Twisted bialgebroids as models for quantum covariant phase space

Andrzej Borowiec
joint work with Anna Pachoł

Wroclaw University

Bayrischzell Workshop 2018:
On Noncommutativity and Physics: Hopf algebras in Noncommutative Geometry
Bayrischzell, April 20 - 23, 2018

Plan:

I. Motivation and general framework: Weyl-Heisenberg algebras and their extensions.

II. Bi-/Hopf algebroids.

III. Twisting bi-/Hopf algebroids by Drinfeld twist.

IV. Yetter-Drinfeld modules and braided categories.

V. Smash product bi-/Hopf algebroids and their twistings by Drinfeld twists.

VI. Quasi-triangular examples.

VII. Hopf-Galois context.

Plan:

I. Motivation and general framework: Weyl-Heisenberg algebras and their extensions.
II. Bi-/Hopf algebroids.
III. Twisting bi-/Hopf algebroids by Drinfeld twist.
IV. Yetter-Drinfeld modules and braided categories.
V. Smash product bi-/Hopf algebroids and their twistings by Drinfeld twists.
VI. Quasi-triangular examples.
VII. Hopf-Galois context.

NOTE: All rings are unital. All objects are modules over commutative (background) ring or field $\mathbb{K}$. All maps are $\mathbb{K}$- linear. $\otimes = \otimes_{\mathbb{K}}$ if not stated otherwise.

CCR algebra = Weyl-Heisenberg algebra=quantum phase space

$$[P_\mu, x^\nu] = -i\delta_\mu^\nu 1, \qquad [x^\mu, x^\nu] = 0 \qquad [P_\mu, P_\nu] = 0.$$

where $\mu, \nu = 0, \ldots n-1$.

Hilbert space realization ($\mathbb{K} = \mathbb{C}$)

Providing the algebra of differential operator on $\mathbb{K}^n$

Generating some abstract associative unital algebra (Weyl algebra) $\Rightarrow$ smash product construction

Smash product algebras

- $A \rtimes H$ - is an extension of a (left) module algebra $(A, \star, H\triangleright, 1_A)$ by the corresponding bialgebra H to create a new algebra

- by determining on the $\mathbb{K}$-module $A \otimes H = A \rtimes H$ the multiplication $(L \triangleright 1_A = \epsilon(L)1_A)$:

$$(a \rtimes L) \star (b \rtimes J) = a \star (L_{(1)} \triangleright b) \rtimes L_{(2)} J.$$

- the initial algebras are canonically embedded, $A \ni f \to f \otimes 1_A$ and $H \ni L \to 1_A \otimes L$ as subalgebras in $A \rtimes H$.

Example: Trivial action $L \triangleright a = \epsilon(L) a$ makes $A \rtimes H$ isomorphic to the ordinary tensor product algebra $A \otimes H$:
$(f \otimes L)(g \otimes M) = fg \otimes LM$.

Weyl-Heisenberg algebra as a smash product

Weyl-Heisenberg algebra contains two Abelian subalgebras which can be considered as a universal enveloping algebras of two Abelian Lie algebras:

Algebra of coordinates $\mathcal{X} \ni x^\mu$.
$\mathcal{X} \equiv \{\mathbb{C}[x^0, \ldots, x^{n-1}] : [x^\mu, x^\nu] = 0\}$
$\mathcal{X}$ is $\mathcal{T}$-(Hopf) module algebra.

Algebra of translations $\mathcal{T} \ni P_\mu$.
$\mathcal{T} \equiv \{\mathbb{C}[P_0, \ldots, P_{n-1}] : [P_\mu, P_\nu] = 0\}$
$\mathcal{T}$ is Hopf algebra with :
$\Delta(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu$

The action is implemented by a duality map

$$P_\mu \triangleright x^\nu = -i\langle P_\mu, x^\nu\rangle = -i\delta_\mu^\nu, \qquad P_\mu \triangleright 1 = 0$$

and can be extended to whole algebra $\mathcal{X}$ due to the Leibniz rule, e.g., $P_\mu \triangleright (x^\nu x^\lambda) = -i\delta_\mu^\nu x^\lambda - i\delta_\mu^\lambda x^\nu$.

From $\mathcal{X} \rtimes \mathcal{T} = \mathcal{W} \quad \longrightarrow$ standard set of Heisenberg commutation relations:

$$[P_\mu, x^\nu]_\rtimes = -i\delta_\mu^\nu 1, \qquad [x^\mu, x^\nu]_\rtimes = 0 \qquad [P_\mu, P_\nu]_\rtimes = 0.$$

as generating relations.

- Note that $\mathcal{W} = \mathcal{X} \rtimes \mathcal{T}$ cannot be equipped in Hopf algebra structure for two reasons:

  evaluation of the counit ϵ on commutator $[P_\mu, x^\nu] = -i\delta_\mu^\nu 1$ leads to a contradiction since $\epsilon(1) = 1$.

Lie-algebraic formula for the coproduct

$$\Delta_0(y) = y \otimes 1 + 1 \otimes y \quad \text{for} \quad y \in \{x^1 \dots x^n\} \cup \{P_1 \dots P_n\}$$

is incompatible with $[P_\mu, x^\nu] = -\imath\delta^\nu_\mu 1$.

- Instead the structure of **unital non-counital bialgebra** equipped with left or right 'half-primitive' coproducts:

$$\Delta_0^R(x^\mu) = x^\mu \otimes 1; \qquad \Delta_0^L(x^\mu) = 1 \otimes x^\mu$$

  and the standard: $\Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu$ turns out to be compatible.

- In contrast to primitive coproduct which is valid only on generators, the above 'half-primitive' coproducts preserve their form for all elements of the algebra $\mathcal{X}$. It provides a (trivial) comodule algebra structure.

Lie-algebraic formula for the coproduct

$$\Delta_0(y) = y \otimes 1 + 1 \otimes y \quad \text{for} \quad y \in \{x^1 \dots x^n\} \cup \{P_1 \dots P_n\}$$

is incompatible with $[P_\mu, x^\nu] = -\imath \delta_\mu^\nu 1$.

- Instead the structure of **unital non-counital bialgebra** equipped with left or right 'half-primitive' coproducts:

  $$\Delta_0^R(x^\mu) = x^\mu \otimes 1; \qquad \Delta_0^L(x^\mu) = 1 \otimes x^\mu$$

  and the standard: $\Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu$ turns out to be compatible.

- In contrast to primitive coproduct which is valid only on generators, the above 'half-primitive' coproducts preserve their form for all elements of the algebra $\mathcal{X}$. It provides a (trivial) comodule algebra structure.

On the other hand one can show that $\mathcal{X} \rtimes \mathcal{T} = \mathcal{W}$ has the structure of

Hopf algebroid. S. Meljanac+Zagreb group]

Covariant quantum phase space

Consider a Lie algebra $\mathfrak{g}$:

$$[L_a, L_b] = \gamma_{ab}^c L_c$$

and its finite dimensional representation

$$\rho : \mathfrak{g} \to End_{\mathbb{K}} V \quad \Leftrightarrow \quad \triangleright : \mathfrak{g} \otimes V \to V$$

$L \triangleright x \equiv \rho(L)(x) = \hat{L}^{\mu}_{\nu} x^{\nu}$.
This action can be uniquely extended to the action of the entire universal enveloping algebra $\triangleright : U_{\mathfrak{g}} \otimes C^{\infty}(V) \to C^{\infty}(V)$ if we define the Lie algebra generators

$$\hat{\rho}(L) = -\hat{L}^{\beta}_{\alpha} x^{\alpha} \partial_{\beta}$$

in terms of first-order differential operators, which are in fact coordinate independent objects ($\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$). This realization leads to the Weyl-Heisenberg extension of the initial algebra $\mathfrak{g}$.

Resulting (abstract) algebra can be represented by the following set of commutation relations

$$[L_a, L_b] = \gamma_{ab}^{c} L_c, \quad [L_b, p_\nu] = (\hat{L}_b)_\nu^\alpha p_\alpha, \quad [p_\mu, p_\nu] = 0$$

$$[L_a, x^\mu] = -(\hat{L}_a)_\alpha^\mu x^\alpha, \quad [p_\nu, x^\mu] = 1\, \delta_\nu^\mu, \quad [x^\mu, x^\nu] = 0$$

The first line represents a Lie subalgebra which can be recognized as a inhomogeneous extension $\mathfrak{ig}_\rho$ of the initial Lie algebra $\mathfrak{g}$ with respect to the representation ρ.

Thus a unital associative algebra generated by the above commutation relations can be presented as a double smash product:

$$\mathcal{X} \rtimes U_{\mathfrak{ig}_\rho} \equiv \mathcal{W} \rtimes U_{\mathfrak{g}}$$

since $U_{\mathfrak{ig}_\rho} = \mathcal{T} \rtimes U_{\mathfrak{g}}$.

Bi-/Hopf algebroids

J.H. Lu, Intern. Journ. Math. 7, 47 (1996);
P. Xu, Comm. Math. Phys. 216, 539 (2001);
T. Brzezinski, G. Militaru, J. Alg. 251, (2002);
G. Bohm, K. Szlachanyi, J. Algebra 274 (2004).

Hopf algebroids are Hopf algebras over noncommutative rings.

A left Hopf algebroid $\mathcal{M} = (M, A, s, t, \Delta, \epsilon)$ is a left bialgebroid together with an antipode $\lambda : M \to M$.
The bialgebroid $\mathcal{M}$ consists of the following data

- a total algebra M and a base algebra A
- two mappings providing A- bimodule or A^e-ring structure on M:
  - an algebra homomorphism $s : A \to M$ - a source map
  - an algebra anti-homomorphism $t : A \to M$ - a target map
  - such that: $s(a)t(b) = t(b)s(a)$, for all $a, b \in A$ and

$$a.m.b = s(a)t(b)m$$

- coproduct and counit maps, with analogous axioms of a coalgebra but all mappings are A-bimodule homomorphisms and $\otimes \Longrightarrow \otimes_A$.

  - Since $M \otimes_A M$ is not an algebra one assumes a coring structure, i.e. the bialgebroid coproduct map $\Delta : M \to M \otimes_A M$ is a coassociative A-bimodule map :

$$\Delta(s(a)t\,(b)\,m) = s(a)m_{(1)} \otimes_A t(b)m_{(2)}$$

    Notice: $(t(a) \otimes_A 1 - 1 \otimes_A s(a))\Delta(m) = 0$
  - Moreover, the image $Im\Delta \subseteq M \times_A M \subseteq M \otimes_A M$, i.e.

$$\Delta(m)(t(a) \otimes_A 1 - 1 \otimes_A s(a)) = 0$$

    and we can require that $\Delta : M \to M \times_A M$ is an algebra map, i.e.

$$\Delta(mn) = \Delta(m)\Delta(n) \equiv m_{(1)}n_{(1)} \otimes_A m_{(2)}n_{(2)}$$

Note: $M \times_A M$ is known as a Takeuchi product.

- The counit map $\epsilon : M \to A$ has to satisfy:
  $\epsilon(1_M) = 1_A$,
  $\epsilon(mn) = \epsilon(ms(\epsilon(n))) = \epsilon(mt(\epsilon(n)))$,
  $s(\epsilon(m_{(1)}))m_{(2)} = t(\epsilon(m_{(2)}))m_{(1)} = m$
  enables to introduce the anchor map $M \ni m \mapsto m \blacktriangleright \in EndA$
  by $m \blacktriangleright a = \epsilon(ms(a)) = \epsilon(mt(a))$
- In the case of Hopf algebroid one requires, in addition, an antipode as antialgebra map $\lambda : M \to M$
  $\lambda \circ t = s$
  $\lambda(m_{(2)})m_{(1)} = t(\epsilon(\lambda(m)))$
  there exists a section $\gamma : M \otimes_A M \to M \otimes M$ s.t.

$$\mu_M \circ (id \otimes \lambda) \circ \gamma \circ \Delta = s \circ \epsilon$$

Yetter - Drinfeld modules (left-right)

(Yang-Baxter modules, crossed modules, Hopf modules)
A Yetter - Drinfeld (YD) module over a bialgebra $H(\mu, \eta, \Delta, \epsilon)$, is a H-module which is simultaneously a H-comodule

- a left H-module with the action $H \otimes A \to A$, $L \otimes a \mapsto L \triangleright a$
- a right H-comodule with the coaction $\rho : A \mapsto A \otimes H$; $\rho(a) = a_{<0>} \otimes a_{<1>}$

Compatibility condition between action and coaction
is required in the form:

$$\rho(L \triangleright a) = L_{(2)} \triangleright a_{<0>} \otimes L_{(3)} \triangleright a_{<1>} L_{(1)}$$

or

$$L_{(1)} \triangleright a_{<0>} \otimes L_{(2)} a_{<1>} = (L_{(2)} \triangleright a)_{<0>} \otimes (L_{(2)} \triangleright a)_{<1>} L_{(1)}$$

Algebras in Yetter-Drinfeld category

- ${}_H\mathfrak{YD}^H$ - denotes the category of all (left-right) Yetter-Drinfeld modules (braided monoidal category).

$$L \triangleright (a \otimes a') = L_{(1)} \triangleright a \otimes L_{(2)} \triangleright' a'$$

$$(a \otimes a')_{<0>} \otimes (a \otimes a')_{<1>} = a_{<0>} \otimes a'_{<0'>} \otimes a'_{<1'>} a_{<1>}$$

$$A \otimes A' \to A' \otimes A : \quad a \otimes a' \to a'_{<0'>} \otimes (a'_{<1'>} \triangleright a)$$

- A module-comodule algebra $A = (A, \star, 1_A, H\triangleright, \rho)$

$$L \triangleright (a \star b) = (L_{(1)} \triangleright a) \star (L_{(2)} \triangleright b)$$

$$(a \star b)_{<0>} \otimes (a \star b)_{<1>} = (a_{<0>} \star b_{<0>}) \otimes b_{<1>} a_{<1>}$$

is an algebra in ${}_H\mathfrak{YD}^H$ if and only if it is a **braided commutative**, i.e. :

$$a \star b = b_{<0>} \star (b_{<1>} \triangleright a)$$

Drinfeld twist deformation

Deformation- **'quantization'** procedure

- The twist F - invertible element of $H \otimes H$
  1. the two-cocycle condition
  $$(F \otimes 1)(\Delta \otimes id)F = (1 \otimes F)(id \otimes \Delta)F$$
  2. normalization $(id \otimes \epsilon)F = (\epsilon \otimes id)F = 1 \otimes 1,$
- Twisted bi/Hopf algebra

$$\begin{array}{rcl} H(\mu, \eta, \Delta, \epsilon, S) & \longrightarrow & H^F(\mu, \eta, \Delta^F, \epsilon, S^F) \\ \Delta^F(\cdot) & = & F\Delta(\cdot)F^{-1} \\ S(\cdot) \to S^F(\cdot) & = & F_1 S(F_2) S(\cdot) S(F_{1'}) F_{2'} \end{array}$$

Notation:

$$F = F_1 \otimes F_2 \in H \otimes H, \qquad F^{-1} = \bar{F}_{1'} \otimes \bar{F}_{2'} \in H \otimes H$$

$$L_{(1)} \otimes L_{(2)} \to L_{(1^F)} \otimes L_{(2^F)} = F_1 L_{(1)} \bar{F}_{1'} \otimes F_2 L_{(2)} \bar{F}_{2'}$$

Twisted braided commutativity

- (Left-right) Yetter-Drinfeld module over H with the right coaction

$$\rho(a) = a_{<0>} \otimes a_{<1>}$$

becomes automatically a YD module over H^F if the action remains unchanged and the **coaction is modified by the twist**

$$\rho_F(a) = F_1 \triangleright (\bar{F}_{2'} \triangleright a)_{<0>} \otimes F_2 (\bar{F}_{2'} \triangleright a)_{<1>} \bar{F}_{1'} = a_{<0^F>} \otimes a_{<1^F>}$$

for all $a \in A$.

- Particularly, a module-comodule algebra $A = (A, \star, 1_A, H\triangleright, \rho)$ is an algebra in ${}_H\mathfrak{YD}^H$ if and only if its twisted counterpart ${}_F A^F = (A, \star_F, 1_A, H^F \triangleright, \rho_F)$ is braided commutative:

$$a \star_F b = b_{<0^F>} \star_F (b_{<1^F>} \triangleright a),$$

where $a \star_F b = (\bar{F}_1 \triangleright a) \star (\bar{F}_2 \triangleright b)$ denotes modified product.

Twisted smash product

Proposition

[D. Bulacu, F. Panaite, F. M. J. Van Oystaeyen, Comm. Alg. 28 (2000),631]

For any Drinfeld twist F:

$$A \rtimes H \cong_F A \rtimes H^F$$

even though $A \ncong_F A$ and $H \ncong H^F$ (as bialgebras).

- Both algebras are determined on the same $\mathbb{K}$-module $A \otimes H$ but differ by the multiplications:

$$\begin{aligned} (a \rtimes L) \star (b \rtimes J) &= a \star (L_{(1)} \triangleright b) \rtimes L_{(2)} J \\ (a \rtimes L) \star_F (b \rtimes J) &= a \star_F (L_{(1^F)} \triangleright b) \rtimes L_{(2^F)} J \end{aligned}$$

- where $\Delta^F(L) = F\Delta(L)F^{-1} = L_{(1^F)} \otimes L_{(2^F)}$ - twisted coproduct of the bialgebra H^F.

- Both algebras are generated by elements:
$$a \rtimes 1_H, a \in A \text{ and } 1_A \rtimes L, L \in H$$
- The isomorphism $\varphi :_F A \rtimes H^F \to A \rtimes H$ can be defined by the formula
$$\varphi\left(a \rtimes L\right) = \left(\bar{F}_1 \triangleright a\right) \rtimes \bar{F}_2 L$$
such that:
$$\varphi\left(\left(a \rtimes L\right) \star_F \left(b \rtimes J\right)\right) = \varphi\left(a \rtimes L\right) \star \varphi\left(b \rtimes J\right)$$
for all $a, b \in A$ and $L, J \in H$.

- Due to the normalization condition: $\varphi(1_A \rtimes L) = 1_A \rtimes L$, but $\varphi\left(a \rtimes 1_H\right) = \left(\bar{F}_1 \triangleright a\right) \rtimes \bar{F}_2$.
- The inverse map $\varphi^{-1} : A \rtimes H \to A_F \rtimes H^F$ is given by $\varphi^{-1}\left(a \rtimes L\right) = \left(F_1 \triangleright a\right) \rtimes F_2 L$.

II.Cross product algebras as bialgebroids

Theorem [T. Brzezinski, G. Militaru, J. Alg. 251, (2002)]:
Let $H = (H, \Delta, \epsilon)$ be a bialgebra, $A = (A, \star, H\triangleright)$ is a left H-module algebra and (A, ρ) a right H-comodule. Then $(A, \star, H\triangleright, \rho)$ is a braided commutative algebra in ${}_H\mathfrak{YD}^H$ if and only if $\left(A \rtimes H, s, t, \tilde{\Delta}, \tilde{\epsilon}\right)$ is an A-bialgebroid with

- The source, target, coproduct and the counit given by:

$$\begin{aligned} s(a) &= a \rtimes 1_H, \quad t(a) \equiv \rho(a) = a_{<0>} \rtimes a_{<1>} \\ \tilde{\Delta}(a \rtimes L) &= (a \rtimes L_{(1)}) \otimes_A (1_A \rtimes L_{(2)}) \\ \tilde{\epsilon}(a \rtimes L) &= \epsilon(L)a \\ (a \rtimes L) \blacktriangleright b &= a \star (L \triangleright b) \end{aligned}$$

- If H has an antipode $S : H \to H$ then $A \rtimes H$ has the antipode λ_S

$$\lambda_S(1_A \rtimes L) = 1_A \rtimes S(L), \quad \lambda_S(a \rtimes 1_H) = S^2(a_{<1>}) \triangleright a_{<0>} \rtimes S^2(a_{<2>})$$

and $\gamma((a \rtimes L) \otimes_A (b \rtimes N)) = (ab_{<0>} \rtimes b_{<1>} L) \otimes (1_A \rtimes N)$

denotes a canonical section $\gamma : \ (A \rtimes H) \otimes_A (A \rtimes H) \to (A \rtimes H) \otimes (A \rtimes H)$.

Back to the Weyl algebra

$$\mathcal{W} = \mathcal{X} \rtimes \mathcal{T} = \{[P_\mu, x^\nu] = -\imath\delta_\mu^\nu 1, \quad [x^\mu, x^\nu] = [P_\mu, P_\nu] = 0\}$$

is (Hopf) bialgebroid $(\mathcal{W}, \mathcal{X}, s, t, \tilde{\Delta}, \tilde{\epsilon})$

The source and target maps: $s(x) = t(x) = x \otimes 1_\mathcal{T}$.

The coproduct and counit:

$$\tilde{\Delta}(x \rtimes P) = (x \rtimes P_{(1)}) \otimes_\mathcal{X} (1_\mathcal{X} \rtimes P_{(2)})$$

$$\tilde{\epsilon}(x \rtimes P) = \epsilon(P)x$$

for all $x \in \mathcal{X}$ and $P \in \mathcal{T}$.

And $\tilde{\Delta}(x^\mu) = x^\mu \otimes_\mathcal{X} 1 = 1 \otimes_\mathcal{X} x^\mu$; $\tilde{\Delta}(P_\mu) = P_\mu \otimes_\mathcal{X} 1 + 1 \otimes_\mathcal{X} P_\mu$
together with $\tilde{\epsilon}(x^\mu) = x^\mu$; $\tilde{\epsilon}(P_\mu) = 0$; $\tilde{\epsilon}(1) = 1_\mathcal{X}$.

III.Twisted bialgebroids

P. Xu, Comm. Math. Phys. 216, 539 (2001).

- Bialgebroid definition provides a canonical action
  $\blacktriangleright: M \otimes A \to A$
  (also known as an anchor $M \ni m \to m \blacktriangleright \in EndA$):
  $m \blacktriangleright a = \epsilon(ms(a)) = \epsilon(mt(a))$,

Theorem [Xu]:
Assume that $(M, A, s, t, \Delta, \epsilon)$ is bialgebroid over the algebra A and $F = F_1 \otimes_A F_2 \in M \otimes_A M$ is a "twistor" (Hopf algebroid twist). Then $(M, A_F, s_F, t_F, \Delta_F, \epsilon)$ is a bialgebroid over the algebra A_F, where

$$s_F(a) = s\left(\bar{F}_1 \blacktriangleright a\right)\bar{F}_2 \quad ; \qquad t_F(a) = t\left(\bar{F}_2 \blacktriangleright a\right)\bar{F}_1 \quad \forall a \in A.$$

and new twisted coproduct $\Delta_F : M \to M \otimes_{A_F} M$:

$$\Delta_F\left(m\right) = F^{\#}\left(\Delta\left(m\right)F^{-1}\right), \quad \forall \ \ m \in M$$

For the twisted coproduct $\Delta_F : M \to M \otimes_{A_F} M$:

$$\Delta_F\,(m) = F^{\#}\left(\Delta\,(m)\,F^{-1}\right), \quad \forall \;\; m \in M$$

$F^{\#} : M \otimes_A M \to M \otimes_{A_F} M$ is defined by:

$$F^{\#}(m \otimes_A n) = (F_1\, m) \otimes_{A_F} (F_2\, n).$$

The multiplication in M does not change.
In A it changes to:

$$\star \mapsto \star_F = \star \circ (\bar{F}_1 \blacktriangleright \otimes \bar{F}_2 \blacktriangleright)$$

IV.Main result

Goal: To compare two constructions of bialgebroids:

The bialgebroid obtained by bialgebroid twisting of the smash product algebra $(A \rtimes H)^{\tilde{F}}$

and

bialgebroid obtained from the smash product algebra of twisted bialgebra with twisted YD module algebra ${}_F A^F \rtimes H^F$

Main result: Both bialgebroids are equivalent (isomorphic):

$$ {}_F A^F \rtimes H^F \cong (A \rtimes H)^{\tilde{F}} $$

Remainder: As algebras all three are equivalent

$$ {}_F A^F \rtimes H^F \cong A \rtimes H \cong (A \rtimes H)^{\tilde{F}} $$

Corollary: First two are not isomorphic as bialgebras !

$$ {}_F A^F \rtimes H^F \ncong A \rtimes H $$

Where the module-comodule algebra A is braided commutative in the category ${}_H\mathfrak{Y}\mathfrak{D}^H$,
$(a \star b = b_{<0>} \star (b_{<1>} \triangleright a))$.

BM construction: $A \rtimes H$ is a bialgebroid over the algebra A if we define
(shifting $\Delta : H \to H \otimes H$ to $\tilde{\Delta} : A \rtimes H \to (A \rtimes H) \otimes_A (A \rtimes H)$)

$$\tilde{\Delta}(a \rtimes L) = (a \rtimes L_{(1)}) \otimes_A (1_A \rtimes L_{(2)}), \quad s\,(a) = a \rtimes 1_H$$

$$t\,(a) = a_{<0>} \rtimes a_{<1>}$$

$$\tilde{\epsilon}(a \rtimes L) = \epsilon(L)a$$

It is easy to see that any Drinfeld twist $F = F_1 \otimes F_2 \in H \otimes H$ in the bialgebra H
can be shifted to the bialgebroid twistor $\tilde{F} \in (A \rtimes H) \otimes_A (A \rtimes H)$ by

$$F \to \tilde{F} = (1_A \rtimes F_1) \otimes_A (1_A \rtimes F_2)$$

which automatically satisfies bialgebroid cocycle and normalization conditions. Therefore, it can be used to construct new (twisted) bialgebroid $(A \rtimes H)^{\tilde{F}}$ by making use of P. Xu twistor $\tilde{F}$.

Then apply bialgebroid twisting [Xu] to bialgebroid $(A \rtimes H)^{\tilde{F}}$ by means of the shifted twist $\tilde{F}$

$$\tilde{\Delta}_{\tilde{F}}(a \rtimes J) = \tilde{F}^{\#}(\tilde{\Delta}(a \rtimes J)\tilde{F}^{-1}), \quad s_{\tilde{F}}(a) = (\bar{F}_1 \triangleright a) \rtimes \bar{F}_2,$$

$$t_{\tilde{F}}(a) = \left(\bar{F}_2 \triangleright a\right)_{<0>} \rtimes (\bar{F}_2 \triangleright a)_{<1>}\bar{F}_1$$

where

$$\tilde{F}^{\#} : (A \rtimes H) \otimes_A (A \rtimes H) \to (A \rtimes H) \otimes_{A_F} (A \rtimes H)$$

and $F^{\#}(m \otimes_A n) = (F_1\, m) \otimes_{A_F} (F_2\, n)$.

Note: original Xu twistor is an inverse of ours.

Similarly (BM construction)
${}_F A^F \rtimes H^F$ a bialgebroid over the algebra ${}_F A$ if we set

$$\widetilde{\Delta^F}(a \rtimes L) = (a \rtimes L_{(1^F)}) \otimes_{{}_F A} (1_A \rtimes L_{(2^F)})$$

$$s^F(a) = a \rtimes 1_H,$$
$$t^F(a) = a_{<0^F>} \rtimes a_{<1^F>}$$

- where
  $a_{<0^F>} \otimes a_{<1^F>} = F_1 \rhd (\bar{F}_{2'} \rhd a)_{<0>} \otimes F_2(\bar{F}_{2'} \rhd a)_{<1>}\bar{F}_{1'}$
- and the algebra ${}_F A$ is braided commutative as well:
  $a \star_F b = b_{<0^F>} \star_F (b_{<1^F>} \rhd a)$.

A. B., A.Pachol , J. Phys. A. (2016)

Theorem

Let H be a bialgebra and $A \in_H \mathfrak{Y}\mathfrak{D}^H$ stands for braided commutative module algebra in the Yetter-Drinfeld category. Assume that $F = F_1 \otimes F_2 \in H \otimes H$ is a normalized cocycle twist in H. Then

$$_F A^F \rtimes H^F \cong (A \rtimes H)^{\tilde{F}}$$

are isomorphic as bialgebroids, where $\tilde{F}$ denotes bialgebroid cocycle twist

$$F \to \tilde{F} = (1_A \rtimes F_1) \otimes_A (1_A \rtimes F_2)$$

obtained from F.

For the proof: The isomorphism

$$\varphi : A_F \rtimes H^F \to A \rtimes H$$

where $\varphi(a \rtimes L) = (\bar{F}_1 \triangleright a) \rtimes \bar{F}_2 L$
of total algebras makes commuting the following diagram

$$\begin{array}{ccc} {}_F A^F \rtimes H^F & \xrightarrow{\varphi} & A \rtimes H \\ \widetilde{\Delta^F} \downarrow & & \downarrow \tilde{\Delta}_{\tilde{F}} \\ ({}_F A^F \rtimes H^F) \otimes_{{}_F A} ({}_F A^F \rtimes H^F) & \xrightarrow{\varphi \otimes_{{}_F A} \varphi} & (A \rtimes H) \otimes_{{}_F A} (A \rtimes H) \end{array}$$

i.e. $\tilde{\Delta}_{\tilde{F}} \circ \varphi = (\varphi \otimes_{{}_F A} \varphi) \circ \widetilde{\Delta^F}$ as well as

$$\varphi \circ s^F = s_{\tilde{F}}, \quad \varphi \circ t^F = t_{\tilde{F}}, \quad \tilde{\epsilon} \circ \varphi = \tilde{\epsilon}.$$

For antipodes $\tilde{\lambda} = \varphi \circ \lambda_{BM} \circ \varphi^{-1}$.

Quasi-triangular example: bialgebra level

- the universal quantum R-matrix $R = R_1 \otimes R_2 \in H \otimes H$

$$R\Delta(X)R^{-1} = \Delta^{op}(X),$$

"almost cocommutative Hopf algebra"

and $(\Delta \otimes id)R = R_{13}R_{23}$,
$(id \otimes \Delta)R = R_{13}R_{12}$,
$(\epsilon \otimes id)R = (id \otimes \epsilon)R = 1$

which imply quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

(H, R) is quasi-triangular bialgebra if (H, R_{21}^{-1}) is quasi-triangular.

Notation:
$R = R_1 \otimes R_2 \in H \otimes H$ and $R^{-1} = \bar{R}_1 \otimes \bar{R}_2$, $R_{21}^{-1} = \bar{R}_2 \otimes \bar{R}_1$.

Braided commutativity 2

- (Left) module A over (H, R) becomes automatically a (left-right) Yetter-Drinfeld module with the right coaction

$$\rho_R(a) = (R_2 \triangleright a) \otimes R_1$$

  for all $a \in A$.
- Particularly, a module algebra $A = (A, \star, 1_A, H\triangleright)$ is an algebra in ${}_H\mathfrak{YD}^H$ if and only if it is a braided commutative:

$$a \star b = (R_2 \triangleright b) \star (R_1 \triangleright a)$$

Twist of quasi-triangular bialgebra

$$(H, R) \longrightarrow (H^F, R^F)$$

- $(H^F, R^F \equiv F_{21} R F^{-1})$ is quasi-triangular too.
- the module algebra $(A, \star_F, 1_A, H\triangleright) \in {}_{H^F}\mathfrak{YD}^{H^F}$ if and only if $(A, \star, 1_A, H\triangleright) \in {}_{H}\mathfrak{YD}^{H}$, where right coactions are given by the corresponding R-matrices.

Back to the covariant quantum phase space

$$\mathcal{X} \rtimes U_{\mathfrak{ig}_\rho} \equiv \mathcal{W} \rtimes U_{\mathfrak{g}}$$

where $U_{\mathfrak{ig}_\rho} = \mathcal{T} \rtimes U_{\mathfrak{g}}$.

$$[L_a, L_b] = \gamma_{ab}^c L_c, \quad [L_b, p_\nu] = (\hat{L}_b)_\nu^\alpha p_\alpha, \quad [p_\mu, p_\nu] = 0$$

$$[L_a, x^\mu] = -(\hat{L}_a)_\alpha^\mu x^\alpha, \quad [p_\nu, x^\mu] = 1\, \delta_\nu^\mu, \quad [x^\mu, x^\nu] = 0$$

Any Drinfeld twist $F \in U_{\mathfrak{ig}_\rho} \otimes U_{\mathfrak{ig}_\rho}$ allows to perform deformation quantization of the Hopf algebroid $\mathcal{X} \rtimes U_{\mathfrak{ig}_\rho}$ and to construct its Hilbert space (Quantum Mechanical) realization with non-commuting position operators. In such framework specialization of the formal deformation parameter to some numerical value is possible.

Hopf-Galois context

For a comodule M over the Hopf algebra H with the right coaction $\rho : M \to M \otimes H$ one defines a subalgebra of coinvariant elements, $M^{coH} = \{h \in M : \rho(m) = m \otimes 1_H\}$.
We say that the extension $M^{coH} \subset M$ is H-Hopf-Galois if the map

$$M \otimes_{M^{coH}} M \to M \otimes H$$

given by $m \otimes n \mapsto (m \otimes 1_H)\rho(n)$, is bijective.
A smash product $\mathcal{A} \rtimes \mathcal{H}$ is a particular kind of a crossed product algebra $\mathcal{A} \rtimes_\sigma \mathcal{H}$, where a convolution invertible map $\sigma : \mathcal{H} \otimes \mathcal{H} \to \mathcal{A}$ has to satisfy (in $\mathcal{A}$) the so-called 2-cocycle

$$[L_{(1)} \triangleright \sigma(J_{(1)}, K_{(1)})]\sigma(L_{(2)}, J_{(2)}K_{(2)}) \, = \, \sigma(L_{(1)}, J_{(1)})\sigma(L_{(2)}J_{(2)}, K)$$

where $\sigma(J, 1_H) = \sigma(1_H, J) = \epsilon(J)1_A$ as well as twisted module

$$[L_{(1)} \triangleright (J_{(1)} \triangleright a)]\sigma(L_{(2)}, J_{(2)}) \, = \, \sigma(L_{(1)}, J_{(1)})[(L_{(2)}J_{(2)}) \triangleright a]$$

conditions for any $a \in \mathcal{A}$ and $L, J, K \in H$.

These properties allow to establish on the vector space $\mathcal{A} \otimes \mathcal{H}$ the structure of unital, associative algebra with the multiplication

$$(a \otimes L)(b \otimes J) = a(L_{(1)} \triangleright b)\sigma(L_{(2)}, J_{(1)}) \otimes L_{(3)} J_{(2)}$$

This algebra is denoted as $\mathcal{A} \rtimes_\sigma \mathcal{H}$. It has a natural left $\mathcal{A}$ module and right $\mathcal{H}$ comodule structures (the so-called normal basis property), which makes it a H-comodule algebra (a coring) with the subalgebra $\mathcal{A} \otimes 1_H = (\mathcal{A} \rtimes_\sigma \mathcal{H})^{coH}$ composed of coinvariants of the coaction.
Due to this fact it provides a canonical example of Hopf-Galois extension which, in turn, is an algebraic counterpart of a quantum principal bundle.
Taking the trivial cocycle $\sigma_0(L, J) = \epsilon(L)\epsilon(J)\, 1_A$ one reconstructs the smash product. A natural question which appears now is whether the result of the present section can be extended to the case of nontrivial cocycle $\sigma : \mathcal{H} \otimes \mathcal{H} \to \mathcal{A}$?

Conclusions

- The covariant phase space of Quantum Mechanics has a Hopf algebroid structure
- Deformed phase space treated as an algebra does not distinguish between commuting and non-commuting space(time) variables ${}_{F}A \rtimes H^{F} \cong A \rtimes H$
- ... unless we consider ${}_{F}A^{F} \rtimes H^{F} \cong (A \rtimes H)^{F}$ = bialgebroid over ${}_{F}A$ while $A \rtimes H$ = bialgebroid over A.

Conclusions

- The covariant phase space of Quantum Mechanics has a Hopf algebroid structure
- Deformed phase space treated as an algebra does not distinguish between commuting and non-commuting space(time) variables ${}_FA \rtimes H^F \cong A \rtimes H$
- ... unless we consider ${}_FA^F \rtimes H^F \cong (A \rtimes H)^F =$ bialgebroid over ${}_FA$ while $A \rtimes H =$ bialgebroid over A.

Thank you for your attention!