a \mapsto t(a_1) \otimes a_2$ are coalgebra maps
- morphisms are the maps of coalgebra spans
- monoidal product is the cotensor product over $B$ with the unit $B = B = B$

**definition** for $\mathcal{S}$ and $B$ as in the theorem, an $\mathcal{S}$-relative category — with object of objects $B$ — is a monoid in the above monoidal category

$$\begin{tikzcd} B & A \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & A\underset{B}{\square}A \arrow[l, "c"'] \\ \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, "i"] & A \end{tikzcd}$$

# Relative categories vs crossed modules of monoids

**definition** a class of spans in a monoidal category is monoidal if

- $X \xleftarrow{f} I \xrightarrow{g} Y \ \in \mathcal{S} \ \forall f, g$
- $X \xleftarrow{f} A \xrightarrow{g} Y, \ X' \xleftarrow{f'} A' \xrightarrow{g'} Y' \ \in \mathcal{S} \Rightarrow \ XX' \xleftarrow{ff'} AA' \xrightarrow{gg'} YY' \ \in \mathcal{S}$

**example** $\mathcal{S}_{\mathrm{clg}}$ is monoidal

- any admissible class of spans in a monoidal category lifts to an admissible class in the category of monoids
- for a monoidal admissible class $\mathcal{S}$, relative pullbacks of monoids are monoids

$\Rightarrow$ $\mathcal{S}$-relative category of monoids is meaningful

**theorem** Let $\mathcal{S}$ be a monoidal admissible class of spans in a monoidal category s.t. for $A \xrightarrow{s} B \xleftarrow{t} C$ for which $A = A \xrightarrow{s} B$, $B \xleftarrow{t} C = C$ $\in \mathcal{S}$ there exists the $\mathcal{S}$-relative pullback $A\underset{B}{\square}C$. Then an analogous application of the **factorization theory of monoids** that we saw in set leads to a notion of crossed module of monoids, together with an equivalence between their category and the category of those $\mathcal{S}$-relative categories of monoids

$$\begin{tikzcd} B & A \arrow[l, shift right=2, "s"'] \arrow[l, shift left=2, "t"] & A\underset{B}{\square}A \arrow[l, "c"'] \\ B \arrow[r, phantom, ""] \end{tikzcd}$$

with $B \xrightarrow{i} A$, for which

$$q_n := (A\underset{B}{\square}I)A^{\underset{B}{\square} n} \xrightarrow{p_A 1} AA^{\underset{B}{\square} n} \xrightarrow{(1\square i\square\cdots\square i)(i\square 1)} A^{\underset{B}{\square} n+1}A^{\underset{B}{\square} n+1} \xrightarrow{m} A^{\underset{B}{\square} n+1}$$

is invertible for all $n \geq 0$.

for the explicit description of crossed module of monoids see [GB arXiv:1803.04124]

## example

- monoids in clg are bialgebras
- an $\mathcal{S}_{\mathrm{clg}}$-relative category of bialgebras consists of
  - a cocommutative bialgebra $B$ and an arbitrary bialgebra $A$
  - bialgebra maps $\begin{tikzcd} B \arrow[r, "i" description] & A \arrow[l, shift right=1.5ex, "s"'] \arrow[l, shift left=1.5ex, "t"] & A\underset{B}{\square}A \arrow[l, "c"'] \end{tikzcd}$ s.t.

  $a \mapsto s(a_1) \otimes a_2$, $a \mapsto a_1 \otimes t(a_2)$ are coalgebra maps
  $i$ is a common section of $s$ and $t$
  $c$ is an associative composition with the unit $i$
- $q_n : (A\underset{B}{\square}I)A^{\square_B n} \to A^{\square_B n+1}$, $(y, a', a'', \ldots, a^{(n)}) \mapsto (yit(a'_1), a'_2, a'' \ldots, a^{(n)})$ has the inverse $(a, a', \ldots, a^{(n)}) \mapsto (aizt(a'_1), a'_2, a'', \ldots, a^{(n)})$ if $B$ is a **Hopf algebra** with the antipode $z$

**corollary** The category of those $\mathcal{S}_{\text{clg}}$-relative categories whose object of objects is a (cocommutative) Hopf algebra is equivalent to the category of crossed modules of bialgebras which consist of

- a cocommutative Hopf algebra $B$ and an arbitrary bialgebra $Y$
- an action $B \otimes Y \xrightarrow{\triangleright} Y$ making $Y$ a $B$-module algebra and $B$-module coalgebra
- a bialgebra map $Y \xrightarrow{k} B$ s.t. $y \mapsto k(y_1) \otimes y_2$ is a coalgebra map and

$$k(b_1 \triangleright y)b_2 = bk(y) \qquad (k(y_1) \triangleright y')y_2 = yy'$$

if also $Y$ is a Hopf algebra $\rightsquigarrow$ [Fernández Vilaboa, López López, Villanueva Novoa]

**explains the equivalence of the categories of**
**relative categories of monoids and crossed modules of monoids**

**¿ simplicial monoids ?**

# Simplicial monoid and its Moore complex

**definition**

a simplicial monoid in a monoidal category consists of monoid morphisms

$$\begin{tikzcd}[column sep=huge]
M_0 \arrow[r, "\sigma_0" description] & M_1 \arrow[l, shift right=3, "\partial_0"'] \arrow[l, shift left=3, "\partial_1"] \arrow[r, shift left=1.5, "\sigma_0" description] \arrow[r, shift right=1.5, "\sigma_1" description] & M_2 \arrow[l, shift right=4.5, "\partial_0"'] \arrow[l, "\partial_1" description] \arrow[l, shift left=4.5, "\partial_2"] \cdots M_{n-1} \arrow[r, shift left=4.5, "\sigma_0" description] \arrow[r, phantom, shift right=1.5, "\vdots"] \arrow[r, shift right=4.5, "\sigma_{n-1}" description] & M_n \cdots \arrow[l, shift right=7.5, "\partial_0"'] \arrow[l, shift right=1.5, "\partial_1" description] \arrow[l, shift left=7.5, "\partial_n"]
\end{tikzcd}$$

satisfying the usual simplicial identities.

**proposition** For a simplicial monoid $M$ and a monoidal admissible class of spans $\mathcal{S}$ assume the existence of the $\mathcal{S}$-relative pullbacks; and hence the morphisms in

$$\begin{tikzcd}
M^{(1)}_{n+1} \arrow[rd, "\partial^{(1)}_k"] \arrow[rrd, bend left, "p_I"] \arrow[dd, "p_{M_{n+1}}"'] & & \\
& M^{(1)}_n \arrow[r, "p_I"] \arrow[d, "p_{M_n}"'] & I \arrow[d, "u"] \\
M_{n+1} \arrow[r, "\partial_k"'] & M_n \arrow[r, "\partial_n"'] & M_{n-1}
\end{tikzcd}
\qquad \cdots \qquad
\begin{tikzcd}
M^{(i)}_{n+1} \arrow[rd, "\partial^{(i)}_k"] \arrow[rrd, bend left, "p_I"] \arrow[dd, "p_{M^{(i-1)}_{n+1}}"'] & & \\
& M^{(i)}_n \arrow[r, "p_I"] \arrow[d, "p^{(i-1)}_{M_n}"'] & I \arrow[d, "u"] \\
M^{(i-1)}_{n+1} \arrow[r, "\partial^{(i-1)}_k"'] & M^{(i-1)}_n \arrow[r, "\partial^{(i-1)}_{n-(i-1)}"'] & M^{(i-1)}_{n-1}
\end{tikzcd}$$

for $0 \leq k \leq n$ $\quad \cdots \quad$ $0 \leq k \leq n-i+1.$

Then for $\{D_{n-1} := \ M^{(n)}_n \xrightarrow{p_{M^{(n-1)}_n}} M^{(n-1)}_n \xrightarrow{\partial^{(n-1)}_0} M^{(n-1)}_{n-1} \ \}_{n>0}$,

$$M^{(n+1)}_{n+1} \xrightarrow{D_n} M^{(n)}_n \xrightarrow{D_{n-1}} M^{(n-1)}_{n-1} \ = \ M^{(n+1)}_{n+1} \xrightarrow{p_I} I \xrightarrow{u} M^{(n-1)}_{n-1}.$$

(If $I$ is terminal then $\cdots \xrightarrow{D_n} M^{(n)}_n \xrightarrow{D_{n-1}} \cdots \xrightarrow{D_1} M^{(1)}_1 \xrightarrow{D_0} M_0$ is a chain complex.)

**definition** $M$ has Moore length $\ell$ if all $M^{(i)}_n$ exist and ( $I \xrightarrow{u} M^{(n-i)}_n$ , $M^{(n-i)}_n \xrightarrow{p_I} I$ ) are mutually inverse isomorphisms for all $0 \leq i$ and $n > i + \ell$.

# Simplicial monoids versus crossed modules

**theorem** Let $\mathcal{S}$ be a monoidal admissible class of spans in a monoidal category s.t. for $A \xrightarrow{s} B \xleftarrow{t} C$ for which $A = A \xrightarrow{s} B$, $B \xleftarrow{t} C = C \in \mathcal{S}$ there exits the $\mathcal{S}$-relative pullback $A \underset{B}{\square} C$. Then the category of crossed modules of monoids is equivalent to the category of those simplicial monoids $M$ for which

- the Moore length is 1
- $M_1 = M_1 \xrightarrow{\partial_0} M_0 \, , \; M_0 \xleftarrow{\partial_1} M_1 = M_1 \in \mathcal{S}$
- $y_{(n,k)} := M_n^{(k+1)} M_{n-1}^{(k)} \xrightarrow{p_{M_n^{(k)}} \sigma_{n-1}^{(k)}} M_n^{(k)} M_n^{(k)} \xrightarrow{m} M_n^{(k)}$ and

  $$q_n := M_1^{(1)} M_1^{\square_{M_0} n-1} \xrightarrow{p_{M_1} 1} M_1 M_1^{\square_{M_0} n-1} \xrightarrow{(1 \square \sigma_0 \square \cdots \square \sigma_0)(\sigma_0 \square 1)} M_1^{\square_{M_0} n} M_1^{\square_{M_0} n} \xrightarrow{m} M_1^{\square_{M_0} n}$$

  are invertible for all $n > 0$ and $0 \leq k < n$.

**obtained the equivalences**

$$\begin{tikzcd} M_0 \arrow[r, "\sigma_0" description] & M_1 \arrow[l, shift right=2, "\partial_0"'] \arrow[l, shift left=2, "\partial_1"] & M_2 \cong M_1 \underset{M_0}{\square} M_1 \arrow[l, "\partial_1"'] \end{tikzcd} \mapsfrom M$$

$$M \mapsto M_1^{(1)} \xrightarrow{D_0} M_0$$

**relative categories** $\xleftarrow[\cong]{}$ **simplicial monoids** $\xrightarrow[\cong]{}$ **crossed modules**
**of Moore length 1**

**example** in clg,

- $I$ is the base field thus terminal (the only coalgebra map $C \to I$ is the counit)
- the simplicial monoids are the simplicial bialgebras $M$
- for any simplicial monoid (=simplicial bialgebra) $M_n^{(i)}$ is the joint equalizer

$$\begin{tikzcd}
M_n^{(i)} \arrow[r] & M_n \arrow[rr, bend left=50, "\widehat{\partial}_n"] \arrow[rr, bend left=25, "\widehat{\partial}_{n-1}" description] \arrow[rr, phantom, "\vdots"] \arrow[rr, bend right=10, "\widehat{\partial}_{n-i+1}" description] \arrow[rr, bend right=40, "\widehat{1\varepsilon(-)}:x\mapsto x_1\otimes 1\otimes x_2"'] & & M_n \otimes M_{n-1} \otimes M_n
\end{tikzcd}$$

where $\widehat{\partial}_k(x) = x_1 \otimes \partial_k(x_2) \otimes x_3$

- if each $M_n$ is a **cocommutative** Hopf algebra then $y_{(n,k)}$, $q_n$ are invertible

$\Rightarrow$ the categories of

- $\mathcal{S}_{\text{clg}}$-relative categories of cocommutative Hopf algebras
- crossed modules of cocommutative Hopf algebras
- simplicial cocommutative Hopf algebras of length 1 Moore complex

are equivalent

# thank you!