Crossed modules of Hopf algebras: an approach via monoids

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Crossed module

of groups [Whitehead 1941]

- generalization of a normal subgroup $N \triangleleft G$ to non-injective $N \rightarrow G$
- diverse applications
- equivalent to: ➤ strict 2-group (= category object in the category of groups)
 - ► simplicial group whose Moore complex has length 1 concise categorical proof [G Janelidze 2003]

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of groupoids [Brown, İçen 2003]
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of Hopf algebras [Fernández Vilaboa, López López, Villanueva Novoa 2006]

- working definitions
- bits of the equivalent forms
- no abstract categorical treatment

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of monoids in monoidal categories ?
a simplified review of [GB arXiv:1803.03418 1803.04124 1803.04622]
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Monoids in monoidal categories

a monoidal category consists of

- a category C
- functors $C \times C \xrightarrow{\text{juxtaposition}} C \stackrel{/}{\leftarrow} 1$
- coherent natural isomorphisms $(--) \longrightarrow -(--)$

$$I - \rightarrow - \leftarrow -I$$
 (omitted throughout)

examples: (set, \times), (span, \square), (vec, \otimes), (clg, \otimes).

a monoid in a monoidal category consists of

$$AAA \xrightarrow{m1} AA \qquad A \xrightarrow{u1} AA$$

• an object A $AAA \xrightarrow{m1} AA$ $A \xrightarrow{u1} AA$ • morphisms $AA \xrightarrow{m} A \xleftarrow{u} I$ s.t. $1m \downarrow \qquad \qquad \downarrow m$ $1u \downarrow \qquad \downarrow m$ commute

examples: ordinary monoids, small categories, algebras, bialgebras.

a monoid morphism is $A \xrightarrow{f} A'$ s.t. f.m = m'.ff and f.u = u'

Factorization of monoids

multiplicative iff (*) commutes

for monoid morphisms
$$A \xrightarrow{f} C \overset{g}{\rightleftharpoons} B$$
 s.t. $q := AB \xrightarrow{fg} CC \xrightarrow{m} C$ is invertible monoid morphism \Leftrightarrow monoid morphisms s.t.
$$A \xrightarrow{f} C \overset{g}{\rightleftharpoons} B \qquad A \xrightarrow{f} C \overset{g}{\rightleftharpoons} B \qquad BA \xrightarrow{ba} DD \\ CC \qquad D \qquad W \qquad (*) \qquad M \\ CC \qquad D \qquad W \qquad (*) \qquad M \\ C \qquad Q^{-1} \downarrow \qquad (*) \qquad M \\ C \qquad Q^{-1} \downarrow \qquad (*) \qquad M \\ C \qquad Q^{-1} \downarrow \qquad (*) \qquad M \\ C \qquad Q^{-1} \downarrow \qquad (*) \qquad M \\ C \qquad Q^{-1} \downarrow \qquad (*) \qquad M \\ C \qquad Q^{-1} \downarrow \qquad (*) \qquad M \\ C \qquad Q^{-1} \downarrow \qquad (*) \qquad M \\ C \qquad Q^{-1} \downarrow \qquad (*) \qquad M \\ C \qquad Q^{-1} \downarrow \qquad (*) \qquad (*) \qquad M \\ C \qquad Q^{-1} \downarrow \qquad (*) \qquad ($$

Idea

view	groups	groupoids	Hopf algebras
as distinguished	monoids	categories	bialgebras
i.e. distinguished			
monoids in the category of	sets	spans	coalgebras

and **apply** the **factorization theory** of monoids to relate category objects and crossed modules



Idea

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and **apply** the **factorization theory** of monoids to relate category objects and crossed modules

Split epimorphisms versus actions

category object in the category of ordinary monoids:

$$B \underset{t}{\overset{s}{\underset{i \to c}{\longleftarrow}}} A \underset{B}{\overset{c}{\longleftarrow}} A \underset{B}{\square} A = \{(a, a') \in A \times A \mid s(a) = t(a')\}_{\downarrow}$$
pullback monoid

for
$$A \square I = \{ a \in A \mid s(a) = 1 \} \xrightarrow{j} A \xleftarrow{i} B$$
,
 $q = (A \square I) \times B \xrightarrow{j \times i} A \times A \xrightarrow{m} A$, $(y, b) \mapsto yi(b)$

has the inverse $a \mapsto (ai(s(a)^{-1}), s(a))$ whenever B is a **group**

$$B \underset{t}{\overset{s}{\rightleftharpoons}} A \overset{c}{\longleftarrow} A \underset{B}{\square} A \iff B \times Y \xrightarrow{\triangleright} Y \text{ s.t.}$$

$$b \triangleright (yy') = (b \triangleright y)(b \triangleright y') \quad b \triangleright 1 = 1$$

$$(bb') \triangleright y = b \triangleright (b' \triangleright y) \quad 1 \triangleright y = 1$$

$$B \underset{i}{\overset{s}{\rightleftharpoons}} A \implies B \times (A \underset{B}{\square} I) \xrightarrow{i \times j} A \times A \xrightarrow{m} A \xrightarrow{q^{-1}} (A \underset{B}{\square} I) \times B \xrightarrow{p_{1}} A \underset{B}{\square} I$$

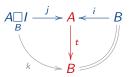
$$(b, y) \mapsto i(b)yi(b^{-1})$$

$$B \underset{(1, -)}{\overset{p_{2}}{\rightleftharpoons}} Y \times B \iff B \times Y \xrightarrow{\triangleright} Y$$

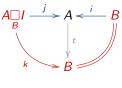
Reflexive graphs versus pre-crossed modules

$$B \underset{\leftarrow}{\overset{s}{\underset{i}{\longrightarrow}}} A \overset{c}{\longleftarrow} A \underset{B}{\square} A$$

monoid morphism



monoid morphisms



s.t.

$$k(b \triangleright y)b = bk(y)$$
1st Peiffer condition

Category objects versus crossed modules

for
$$A \square I \xrightarrow{j} A \xrightarrow{1 \square i} A \square A \xrightarrow{j \square 1} A$$

 $q_2 = (A \square I) \times A \xrightarrow{j \times 1} A \times A \xrightarrow{j \times 1} A \times A \xrightarrow{j \times 1} (A \square A) \times (A \square A) \xrightarrow{m} A \square A, \ (y, a) \mapsto (yit(a), a)$
has the inverse $(a', a) \mapsto (a'i(t(a)^{-1}), a)$ whenever B is a group

 $A \bigsqcup_{R} I \xrightarrow{(1 \sqcup i), j} A \bigsqcup_{R} A \xrightarrow{i \square 1} A \Leftrightarrow A \bigsqcup_{R} I \xrightarrow{(1 \sqcup i), j} A \bigsqcup_{R} A \xrightarrow{i \square 1} A \xrightarrow{R} \text{ s.t. } A \times (A \bigsqcup_{R} I) \xrightarrow{1 \times j} A \times A$ $A \times A$ $(i\Box 1)\times(1\Box i)$ $(A\square A)\times (A\square A)$ m $A \square A$ m $(A \square I) \times A \xrightarrow[i \times 1]{} A \times A$

i.e. $y'y = (k(y') \triangleright y)y'$ 2nd Peiffer condition

yields an equivalence of the categories of strict 2-groups and crossed modules

generalizes to any monoidal category with pullbacks (e.g. set → span)

Problem: in some examples — e.g. in clg — there are no pullbacks

for coalgebra maps $A \stackrel{s}{\longrightarrow} B \stackrel{t}{\longleftarrow} C$

cotensor product

- $A \square C := \{ \sum_i a^i \otimes c^i \in A \otimes C \mid \sum_i a_1^i \otimes s(a_2^i) \otimes c^i = \sum_i a^i \otimes t(c_1^i) \otimes c_2^i \}^{\checkmark}$ is a subcoalgebra iff $a \mapsto a_1 \otimes s(a_2)$ and $c \mapsto t(c_1) \otimes c_2$ are coalgebra maps (then the counits ε induce coalgebra maps $A \xrightarrow{1 \otimes \varepsilon} A \otimes C \xrightarrow{\varepsilon \otimes 1} C$)
- a factorization $A \underset{1 \otimes \varepsilon}{\longleftarrow} A \otimes C \underset{j}{\longleftarrow} A \underset{B}{\square} C \underset{j}{\longrightarrow} A \otimes C \underset{\varepsilon \otimes 1}{\longrightarrow} C$ exists

iff $d \mapsto a(d_1) \otimes c(d_2)$ is a coalgebra map

Idea: only **relative** pullbacks wrt a suitable class of spans

Admissible class of spans

definition a class ${\cal S}$ of spans in any category is admissible if

$$X \stackrel{f}{\leftarrow} A \stackrel{g}{\Rightarrow} Y \in \mathcal{S} \Rightarrow X' \stackrel{f'}{\leftarrow} X \stackrel{f}{\leftarrow} A \stackrel{g}{\Rightarrow} Y \stackrel{g'}{\Rightarrow} Y' \in \mathcal{S} \ \forall f', g' \ \text{and}$$

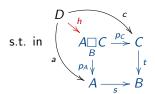
$$\Rightarrow X \stackrel{f}{\leftarrow} A \stackrel{h}{\leftarrow} B \stackrel{h}{\Rightarrow} A \stackrel{g}{\Rightarrow} Y \in \mathcal{S} \ \forall h.$$

examples

- the class of all spans in any category is admissible
- in clg the class $S_{\text{clg}} := \{ X \overset{f}{\leftarrow} A \overset{g}{\Rightarrow} Y \mid a \mapsto f(a_1) \otimes g(a_2) \text{ is a coalgebra map} \}$ is admissible

Relative pullback

definition the S-relative pullback of any $A \stackrel{s}{\rightarrow} B \stackrel{t}{\leftarrow} C$ is $A \stackrel{p_A}{\leftarrow} A \square C \stackrel{p_C}{\rightarrow} C \in S$



- the blue square commutes
- if $A \stackrel{a}{\leftarrow} D \stackrel{c}{\rightarrow} C \in S$ and the exterior commutes then $\exists ! h$
- $A \stackrel{p_A}{\leftarrow} A \square C \stackrel{f}{\leftarrow} D \stackrel{g}{\rightarrow} E \& C \stackrel{p_C}{\leftarrow} A \square C \stackrel{f}{\leftarrow} D \stackrel{g}{\rightarrow} E \in S \Rightarrow A \square C \stackrel{f}{\leftarrow} D \stackrel{g}{\rightarrow} E \in S$ $E \stackrel{g}{\leftarrow} D \stackrel{f}{\rightarrow} A \square C \stackrel{p_A}{\rightarrow} A \& E \stackrel{g}{\leftarrow} D \stackrel{f}{\rightarrow} A \square C \stackrel{p_C}{\rightarrow} C \in S \Rightarrow E \stackrel{g}{\leftarrow} D \stackrel{f}{\rightarrow} A \square C \in S$

example if $A \stackrel{s}{\Rightarrow} B \stackrel{t}{\Leftarrow} C$ are coalgebra maps s.t. $a \mapsto a_1 \otimes s(a_2), c \mapsto t(c_1) \otimes c_2$

 $A \underset{B}{\square} C \xrightarrow{(\varepsilon \otimes 1).j} C$ are coalgebra maps then $(1 \otimes \varepsilon).j \bigvee_{t} \text{ is an } \mathcal{S}_{\text{clg}}\text{-relative pullback}$ $A \xrightarrow{\qquad \qquad } B$

Relative category

theorem Let S be an admissible class of spans in a category s.t. for $A \stackrel{s}{\Rightarrow} B \stackrel{t}{\Leftarrow} C$ for which $A = A \stackrel{s}{\Rightarrow} B$, $B \stackrel{t}{\Leftarrow} C = C \in S$, there exists the S-relative pullback $A \square C$. Then for any B for which $B = B = B \in S$, there is a monoidal category:

- objects are the spans $B \stackrel{t}{\leftarrow} A \stackrel{s}{\Rightarrow} B$ s.t. $A = A \stackrel{s}{\Rightarrow} B$, $B \stackrel{t}{\leftarrow} C = C \in S$
- morphisms are the span morphisms
- monoidal product is the S-relative pullback with the unit B = B = B

example for a cocommutative coalgebra B there is a monoidal category:

- objects are the spans $B \stackrel{t}{\leftarrow} A \stackrel{s}{\rightarrow} B$ of coalgebras s.t. $a \mapsto a_1 \otimes s(a_2)$ and $a \mapsto t(a_1) \otimes a_2$ are coalgebra maps
- morphisms are the maps of coalgebra spans
- monoidal product is the cotensor product over B with the unit B = B = B

definition for S and B as in the theorem, an S-relative category — with object of objects B — is a monoid in the above monoidal category

$$B \underset{t}{\overset{s}{\underset{i \to \infty}{\longleftarrow}}} A \overset{c}{\underset{B}{\longleftarrow}} A \square A$$

Relative categories vs crossed modules of monoids

definition a class of spans in a monoidal category is monoidal if

- $X \stackrel{f}{\longleftarrow} I \stackrel{g}{\Longrightarrow} Y \in \mathcal{S} \ \forall f, g$
- $\bullet \quad X \overset{f}{\longleftarrow} A \overset{g}{\Longrightarrow} Y, \ X' \overset{f'}{\longleftarrow} A' \overset{g'}{\Longrightarrow} Y' \ \in \mathcal{S} \Rightarrow \ XX' \overset{ff'}{\longleftarrow} AA' \overset{gg'}{\Longrightarrow} YY' \ \in \mathcal{S}$

example S_{clg} is monoidal

- any admissible class of spans in a monoidal category lifts to an admissible class in the category of monoids
- \bullet for a monoidal admissible class $\mathcal{S},$ relative pullbacks of monoids are monoids
- \Rightarrow S-relative category of monoids is meaningful

theorem Let \mathcal{S} be a monoidal admissible class of spans in a monoidal category s.t. for $A \stackrel{s}{\Rightarrow} B \stackrel{t}{\Leftarrow} C$ for which $A = A \stackrel{s}{\Rightarrow} B$, $B \stackrel{t}{\Leftarrow} C = C \in S$ there exists the \mathcal{S} -relative pullback $A\square C$. Then an analogous application of the **factorization** theory of monoids that we saw in set leads to a notion of crossed module of monoids, together with an equivalence between their category and the category of those S-relative categories of monoids $B \stackrel{s}{\underset{\leftarrow}{\longleftarrow}} A \stackrel{c}{\underset{\leftarrow}{\longleftarrow}} A \square A$ for which

$$q_n := (A \square I) A^{\square n}_B \xrightarrow{p_A 1} A A^{\square n}_B \xrightarrow{(1 \square i \square \dots \square i)(i \square 1)} A^{\square n+1}_B A^{\square n+1}_B \xrightarrow{m} A^{\square n+1}_B$$
is invertible for all $n > 0$

is invertible for all n > 0.

for the explicit description of crossed module of monoids see [GB arXiv:1803.04124]

example

- monoids in clg are bialgebras
- ullet an $\mathcal{S}_{ exttt{clg}} exttt{-relative category of bialgebras consists of}$
 - ► a cocommutative bialgebra B and an arbitrary bialgebra A
 - ▶ bialgebra maps $B \stackrel{\leq}{\underset{t}{\rightleftharpoons}} A \stackrel{c}{\longleftrightarrow}
 - $a\mapsto s(a_1)\otimes a_2,\ a\mapsto a_1\otimes t(a_2)$ are coalgebra maps i is a common section of s and t
 - c is an associative composition with the unit i
- $q_n: (A \square I)A^{\square n}_B \to A^{\square n+1}_B$, $(y, a', a'', \dots, a^{(n)}) \mapsto (yit(a'_1), a'_2, a'', \dots, a^{(n)})$ has the inverse $(a, a', \dots, a^{(n)}) \mapsto (aizt(a'_1), a'_2, a'', \dots, a^{(n)})$ if B is a **Hopf** algebra with the antipode z

corollary The category of those \mathcal{S}_{clg} -relative categories whose object of objects is a (cocommutative) Hopf algebra is equivalent to the category of crossed modules of bialgebras which consist of

- a cocommutative Hopf algebra B and an arbitrary bialgebra Y
- an action $B \otimes Y \xrightarrow{\triangleright} Y$ making Y a B-module algebra and B-module coalgebra
- a bialgebra map $Y \stackrel{k}{\Rightarrow} B$ s.t. $y \mapsto k(y_1) \otimes y_2$ is a coalgebra map and

$$k(b_1 \triangleright y)b_2 = bk(y)$$
 $(k(y_1) \triangleright y')y_2 = yy'$

if also Y is a Hopf algebra → [Fernández Vilaboa, López López, Villanueva Novoa]

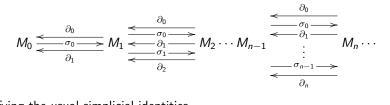
explains the equivalence of the categories of relative categories of monoids and crossed modules of monoids

¿ simplicial monoids ?

Simplicial monoid and its Moore complex

definition

a simplicial monoid in a monoidal category consists of monoid morphisms



satisfying the usual simplicial identities.

proposition For a simplicial monoid M and a monoidal admissible class of spans S assume the existence of the S-relative pullbacks; and hence the morphisms in

$$M_{n+1}^{(1)} \xrightarrow{\partial_{k}^{(1)}} P_{l} \cdots P_{M_{n+1}^{(i-1)}} M_{n}^{(i)} \xrightarrow{p_{l}} M_{n}^{(i)} \downarrow u$$

$$M_{n+1} \xrightarrow{\partial_{k}} M_{n} \xrightarrow{\partial_{n}} M_{n-1} \cdots M_{n+1}^{(i-1)} \xrightarrow{\partial_{k}^{(i-1)}} M_{n}^{(i-1)} \xrightarrow{\partial_{n-(i-1)}} M_{n-1}^{(i-1)}$$

$$\text{for } 0 \leq k \leq n \qquad \cdots \qquad 0 \leq k \leq n-i+1.$$
Then for $\{D_{n-1} := M_{n}^{(n)} \xrightarrow{P_{M_{n}^{(n-1)}}} M_{n}^{(n-1)} \xrightarrow{\partial_{0}^{(n-1)}} M_{n-1}^{(n-1)} \}_{n>0},$

$$M_{n+1}^{(n+1)} \xrightarrow{D_{n}} M_{n}^{(n)} \xrightarrow{D_{n-1}} M_{n-1}^{(n-1)} = M_{n+1}^{(n+1)} \xrightarrow{P_{l}} I \xrightarrow{u} M_{n-1}^{(n-1)}.$$

$$\text{(If l is terminal then } \cdots \xrightarrow{D_{n}} M_{n}^{(n)} \xrightarrow{D_{n-1}} \cdots \xrightarrow{D_{1}} M_{1}^{(1)} \xrightarrow{D_{0}} M_{0} \text{ is a chain complex.})$$

definition M has Moore length ℓ if all $M_n^{(i)}$ exist and $(I \stackrel{u}{\to} M_n^{(n-i)}, M_n^{(n-i)} \stackrel{p_l}{\to} I)$ are mutually inverse isomorphisms for all $0 \le i$ and $n > i + \ell$.

Simplicial monoids versus crossed modules

theorem Let S be a monoidal admissible class of spans in a monoidal category s.t. for $A \stackrel{s}{\Rightarrow} B \stackrel{t}{\Leftarrow} C$ for which $A = A \stackrel{s}{\Rightarrow} B$, $B \stackrel{t}{\Leftarrow} C = C \in S$ there exits the S-relative pullback $A \stackrel{c}{\sqsubseteq} C$. Then the category of crossed modules of monoids is equivalent to the category of those simplicial monoids M for which

- the Moore length is 1
- $M_1 = M_1 \stackrel{\partial_0}{\Rightarrow} M_0$, $M_0 \stackrel{\partial_1}{\Leftarrow} M_1 = M_1 \in \mathcal{S}$

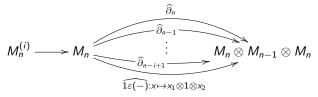
•
$$y_{(n,k)} := M_n^{(k+1)} M_{n-1}^{(k)} \xrightarrow{P_{M_n^{(k)}} \sigma_{n-1}^{(k)}} M_n^{(k)} M_n^{(k)} \xrightarrow{m} M_n^{(k)}$$
 and
$$q_n := M_1^{(1)} M_1^{M_0} \xrightarrow{p_{M_1} 1} M_1 M_1^{M_0} \xrightarrow{m-1} \underbrace{(1 \square \sigma_0 \square \cdots \square \sigma_0)(\sigma_0 \square 1)}_{m-1} M_1^{M_0} M_1^{M_0} \xrightarrow{m} M_1^{M_0}$$

are invertible for all n > 0 and $0 \le k < n$.

obtained the equivalences

example in clg,

- I is the base field thus terminal (the only coalgebra map $C \rightarrow I$ is the counit)
- the simplicial monoids are the simplicial bialgebras M
- for any simplicial monoid (=simplicial bialgebra) $M_n^{(i)}$ is the joint equalizer



where
$$\widehat{\partial}_k(x) = x_1 \otimes \partial_k(x_2) \otimes x_3$$

• if each M_n is a **cocommutative** Hopf algebra then $y_{(n,k)}$, q_n are invertible

\Rightarrow the categories of

- $ightharpoonup \mathcal{S}_{\mathsf{clg}}$ -relative categories of cocommutative Hopf algebras
- ► crossed modules of cocommutative Hopf algebras
- ► simplicial cocommutative Hopf algebras of length 1 Moore complex

are equivalent

thank you!

