L-infinity (non)formality and a generalization of T. Voronov's higher brackets

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- M.B., A.Makhlouf, Int.J.Theor Phys (2008) 47: 311-332
- O.Elchinger, thèse de doctorat UHA Mulhouse, 2012
- M.B., O.Elchinger, S.Gutt, A.Makhlouf: *L-infinity-Formality check for the Hochschild Complex of certain Universal Enveloping Algebras*, preprint 2018
- M.B.: *An unabelian version of the Voronov higher bracket construction*, Georgian Math.J. 22 (2015), 189-204.

Plan of the talk

MOTIVATION

- **differential graded Lie algebra** $(\mathfrak{g}, \delta, [\ ,\ ] = c_2)$:
  - *graded antisymmetry*: $[y, x] = -(-1)^{|x||y|}[x, y]$,
  - $\delta^2 = 0$ and δ graded derivation of degree 1
    $[\delta, c_2]_{SNR}(x, y) := \delta[x, y] - [\delta(x), y] - (-1)^{|x|}[x, \delta(y)] = 0$,
  - *graded Jacobi identity*

$$[[x, y], z] + (-1)^{|x|(|y|+|z|)}[[y, z], x] + (-1)^{|z|(|x|+|y|)}[[z, x], y] =: [c_2, c_2]_{SNR}(x, y, z) \stackrel{!}{=} 0.$$

  - Example: *Hochschild cohomology complex* $(CH(A, A), b, [\ ,\ ]_G)$ of an associative algebra A equipped with the **Gerstenhaber bracket**
  - Example: *Polyvectorfields* $\Gamma^\infty(M, \Lambda TM)$ with the *Schouten bracket* and Poisson structure $[P, P]_S = 0$, $\delta = [P,\ ]_S$.

- (differential graded Lie algebra)
  - *Morphisms*: linear maps $\phi_1 : \mathfrak{g} \to \mathfrak{h}$ of degree 0 preserving graded Lie brackets and differentials
  - *Maurer-Cartan elements*: $\mu' \in \lambda\mathfrak{g}[[\lambda]]$ of degree 1 with
    $$\delta(\mu') + \frac{1}{2}[\mu', \mu'] = 0.$$
  - *quasi-isomorphisms* induce isomorphisms on the cohomology Lie algebras, BUT do not have quasi-inverses
  - *weak equivalence*: zig-zag of quasi-isomorpisms
    $\mathfrak{g} \to \mathfrak{g}_1 \leftarrow \mathfrak{g}_2 \to \cdots \to \mathfrak{g}_N \to \mathfrak{g}'$
  - L_∞ *quasi-isomorphism*: embed dgLas in bigger category where weak equivalences are quis with quasi-inverse

- L_∞**-algebra** $(\mathfrak{g}, \delta =: c_1, [\ ,\ ] =: c_2, c_3, c_4, \ldots)$ (Lada, Stasheff 1993)
  - graded antisymmetry, $\delta^2 = 0$, and δ graded derivation, BUT
  - *graded Jacobi identity up to a coboundary*

$$[c_2, c_2]_{SNR} + 2[\delta, c_3]_{SNR} = 0$$

  - c_n is graded antisymmetric n-linear map of degree $2 - n$
  - *higher order identities* for all integers $n \geq 1$

$$\sum_{r=1}^{n-1} [c_r, c_{n-r}]_{SNR} = 0$$

  - *Morphisms*: sequence of graded antisymmetric n-linear maps $(\phi_n)_{n \in \mathbb{N} \setminus \{0\}}$ of degree $1 - n$ satisfying a series of identities '*Lie algebra morphism up to a coboundary*'

- (L_∞-algebra)
  - *Maurer-Cartan elements*: $\mu' \in \lambda\mathfrak{g}^1[[\lambda]]$ of degree 1 with
    $$\delta(\mu') + \sum_{r=2}^{\infty} \frac{1}{r!} c_r(\mu', \ldots, \mu') = 0.$$
  - *Construction of L_∞.structures* difficult, but
    - Homotopy transfer
      $(A, d_A) \overset{\leftarrows}{\rightarrow} (B, d_B)$
      with
      $ip = id_A, \quad pi = id_B - [h, d_B]$
    - T.Voronov constructions (*to* later)

- **Significance of L_∞-structures**
  - **M. Kontsevich**'s ingenious trick (1997): phrase the *deformation quantization problem* of Poisson manifolds as L_∞-morphism $\mathcal{U}$ between dg-Lie algebras

$$\Gamma^\infty(M, \Lambda^\bullet TM) \quad - - > \quad \mathrm{CH}^\bullet_{\mathrm{diff,nc}}(\mathcal{C}^\infty(M,\mathbb{R}), \mathcal{C}^\infty(M,\mathbb{R}))$$

    in general NO morphism of differential graded Lie algebras !! *Maurer-Cartan elements*: Poisson structures (left) and deformations of associative multiplications (right).
  - Formulation of *algebraic identities of polynomial degree higher than quadratic* in terms of Maurer-Cartan elements of L_∞-structures:
    - *Bialgebras* (**M.Markl**)
    - Complex of *simultaneous formal deformation of (associative) algebras and their morphisms* (**Y.Frégier**, **M.Zambon**, 2013). *Maurer-Cartan elements*: deformations of two associative structures and a deformation of a morphism between them.

SOME GRADED STRUCTURES

- **Graded K-modules**
  - K associative commutative unital ring, $K \supset \mathbb{Q}$,
  - *graded K-module* $V = \oplus_{i\in\mathbb{Z}} V^i$, each V^i is a K-module. Notation for homogeneous elements: $x \in V^i$ then $i =: |x|$.
  - *tensor product*: $(V \otimes_K W)^i = \oplus_{j\in\mathbb{Z}} V^j \otimes_K W^{i-j}$
  - *graded transposition*: $\tau_{V,W}(x \otimes_K y) = (-1)^{|x||y|} y \otimes_K x$
  - *graded homs*:
    $\mathrm{Hom}^i_K(V, W) := \{K\text{−linear maps of degree } i\}$,
    $$\mathrm{Homgr}_K(V, W) := \oplus_{i\in\mathbb{Z}} \mathrm{Hom}^i_K(V, W)$$
- **Sign rule**: for any homogeneous $\phi \in \mathrm{Homgr}_K(V, W)$, $\psi \in \mathrm{Homgr}_K(V', W')$, $x \in V$, and $x' \in V'$

$$(\phi \otimes_K \psi)(x \otimes x') = (-1)^{|\psi||x|} (\phi(x)) \otimes_K (\psi(x')).$$

- **Suspension**
  - $V[i]^j := V^{i+j}$: new graded K-module $V[i]$, same underlying K-module
  - $\mathsf{s}^i_V = \mathsf{s}^i : V[i] \to V$ *suspension* is identity map of underlying K-module, BUT: of degree i
  - $\mathsf{s}^1 =: \mathsf{s} : V[1] \to V$
  - *shifting n-multilinear maps* $\phi : V \otimes_K \cdots \otimes_K V \to W$ by $\phi[i] : V[i] \otimes_K \cdots \otimes_K V[i] \to W[i]$ defined by

$$\phi[i] := \mathsf{s}^{-i}_W \circ \phi \circ (\mathsf{s}^i_V \otimes_K \cdots \otimes_K \mathsf{s}^i_V$$

    - $|\phi[i]| = |\phi| + (n-1)i$.
    - Sign rule generates sign differences between ϕ and $\phi[i]$.
    - Graded symmetry changes to graded antisymmetry if i is odd.

Sign rule and suspension help to hide signs.

- **Categorical Remarks**
  - $\mathcal{C} := K$-**modgr**: category of all graded K-modules, morphisms: $\mathrm{Hom}_{\mathcal{C}}(V,W) = \mathrm{Hom}^0_K(V,W)$, DEGREE 0!! What about K-linear maps of other degrees??
  - $(\mathcal{C}, \otimes_K, K, \tau)$ *symmetric monoidal category*, and *closed* i.e. $\mathrm{Homgr}_K(W,?)$ adjoint functor to $? \otimes_K W$

$$\mathrm{Hom}^0_K\big(V \otimes_K W, X\big) \cong \mathrm{Hom}^0_K\big(V, \mathrm{Homgr}_K(W,X)\big)$$

  - Problem: $\mathrm{Homgr}_K(W,X)$ is **object** of $\mathcal{C}$, how can it **act**? There is abstract evaluation $\mathrm{Homgr}_K(W,X) \otimes_K W \to X$, as in any *closed monoidal category*, but suspension?
  - Family $(K[i])_{i\in\mathbb{Z}}$ with $K[i] \otimes_K K(j] \cong K[i+j]$ in $\mathcal{C}$, define

$$V^i := \mathrm{Hom}_{\mathcal{C}}(K[-i], V) \text{ (sets)}, \quad V[j] := K[j] \otimes_K V \quad \text{(objects)},$$
$$s^i \in \mathrm{Hom}_{\mathcal{C}}\big(K[-i], \mathrm{Homgr}_K(K[i] \otimes_K V, V)\big) \quad \text{(natural tr.)},$$

and all ends well (*enriched categories*).

- **Graded (co)algebras** (e.g. D.Quillen, 1969)
  - *Graded coassociative counital connected (**CCC**) coalgebra* $(C, \Delta_C, \epsilon_C, 1_C)$: all K-linear maps of degree 0
    - connected: 1_C grouplike and $\epsilon_C(1_C) = 1$;
      the ascending subcoalgebra filtration $(C_{(r)})_{r\in\mathbb{N}}$ of C with $C_{(0)} = K1_C$ and for all $r \in \mathbb{N}$
      $$C_{(r+1)} = \{x \in C \mid \Delta_C(x) - x \otimes_K 1_C - 1_C \otimes_K x \in C_{(r)} \otimes_K C_{(r)}\}$$
      is exhaustive; $\Rightarrow (C/(K1_C), \Delta')$ is conilpotent.
    - *graded cocommutative (**CCCC**) coalgebra*: $\tau \circ \Delta_C = \Delta_C$
  - Example: *graded symmetric coalgebra*: $\mathsf{S}(V) = \oplus_{r\in\mathbb{N}}\mathsf{S}^r(V)$
    - graded commutative and cocommutative *bialgebra*:
      $\bullet : \mathsf{S}(V) \otimes_K \mathsf{S}(V) \to \mathsf{S}(V)$ graded commutative multiplication,
      $\Delta : \mathsf{S}(V) \to \mathsf{S}(V) \otimes_K \mathsf{S}(V)$ algebra morphism,
      $$\Delta(x_1 \bullet \cdots \bullet x_n) := \sum_{I \dot\cup J = \{1,\ldots,n\}} \epsilon_{I,J}(x) x_I \otimes_K x_J$$
      determined by $\Delta(x) = x \otimes_K 1 + 1 \otimes_K x$ for all $x \in V$

- (graded (co)algebras)
  - *Coalgebra morphisms*: $\phi : C \to C'$ K-linear of degree 0 s.t.

$$\Delta_{C'} \circ \phi = (\phi \otimes_K \phi) \circ \Delta_C, \quad \epsilon_{C'} \circ \phi = \epsilon_C, \quad \phi(1_C) = 1_{C'}.$$

  - *graded coderivations along a morphism* ϕ : $d : C \to C'$ K-linear of any degree, s.t.

$$\Delta_{C'} \circ d = (d \otimes_K \phi + \phi \otimes_K d) \circ \Delta_C \quad \Rightarrow \quad \epsilon_{C'} \circ d = 0$$

    - d is called *flat* iff $d(1_C) = 0$.
    - If $C = C'$, $\phi = \mathrm{id}_C$:
      $(\mathrm{Coder}(C), [\ ,\ ])$ is a **graded Lie algebra**.

- (graded (co)algebras)
  - *Graded bialgebra* $(B, \mu_B, 1_B, \Delta_B, \epsilon_B)$:
    $(B, \mu_B, 1_B)$ graded associative unital algebra
    $(B, \Delta_B, \epsilon_B)$ graded coassociative counital coalgebra
    such that

    $$\Delta_B \circ \mu_B = (\mu_B \otimes_K \mu_B) \circ (\mathrm{id}_B \otimes_K \tau_{B,B} \otimes_K \mathrm{id}_B) \circ (\Delta_B \otimes_K \Delta_B)$$

    and ϵ_B morphism of graded unital algebras, 1_B grouplike.
  - *primitive elements*:

    $$\mathfrak{b} := \{b \in B \mid \Delta_B(x) = x \otimes_K 1_B + 1_B \otimes_K b\}.$$

    $\Rightarrow$: $\mathfrak{b}$ is a graded *sub-Lie algebra* of (B, μ_B^-).
    $\Rightarrow$: left or right multiplications with primitive elements are coderivations
  - Example: *universal envelopping algebra* $\mathrm{U}(\mathfrak{g})$ of a graded Lie algebra $(\mathfrak{g}, [\ ,\ ])$ over K

- **Vive la convolution!!**
  - $(C, \Delta_C, \epsilon_C)$ graded coassociative counital coalgebra, $(A, \mu_A, 1_A)$ graded associative unital algebra $\phi, \psi \in \mathrm{Homgr}_K(C, A)$ then define the convolution

    $$\phi * \psi := \mu_A \circ (\phi \otimes_K \psi) \circ \Delta_C$$

    hence

    $$(\mathrm{Homgr}_K(C, A), *, 1_A\epsilon_C) \quad \text{graded associative unital algebra}$$

  - In case $(C, \Delta_C, \epsilon_C, 1_C)$ *connected*, then $\mathrm{Homgr}_K(C, A)$ carries a *complete descending filtration*:
  - If $\phi \in \mathrm{Homgr}_K(C, A)$ with $\phi(1_C) = 0$, and $|\phi| = 0$ then

    $$a_0 1_A \epsilon_C + \sum_{r=1}^{\infty} a_r \phi^{*r} \quad \text{converges for all } a_0, a_1, a_2, \ldots \in K.$$

  - In case $C = \mathsf{S}(K) = K[\lambda]$ then $(\mathrm{Homgr}_K(C, K), *) \cong K[[\lambda]]$.

- (Convolution)
  - **Theorem**: let $(C, \Delta_C, \epsilon_C, 1_C)$ a graded CCCC coalgebra. Let $(B, \mu_B, 1_B, \Delta_B, \epsilon_B)$ graded bialgebra, $\mathfrak{b}$ its Lie algebra of primitive elements. Then
    - (**J.Helmstetter,1989**) If $\varphi : C \to \mathfrak{b}$ is K-linear, of degree 0, and $\varphi(1_C) = 0$ then

$$\overline{\varphi} := e^{*\varphi} := 1_B\epsilon_C + \sum_{r=1}^{\infty} \frac{1}{r!}\varphi^{*r}$$

      is a morphism of graded augmented counital coalgebras $C \to B$ with $\overline{\varphi}(1_C) = 1_B$, and any such morphism is of this form.
    - If $d \in \mathrm{Homgr_K(C, \mathfrak{b})}$ of any degree, and φ as above then

$$\overline{d} = d * e^{*\varphi}$$

      is a graded coderivation along $\overline{\varphi} := e^{*\varphi}$, and any such coderivation is of that form.

- (Convolution)
  - (Theorem)
    - If $\Phi, \Psi : C \to B$ are morphisms of counital coalgebras, then
      $$\Phi * \Psi$$
      is a morphism of counital coalgebras.
    - If $D : C \to B$ is a graded coderivation along the morphism $\Xi : C \to B$ then
      $$\Phi * d * \Psi \quad \text{is a graded coderivation along } \Phi * \Xi * \Psi.$$

Coalgebraic structures and convolution help to hide combinatorics.

- **Cofree coalgebras**
  Graded symmetric bialgebra $\mathsf{S}(W)$ **cofree** in the category of graded CCCC coalgebras:
  - $\varphi : C \to W$ K-linear map of degree 0 with $\varphi(1_C) = 0$

$$\begin{tikzcd} \mathsf{S}(W) \arrow[dr, "\mathrm{pr}_W"'] & & C \arrow[ll, "\overline{\varphi}"'] \arrow[dl, "\varphi"] \\ & V & \end{tikzcd} \qquad \text{with } \overline{\varphi} = e^{*\varphi}$$

  where $\overline{\phi} : C \to \mathsf{S}(W)$ **morphism** of connected coalgebras. V is the submodule of primitive elements of $\mathsf{S}(W)$.
  - Any such morphism $\Phi : C \to \mathrm{W}$ is *uniquely determined* by its **Taylor coefficients** $\mathrm{pr}_W \circ \Phi : C \to W$.
  - Any CCCC coalgebra C can be embedded in $\mathsf{S}(\mathrm{Ker}\ \epsilon_C)$ as a graded CCCC subcoalgebra.

- (Cofree coalgebras)
  - $d : C \to V$ K-linear map of any degree, $\Phi : C \to \mathsf{S}(W)$ morphism of graded CCCC coalgebras:

$$\begin{tikzcd} \mathsf{S}(W) \arrow[dr, "\mathrm{pr}_W"'] & & C \arrow[ll, "\overline{d}"'] \arrow[dl, "d"] \\ & V & \end{tikzcd} \quad \text{with } \overline{d} = d * \Phi$$

    where $\overline{d} : C \to \mathsf{S}(W)$ graded **coderivation** of CCCC coalgebras along Φ.
  - Any such derivation $D : C \to \mathrm{W}$ along Φ is *uniquely determined* by its **projection to** W, $\mathrm{pr}_W \circ D : C \to W$.

- **L_∞-structures on V: Stasheff's shifted version**:
  - $\big(\mathsf{S}(V[1]), \Delta, \epsilon, 1, \overline{d}\big)$ *differential graded CCCC coalgebra*

$$\overline{d} \in \mathrm{Coder}^1\big(\mathsf{S}(V[1])\big) \;\; s.t. \;\; \overline{d}^2 = 0, \;\; \overline{d}(1) = 0.$$

    and

$$d = \mathrm{pr}_{V[1]} \circ \overline{d} = \sum_{r=1}^{\infty} d_r, \quad \text{and} \quad c_r = d_r[-1]$$

    d_r: graded *symmetric*; c_r: graded *antisymmetric* !
  - *L_∞-morphism $V - - > V'$*: morphism of dg CCCC coalgebras

$$\overline{\mathfrak{U}} = e^{*\mathfrak{U}} : \mathsf{S}(V[1]) \to \mathsf{S}(V'[1]) \;\; \text{with} \;\; \overline{\mathfrak{U}} \circ \overline{d} = \overline{d'} \circ \overline{\mathfrak{U}}$$

    with Taylor coefficients $\mathfrak{U} : \mathsf{S}(V[1]) \to V'[1]$, $\mathfrak{U}(1) = 0$.
  - *Maurer-Cartan elements*: $\nu' = s^{-1}(\mu') \in \mathrm{Hom}^0_K\big(\lambda K[\lambda], V[1]\big)$

$$e^{*\nu'} : \mathsf{S}(K) = K[\lambda] \to \mathsf{S}(V[1]) \;\; \text{with} \;\; \overline{d} \circ e^{*\nu'} = 0$$

    (morphisms of dg CCCC coalgebras).

FORMALITY CHECK FOR THE HOCHSCHILD COMPLEX OF UNIVERSAL ENVELOPING ALGEBRAS OF LIE ALGEBRAS

Let $(\mathfrak{g}, [\ ,\ ])$ be a (graded) Lie algebra over K.

- *Universal Enveloping Algebra*:
  - *Definition*: let $\mathsf{T}(\mathfrak{g})$ be the tensor algebra generated by the K-module $\mathfrak{g}$, set

$$\mathsf{U}(\mathfrak{g}) := \mathsf{T}(\mathfrak{g})/I,$$

    where I is the two-sided ideal in the tensor algebra $T(\mathfrak{g})$ generated by (for any $\xi, \eta \in \mathfrak{g}$)

$$\xi \otimes \eta - (-1)^{|\xi||\eta|}\eta \otimes \xi - [\xi, \eta].$$

  - *Universal property*: A graded associative algebra, $\varphi : \mathfrak{g} \to A^-$ morphism of graded Lie algebras of degree 0, then there is a unique morphism of graded associative algebras

$$\overline{\varphi} : \mathsf{U}(\mathfrak{g}) \to A \ \text{ with } \ \overline{\varphi} \circ \iota_{\mathfrak{g}} = \varphi.$$

- (Universal Enveloping Algebras)
  - $\mathsf{U}(\mathfrak{g})$ is a *graded bialgebra* such that $\big(\mathsf{U}(\mathfrak{g}), \Delta, \epsilon, 1\big)$ is a graded CCCC coalgebra:
    PBW: $\iota_{\mathfrak{g}} : \mathfrak{g} \to \mathsf{U}(\mathfrak{g})$ injection; *graded symmetrization map*

    $$\omega_{\mathfrak{g}} = e^{*\iota_{\mathfrak{g}} \circ \mathrm{pr}_{\mathfrak{g}}} : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$$

    is an isomorphism of graded CCCC coalgebras (Quillen, 1969).
  - **Eulerian idempotent**: (e.g. Barr '65,...,Helmstetter '89, Loday '92,...)

    $$e_{\mathfrak{g}}^{(1)} = \iota_{\mathfrak{g}} \circ \mathrm{pr}_{\mathfrak{g}} \circ \omega_{\mathfrak{g}}^{-1} = \ln_*\big(\mathrm{id}_{\mathsf{U}(\mathfrak{g})}\big) := \ln_*\Big(1\epsilon + \big(\mathrm{id}_{\mathsf{U}(\mathfrak{g})} - 1\epsilon\big)\Big)$$

  - *universal property* reads (where $\varphi : \mathfrak{g} \to A^-$)

    $$\overline{\varphi} = e^{*(\varphi \circ \mathrm{pr}_{\mathfrak{g}} \circ \omega_{\mathfrak{g}}^{-1})}$$

- (Universal Enveloping Algebras)
  - **BCH formula**:
    - for any nonnegative integer n define

$$\mathfrak{G}^{\langle n\rangle} := \mathrm{Homgr}^0_K\Big(\big(\mathsf{S}(\mathfrak{g})\big)^{\otimes n}, \mathfrak{g}\Big)$$

      Equipped with the obvious **convolution Lie bracket** this is a *rational Lie algebra equipped with a complete descending filtration*: $\big(\mathfrak{G}^{\langle n\rangle}_{(k)}\big)_{k\in\mathbb{N}}$
    - Take the usual *Baker-Campbell-Hausdorff* formal group law

$$\mathrm{BCH} : \mathfrak{G}^{\langle 2\rangle}_{(1)} \times \mathfrak{G}^{\langle 2\rangle}_{(1)} \to \mathfrak{G}^{\langle 2\rangle}_{(1)}$$

      and define on the graded K-module $\mathsf{S}(\mathfrak{g})$

$$\mu_{\mathsf{U}(\mathfrak{g})} = e^{*\mathrm{BCH}(\mathrm{pr}_{\mathfrak{g}}\otimes_K \epsilon\ ,\ \epsilon\otimes_K \mathrm{pr}_{\mathfrak{g}})}$$

      graded K-bialgebra version of star-product formulas by **S.Gutt (1983)** and **V.G.Drinfel'd (1983)**

PROBLEM: L_∞-formality of the Hochschild complex of $\mathrm{U}(\mathfrak{g})$??

Slight generalization of Kontsevich's formality for $\mathrm{S}(V)$, V finite-dimensional; V seen as abelian Lie algebra.

- *Results* (MB, O.Elchinger, S.Gutt, A.Makhlouf 2018)
  - $\mathfrak{g}$ **abelian**: FORMALITY,
  - $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K}) \oplus \mathbb{K}^n$: FORMALITY,
  - $\Rightarrow$ $\mathfrak{g} =$ "$ax + b$": FORMALITY,
  - $\mathfrak{g}$ **Cartan-3-regular quadratic** : NO FORMALITY,
  - $\Rightarrow$ $\mathfrak{g}$ **semisimple** : NO FORMALITY,
  - $\Rightarrow$ $\mathfrak{g}$ **nonabelian reductive** : NO FORMALITY,
  - $\mathfrak{g}$ **3-dim Heisenberg**: NO FORMALITY,
  - $\mathfrak{g}$ **free Lie algebra over** V, $\dim(V) \geq 2$: NO FORMALITY.
- *Cartan-3-regular quadratic Lie algebra* $\mathfrak{g}$
  - $\exists\ B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ nondegenerate symmetric invariant bilinear form, i.e. $B([x, y], z) = B(x, [y, z])$, see e.g. MB 1997,
  - and *Cartan* 3*-cocycle* $\Omega(x, y, z) = B([x, y], z)$ defines nontrivial cohomology class.

- *Methods of proof*
  - for any dgLA $(\mathfrak{G}, [\ ,\ ], T)$ with cohomology $(\mathfrak{H}, [\ ,\ ])$ its **characteristic** 3-**class** c_3 in $H^3_{CE}(\mathfrak{H}, \mathfrak{H})$: computation of c_3 for $\mathfrak{G}$ the Hochschild complex of $\mathrm{U}(\mathfrak{g})$: first obstruction to L_∞-formality.
  - for finite-dimensional $\mathfrak{g}$: replace the inhuman Hochschild complex by the **Chevalley-Eilenberg complex** $C_{CE}(\mathfrak{g}, \mathrm{S}(\mathfrak{g}))$ with the Schouten bracket.
  - for Cartan-3-regular $\mathfrak{g}$: $\quad z_3(B^{-1}, B^{-1}, \Omega) = 8B^{-1} \neq 0$.
  - free Lie algebra: $HH(\mathrm{T}(V), \mathrm{T}(V)) \cong \mathbb{K} \oplus \mathfrak{outder}(\mathrm{T}(V))$, first finite-dimensional case, elementary, but long.

CODERIVATIONAL ACTIONS OF DG LIE ALGEBRAS

- Let $(\mathfrak{g}, [\ ,\ ], T)$ be a **DG Lie algebra**,
  i.e. T graded derivation of $\mathfrak{g}$ of degree 1 and $T^2 = 0$.
- Let $C = \big(\mathsf{S}(W), \Delta, \epsilon, 1\big)$ be the cofree graded CCCC coalgebra cogenerated by the graded K-module W.
- Suppose that there is a **representation ρ of the dg Lie algebra $\mathfrak{g}$ in C by coderivations**, i.e. a K-linear map

$$\rho : \mathfrak{g} \to \mathrm{Coder}\big(\mathsf{S}(W)\big) : \xi \mapsto \rho_\xi$$

  of degree 0, and a coderivation ρ_T of degree 1 of C satisfying for all $\xi, \eta \in \mathfrak{g}$

$$\begin{aligned} \rho_{[\xi,\eta]} &= \rho_\xi \circ \rho_\eta - (-1)^{|\xi||\eta|} \rho_\eta \circ \rho_\xi, \\ \rho_{T(\xi)} &= \rho_T \circ \rho_\xi - (-1)^{|\xi|} \rho_\xi \circ \rho_T, \\ \rho_T^2 &= 0. \end{aligned}$$

- Important particular case: $(\mathfrak{g}, [\ ,\ ])$ graded Lie algebra $\xi_0 \in \mathfrak{g}$ such that $|\xi_0| = 1$, and $[\xi_0, \xi_0] = 0$. Then with

$$T := \mathrm{ad}_{\xi_0} \quad \text{inner derivation}$$

the triple $(\mathfrak{g}, [\ ,\ ], T)$ is a dg Lie algebra.
If $\rho : \mathfrak{g} \to \mathrm{Coder}(\mathsf{S}(W))$ is a graded representation, then with $\rho_T := \rho_{\xi_0}$ there is a coderivational dg Lie algebra action on C.
- **Easy consequence**: if there is a coderivational action of the DG Lie algebra $(\mathfrak{g}, [\ ,\ ], T)$ on $\mathsf{S}(W)$ then

$$(\mathsf{S}(W), T)$$

is a dg coalgebra, hence **defines an L_∞-structure on** $V := W[-1]$.

- Two natural questions: given a DG Lie algebra $(\mathfrak{g}, [\ ,\ ], T)$,

  1. **T.Voronov I**: What are appropriate graded CCCC cofree coalgebras $\mathsf{S}(W)$ *associated to* $\mathfrak{g}$ *and its subalgebras* on which $\mathfrak{g}$ acts by coderivations ?
  2. **T.Voronov II**: How can Voronov I be *extended by* $\mathfrak{g}$ in order to get L_∞-structures on $\mathfrak{g} \oplus W[-1]$?

T.Voronov, 2005 I and **T.Voronov, 2005 II**

VORONOV I: UNABELIAN CONSTRUCTION

- *DG Lie algebra inclusions*
  - Let $(\mathfrak{g}, [\ ,\ ], T)$ be a DG Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra.
    $(\mathfrak{g}, [\ ,\ ], T, \mathfrak{h})$ *DG Lie algebra inclusion* iff $T(\mathfrak{h}) \subset \mathfrak{h}$.
  - Example: $(\mathfrak{g}, [\ ,\ ], \mathrm{ad}_{\eta_0}, \mathfrak{h})$ where $\mathfrak{h} \subset \mathfrak{g}$ subalgebra, and $\eta_0 \in \mathfrak{h}$ with $|\eta_0| = 1$ and $[\eta_0, \eta_0] = 0$.

- *The CCCC coalgebra* $\mathcal{M}$:
  - Define for any DG Lie algebra inclusion $(\mathfrak{g}, [\ ,\ ], T, \mathfrak{h})$

$$\mathcal{M} := \frac{\mathsf{U}(\mathfrak{g})}{\mathsf{U}(\mathfrak{g})\mathfrak{h}}, \quad \Pi : \mathsf{U}(\mathfrak{g}) \to \mathcal{M} \ \ \textit{natural projection.}$$

  - **Theorem Ia**
    - $\mathcal{M}$ is a graded CCCC coalgebra over K: since $\mathsf{U}(\mathfrak{g})\mathfrak{h}$ is a *graded coideal of* $\mathsf{U}(\mathfrak{g})$
    - The DG Lie algebra $(\mathfrak{g}, [\ ,\ ], T)$ acts by coderivations on $\mathcal{M}$ via

$$\rho_\xi\big(\Pi(u)\big) := \Pi(\xi u), \quad \rho_T\big(\Pi(u)\big) := \Pi\big(\hat{T}(u)\big)$$

      where $\hat{T} : \mathsf{U}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$ natural extension of T to *graded biderivation of* $\mathsf{U}(\mathfrak{g})$

- (The CCCC coalgebra $\mathcal{M}$)
  - Let G Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ closed subgroup with Lie algebra $\mathfrak{h}$, consider the **homogeneous space**

    $$M := G/H, \qquad \pi : G \to M \ \ \textit{natural projection}$$

    then H acts on $\mathcal{M} = \mathsf{U}(\mathfrak{g})/\big(\mathsf{U}(\mathfrak{g})\mathfrak{h}\big)$ via the induced adjoint representation, and the associated bundle

    $$G \times_H \left(\frac{\mathsf{U}(\mathfrak{g})}{\mathsf{U}(\mathfrak{g})\mathfrak{h}}\right)$$

    is a filtered vector bundle over M whose (filtered) smooth sections are *differential operators on* $\mathcal{C}^\infty(M, \mathbb{R})$.
    Fibre $\mathcal{M}$: *distributions on M supported in* $\{\pi(e)\}$.
    (**Alekseev-Lakhowska, 2005**; **Calaque-Caldararu-Tu 2011**; **MB 2012**)

- (The CCCC coalgebra $\mathcal{M}$)
  - **Cofreeness of $\mathcal{M}$:**
    - Let $V \subset \mathfrak{g}$ be a *graded K-submodule complement* to $\mathfrak{h}$:

      $$\mathfrak{g} = V \oplus \mathfrak{h}; \quad \text{decomposition}: \quad \xi = \xi_V + \xi_{\mathfrak{h}}$$

      (always possible if K is a field).
      Projections $\pi_V : \mathfrak{g} \to V$, $\pi_{\mathfrak{h}} : \mathfrak{g} \to \mathfrak{h}$;
      Inclusions $i_V : V \to \mathfrak{g}$, $i_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{g}$.
    - Define the *projections* $p_V, p_{\mathfrak{h}} : \mathsf{S}(\mathfrak{g}) \to \mathfrak{g}$ by

      $$p_V(f) := \big(\mathrm{pr}_{\mathfrak{g}}(f)\big)_V, \qquad p_{\mathfrak{h}}(f) := \big(\mathrm{pr}_{\mathfrak{g}}(f)\big)_{\mathfrak{h}}$$

    - define the *coalgebra isomorphism* $\tilde{\Phi} : \mathsf{S}(\mathfrak{g}) \to \mathsf{U}(\mathfrak{g})$

      $$\tilde{\Phi} := e^{*\iota_{\mathfrak{g}} \circ p_V} * e^{*\iota_{\mathfrak{g}} \circ p_{\mathfrak{h}}} =: \tilde{\Phi}_V * \tilde{\Phi}_{\mathfrak{h}} \;\neq\; \omega_{\mathfrak{g}}$$

- (The CCCC coalgebra $\mathcal{M}$)
  - **Theorem Ib**
    - $\tilde{\Phi}\big(\mathsf{S}(\mathfrak{g})\bullet\mathfrak{h}\big) = \big(\mathsf{U}(\mathfrak{g})\big)\mathfrak{h}$.
    - the K-linear map Φ defined by

      $$\Phi := \Pi \circ \tilde{\Phi} \circ \iota_{\mathsf{S}(V)} = \Pi \circ \tilde{\Phi}_V \circ \iota_{\mathsf{S}(V)} : \mathsf{S}(V) \to \mathcal{M}$$

      is an isomorphism of graded CCCC coalgebras, whence $\mathcal{M}$ **is cofree**.
    - There is the pulled-back coderivational $(\mathfrak{g}, [\ ,\ ], T)$-action ρ' on $\mathsf{S}(V)$ given by

      $$\rho'_\xi := \Phi^{-1} \circ \rho_\xi \circ \Phi \text{ and } \rho'_T := \Phi^{-1} \circ \rho_T \circ \Phi$$

- (The CCCC coalgebra $\mathcal{M}$): **Computation of ρ'**
  - for any $\xi \in \mathfrak{g}$ define the **T.Voronov map** $v_\xi : \mathsf{S}(V) \to V$ by $v_\xi(1) := \xi_V$, and for any $f = x_1 \bullet \cdots \bullet x_k \in \mathsf{S}^k(V)$

$$v_\xi(f) := \Big((\pi_V \epsilon_{\mathsf{S}(V)}) *' e^{-*'\mathrm{ad}_{\mathrm{p}_V}} *' (\xi \epsilon_{\mathsf{S}(V)}) \Big)(f) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(|x_1|, \ldots, |x_r|, \sigma) \Big([\cdots[\xi, x_{\sigma(1)}], \ldots, x_{\sigma(k)}] \Big)_V,$$

  - let $E_{VV} : \mathsf{S}(V) \to \mathrm{Homgr}(V, V)$, $E_{V\mathfrak{h}} : \mathsf{S}(V) \to \mathrm{Homgr}(\mathfrak{h}, V)$

$$\begin{aligned} E_{VV} &:= (\pi_V \epsilon_{\mathsf{S}(V)}) *' \frac{\mathrm{id}_{\mathfrak{g}} \epsilon_{\mathsf{S}(V)} - e^{-*'\mathrm{ad}_{\mathrm{p}_V}}}{*'\mathrm{ad}_{\mathrm{p}_V}} *' (i_V \epsilon_{\mathsf{S}(V)}) \\ E_{V\mathfrak{h}} &:= (\pi_V \epsilon_{\mathsf{S}(V)}) *' \frac{\mathrm{id}_{\mathfrak{g}} \epsilon_{\mathsf{S}(V)} - e^{-*'\mathrm{ad}_{\mathrm{p}_V}}}{*'\mathrm{ad}_{\mathrm{p}_V}} *' (i_{\mathfrak{h}} \epsilon_{\mathsf{S}(V)}) \end{aligned}$$

- (The CCCC coalgebra $\mathcal{M}$):
  - where
    - $\mathrm{ad}_{\mathrm{p}_V} : \mathsf{S}(V) \to \mathrm{Homgr}_K(\mathfrak{g}, \mathfrak{g}) : f \mapsto (\xi \mapsto [\mathrm{p}_V(f), \xi])$,
    - for all homogeneous K-linear maps

$$\begin{aligned} \varphi : \mathsf{S}(V) &\to \mathrm{Homgr}_K(W, W'), \\ \psi : \mathsf{S}(V) &\to \mathrm{Homgr}_K(W', W''), \\ \chi : \mathsf{S}(V) &\to W \end{aligned}$$

define *composition/evaluation convolution* $*'$ by

$$\begin{aligned} (\psi *' \varphi)(f) &= \sum_{(f)} (-1)^{|\varphi||f^{(1)}|} \psi(f^{(1)}) \circ \varphi(f^{(2)}) \\ (\varphi *' \chi)(f) &= \sum_{(f)} (-1)^{|\chi||f^{(1)}|} (\varphi(f^{(1)}))(\chi(f^{(2)})) \end{aligned}$$

for all $f \in \mathsf{S}(V)$.

- (The CCCC coalgebra $\mathcal{M}$):
  - **Theorem Ic**:
    - Let $d'_\xi := \mathrm{pr}_V \circ \rho'_\xi$. Then
      $$d'_\xi = E_{VV}^{*'-1} *' v_\xi.$$
    - If $\xi_0 \in \mathfrak{g}$, $|\xi_0| = 1$, $[\xi_0, \xi_0] = 0$.
      $$d'_\xi \ \text{flat} \quad \Leftrightarrow \quad \xi_0 \in \mathfrak{h}.$$
    - Let $d'_T := \mathrm{pr}_V \circ \rho'_T$. Then
      $$d'_T = \pi_V \circ T \circ \mathrm{p}_V \ + \ E_{VV}^{*'-1} *' E_{V\mathfrak{h}} *' (\pi_\mathfrak{h} \circ T \circ \mathrm{p}_V).$$

- (The CCCC coalgebra $\mathcal{M}$):
  - (Theorem Ic)
    - For any $\xi \in \mathfrak{g}$, $x, y \in V$:

$$\begin{aligned}
d'_\xi(1) &= \xi_V, \\
d'_\xi(x) &= \frac{1}{2}[\xi_V, x]_V + [\xi_{\mathfrak{h}}, x]_V, \\
d'_\xi(x \bullet y) &= \frac{1}{2}\big[[\xi, x]_{\mathfrak{h}}, y\big]_V + (-1)^{|x||y|}\frac{1}{2}\big[[\xi, y]_{\mathfrak{h}}, x\big]_V \\
&\quad + \frac{1}{12}\big[[\xi_V, x], y\big]_V + (-1)^{|x||y|}\frac{1}{12}\big[[\xi_V, y], x\big]_V.
\end{aligned}$$

- (The CCCC coalgebra $\mathcal{M}$):
  - (Theorem Ic)
    - For any $\xi \in \mathfrak{g}$, $x, y \in V$:

$$\begin{aligned} d'_T(1) &= 0, \\ d'_T(x) &= \big((T(x)\big)_V, \\ d'_T(x \bullet y) &= \frac{1}{2}\big[\big(T(x)\big)_{\mathfrak{h}}, y\big]_V + (-1)^{|x||y|}\frac{1}{2}\big[\big(T(y)\big)_{\mathfrak{h}}, x\big]_V. \end{aligned}$$

- (The CCCC coalgebra $\mathcal{M}$):
  - **Geometric analogy**: let $\mathfrak{g}$, $\mathfrak{h}$, V real, finite-dimensional, trivially graded, associated to $G \supset H$ Lie groups, H closed subgroup.
    Consider the smooth maps (local diffeomorphisms around e and $\pi(e)$)

$$\begin{aligned} \tilde{\psi} : \mathfrak{g} \to G \quad &: \quad \xi \mapsto \exp(\xi_V) \exp(\xi_{\mathfrak{h}}) \\ \psi : V \to M = G/H \quad &: \quad x \mapsto \pi\big(\exp(x)\big) \end{aligned}$$

    Tangent maps of $\tilde{\psi}$ and ψ lead to above formulas. Recall

$$\big(T_X \exp\big)(Y) = e^X \frac{I - e^{-\mathrm{ad}_X}}{\mathrm{ad}_X}(Y)$$

    Inverse contains *Bernoulli numbers*.

- (The CCCC coalgebra $\mathcal{M}$):
  - **Theorem Id**
    If $T = \mathrm{ad}_{\eta_0}$ is inner, $\eta_0 \in \mathfrak{h}$, and if the **Nguyen-Van Hai class of the Lie algebra inclusion** $\mathfrak{g} \supset \mathfrak{h}$ **vanishes** then d' is reduced to its linear component.
  - Nguyen-Van Hai class (1965) (or Atiyah class): cohomological obstruction against the extension of the natural $\mathfrak{h}$-action

    $$ad'_\eta : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h} : \varpi(\xi) \mapsto \varpi([\eta, \xi])$$

    to an $\mathfrak{h}$-**invariant** bilinear map

    $$\gamma : \mathfrak{g} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h},$$

    (G-invariant linear connection on G/H).
    Calaque-Caldararu-Tu 2011; **MB 2012**)

- (The CCCC coalgebra $\mathcal{M}$):
  - **Abelian Case**: suppose that V is an **abelian Lie subalgebra** of $\mathfrak{g}$. Then

$$\begin{aligned} E_{VV} &= \mathrm{id}_V \epsilon_{S(V)} \\ d'_\xi &= v_\xi \end{aligned}$$

where v_ξ simplifies (no more sum over permutations), and $d'_T(1) = 0$, $d'_T(x) = \big(\big(T(x)\big)_V$, and

$$d'_T(x_1 \bullet \cdots \bullet x_k) = \Big([\ldots[T(x_1), x_2], \ldots, x_k]\Big)_V$$

for all $x, x_1, \ldots, x_k \in V$.
T.Voronov, 2005

- (The CCCC coalgebra $\mathcal{M}$):
  - **Nonabelian Case**: suppose that V is an **Lie subalgebra** of $\mathfrak{g}$. (also obtained by other methods by **R.Bandiera, 2013**)
    - Consider universal envelopping algebra $\mathrm{U}(V)$
    - right $\mathrm{U}(V)$-action on $\mathfrak{g}$:

$$\xi \dashv 1_{\mathrm{U}(V)} := \xi, \qquad \xi \dashv (x_1 \cdots x_k) := [\cdots[\xi, x_1], \ldots, x_k].$$

    - $\mathfrak{g}$ right $\mathrm{U}(V)$-module Lie algebra: $\forall\ u \in \mathrm{U}(V),\ \xi, \xi' \in \mathfrak{g}$

$$[\xi, \xi'] \dashv u = \sum_{(u)} (-1)^{|u^{(1)}||\xi'|} \big[\xi \dashv u^{(1)}, \xi' \dashv u^{(2)}\big].$$

    - *Nonabelian Voronov map* $w_\xi : \mathrm{U}(V) \to V$

$$w_\xi(u) := (\xi \dashv u)_V$$

    - *graded* **dressing action** *of* $\mathfrak{g}$ *on* $\mathrm{U}(V)$ *by coderivations*

$$\overline{w_\xi}^R = \mathrm{id}_{\mathrm{U}(V)} * w_\xi, \quad \text{and} \quad \left[\overline{w_\xi}^R, \overline{w_{\xi'}}^R\right] = \overline{w_{[\xi,\xi']}}^R.$$

- (The CCCC coalgebra $\mathcal{M}$):
  - (Nonabelian Case):
    - Dressing action: analogy $G = \mathcal{V}H$ where $\mathcal{V}$ Lie subgroup whose Lie algebra is V, whence
      $$M = G/H \cong \mathcal{V}$$
      $\overline{w_\xi}^R$ corresponds to infinitesimal generator of natural left G-action on $\mathcal{V}$.
    - Main result:
      $$d'_\xi = e_V^{(1)} \circ \overline{w_\xi}^R \circ \omega_V$$
    - If V abelian: $w_\xi = v_\xi$.

VORONOV II: EXTENSION BY THE LIE ALGEBRA

- Fix the following data:
  - $(\mathfrak{g}, [\ ,\ ], T)$ a DG Lie algebra.
  - $(C := \mathsf{S}(W), \Delta, \epsilon, 1)$ a graded CCCC coalgebra.
  - $\rho : (\mathfrak{g}, [\ ,\ ], T) \to \mathsf{S}(W)$ an action by coderivations.
- Form the **graded Lie homology complex of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\mathsf{S}(W)$**

$$\mathsf{S}(\mathfrak{g}[1] \oplus W) \cong \mathsf{S}(\mathfrak{g}[1]) \otimes_K \mathsf{S}(W) \cong \text{``}\bigwedge \mathfrak{g} \otimes_K \mathsf{S}(W)\text{''}$$

with the 'usual *graded Chevalley-Eilenberg differential D*'

- Define $\kappa : \mathsf{S}^2(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ by the shifted Lie bracket $[\ ,\ ][1]$.

- **Theorem IIa**: Under the above hypotheses, defining
  - $D_{[\ ,\ ]} = \Big(\big(\kappa \circ \mathrm{pr}_{\mathsf{S}^2(\mathfrak{g}[1])}\big) \mathbin{\tilde{*}} \mathrm{id}_{\mathsf{S}(\mathfrak{g}[1])}\Big) \otimes \mathrm{id}_C,$
  - $D_\rho :=$ $\big(\mathrm{id}_{\mathsf{S}(\mathfrak{g}[1])} \otimes \rho\big) \circ \big(\mathrm{id}_{\mathsf{S}(\mathfrak{g}[1])} \otimes (s_{\mathfrak{g}} \circ \mathrm{pr}_{\mathfrak{g}[1]})\big) \otimes \mathrm{id}_C\big) \circ \big(\Delta_{\mathsf{S}(\mathfrak{g}[1])} \otimes \mathrm{id}_C\big),$
  - $D_T := -\hat{T} \otimes \mathrm{id}_C + \mathrm{id}_{\mathsf{S}(\mathfrak{g}[1])} \otimes \rho_T.$

  it follows that
  - $D = -D_{[\ ,\ ]} + D_\rho + D_T$ is a coderivation of degree 1 s.t.

    $$D^2 = 0 \quad \text{and} \quad D(1 \otimes_K 1) = 0$$

    whence defines a **flat** L_∞**-structure on** $\mathfrak{g} \oplus W[-1]$.
  - Linear component d_1 of D is the differential of the *mapping cone* of the map $\mathfrak{g}[1] \to W : a \mapsto \rho_{s_{\mathfrak{g}}(a)}(1)$,

    $$d_1 : \mathfrak{g}[1] \oplus W \to \mathfrak{g}[1] \oplus W : (a, x) \mapsto \big(-T[1](a), \rho_{s_{\mathfrak{g}}(a)}(1) + \rho_T(x)\big).$$

FUNERAL OF A ZOMBIE ?

- M.B., G.Ginot, G.Halbout, H.-C.Herbig, S.Waldmann: *Star-représentations sur des sous-variétés coïsotropes.* `arXiv.math/0309321v1`, 2003
  the good
- M.B., G.Ginot, G.Halbout, H.-C.Herbig, S.Waldmann: *Formalité G_∞ adaptée et star-représentations sur des sous-variétés coïsotropes.* `arXiv.math/0504276`, 2005
  the zombie: main result wrong!!

- Coisotropic: geometry
  - Let (M, Λ) be a Poisson manifold, $C \subset M$ be a submanifold.
  - C **coisotropic** iff

$$\forall\, c \in C : \Lambda_c^\sharp(T_c C^{\mathrm{ann}}) \subset T_c C$$

- Coisotropic: algebra
  - Let $A = \mathcal{C}^\infty(M)$, $B = \mathcal{C}^\infty(C)$, and

$$I := \{g \in A \mid \forall\, c \in C : \; g(c) = 0\} \quad \text{vanishing ideal of } C.$$

  - Hence $B \cong A/I$, and C coisotropic iff

$$\{I, I\}_\Lambda \subset I :$$

- Coisotropic: quantization = representation (e.g. MB 2005)
  - Recall star-product $*$ on M:
    $*$ is an associative $K[[\lambda]]$ bilinear (bidifferential) unital multiplication on $A[[\lambda]]$ which is a formal associative deformation of the pointwise multiplication on $A = \mathcal{C}^\infty(M)$ having Λ as first-order commutator
  - Find star-product $*$ on M such that

$$I[[\lambda]] \text{ is a left ideal of } (A[[\lambda]], *)$$

  - Then $B[[\lambda]] \cong A[[\lambda]]/I[[\lambda]]$ is a left $(A[[\lambda]], *)$-module.
  - **NOT ALWAYS POSSIBLE!** as opposed to main statement in the zombie paper:
    Counterexample by **T.Willwacher, 2007**.