Twisted reality

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Between the idea
And the reality
[...]
Falls the Shadow.

T.S. Eliot, The Hollow Men

Reference:

Aims:

**Primary aim:** To show relationship between two approaches to the twisting of real structure for Dirac operators:

- *Twisted real structure for spectral triples* discussed in [TB, N Ciccoli, L Dąbrowski, A Sitarz ’16]
- *Twisted real spectral triples* of [G Landi, P Martinetti ’16].

**Secondary aim:** To propose a unifying framework for existing (and possible future) reality conditions.
Real spectral triples

**Ingredients:**
- A $\ast$-algebra $A$;
- A Hilbert space $H$ and a $\ast$-representation $\pi : A \rightarrow B(H)$;
- Operators $D, \gamma, J : H \rightarrow H$;
- Signs: $\varepsilon, \varepsilon', \varepsilon'' \in \{+, -\}$.

**Types of relations:**
- between operators representing elements of $A$ and their conjugates through $J$ (the order-zero condition):
  \[ [\pi(a), J\pi(b)J^{-1}] = 0; \]
- between $D$ and operators representing elements of $A$;
- between operators only (no representation), $D$ and $J$ and $\gamma$;
- between $D$, $J$ and operators representing elements of $A$ (the first-order condition).
How to do the twist?

- Relations involving elements of the algebra can be twisted by an algebra automorphism $\rho$;
- Relations between operators can only (?) be twisted by an invertible operator $\nu$;
- The action of $\nu$ should be transferrable to an automorphism $\hat{\nu}$ of $A$ through $\pi$, the *implementation*

$$\pi(\hat{\nu}(a)) := \nu \pi(a) \nu^{-1}.$$  

- The first order condition should be twisted by both $\nu$ and $\rho$. 
Preliminary notation and assumptions:

**Preliminary notation:**
- Conjugate (anti-)representation:
  \[ \pi_J: A \rightarrow B(H), \quad \pi_J(a) = J\pi(a)J^{-1}. \]
- Twisted commutators:
  \[ [T, a]_\pi := T\pi(a) - \pi(\rho(a))T. \]

**Preliminary assumptions:**
- \( \pi \) is faithful;
- \( J \) is an anti-linear isometry, \( J^2 = \varepsilon \);
- the zero-order condition holds, i.e. \( \pi_J(A) \) is in the commutant of \( \pi(A) \);
- \( D \) is a densely defined, self-adjoint operator with compact resolvent, and domain preserved by \( \pi(A) \).
Twisting ingredients

- An algebra automorphism $\rho : A \to A$ such that
  $$\rho \circ * = * \circ \rho^{-1},$$

- A bounded operator $\nu$ on $H$ with bounded inverse, which implements an algebra automorphism $\hat{\nu} : A \to A$ (in representation $\pi$).
(A, H, π, D, J) is a \((\nu, \rho)\)-type twisted real spectral triple if:

(a) for all \(a \in A\), the twisted commutators \([D, a]_\rho^\pi\) are bounded;
(b) \(\nu JD\) preserves the domain of \(D\);
(c) \(DJ\nu = \epsilon' \nu JD\);
(d) \(\nu J\nu = J\);
(e) the \((\nu, \rho)\)-twisted first-order condition:

\[
[[D, a]_\rho^\pi, b]_{\rho_\nu\hat{\nu}-2}^\pi J = 0, \quad \forall a, b \in A,
\]

is satisfied;
(f) in the even case also: \(\nu^2 \gamma = \gamma \nu^2\).
Special cases

- The \((\nu, \text{id})\)-type \(\equiv\) spectral triple (untwisted) with twisted reality of [Brzeziński, Ciccoli, Dąbrowski & Sitarz].

- The \((1, \rho)\)-type \(\equiv\) \(\rho\)-twisted real spectral triple of [Landi & Martinetti].
Conformal rescaling: setup

- $(A, H, D, J)$ is a real spectral triple (representation $\pi$).
- $u \in A$ is invertible and such that $k = \pi(u)$ is positive with bounded inverse.
- $k_J = JkJ^{-1}$ (note $k_J$ commutes with all $\pi(a)$).
Example (i)

Let:

\[ \hat{D}_k = k D k, \quad \nu = k k^{-1}, \quad \rho(a) = \hat{\nu}^2(a) = u^2 a u^{-2}. \]

Then \((A, H, \hat{D}_k, J)\) is a \((\nu, \rho)\)-type twisted real spectral triple.

Note: Since \(\rho = \hat{\nu}^2\) the outer commutators in the first-order condition are not twisted!

Such conformally twisted Dirac operators were studied by Connes & Moscovici (not the real structures, though).
Example (ii)

Let:

\[ D_k = k J D_k J, \quad \nu = k^{-1} k J, \quad \rho = \text{id}. \]

Then \((A, H, D_k, J)\) is a \((\nu, \text{id})\)-type twisted real spectral triple [BCDS'16].
Example (iii)

Let:

\[ \tilde{D}_k = k J k D k J, \quad \nu = 1, \quad \rho(a) = u^2 a u^{-2}. \]

Then \((A, H, \tilde{D}_k, J)\) is a \((1, \rho)\)-type twisted real spectral triple.
Duality:

Theorem

If \((A, H, \pi, D, J)\) is a type \((\nu, \text{id})\)-twisted real spectral triple, then \((A^{\text{op}}, H, \pi^{\circ}, D, J)\), where

\[
\pi^{\circ} = \pi J \circ *,
\]

is of \((\nu^{-1}, \hat{\nu}^{-2})\)-type.

Remarks:

- The outer commutator in the first order condition is ‘un-twisted’.
- Examples (i) and (ii) are related by this duality.
Untwisting:

Theorem

(Sketch) If $(A, H, \pi, D, J)$ is of $(\text{id}, \nu^{-2})$-type, then $(A, H, \tilde{\pi}, \tilde{D})$, where

\[
\tilde{\pi} = \pi \quad \text{or} \quad \tilde{\pi} = \pi_{\nu} := \text{Ad}_{\nu^{-1}} \circ \pi, \quad \tilde{D} = \nu D\nu,
\]

is of $(\nu, \text{id})$-type.

Examples (ii) and (iii) are related by this duality.
The minimal twist: First data

- An even dimensional spin manifold $M$;
- Hilbert space of square integrable sections of the spinor bundle, $H = H_+ \oplus H_-$;
- Dirac operator $\tilde{D}$, representation $\tilde{\pi}_0$ of $C^\infty(M)$, and the grading $\gamma$:

$$\tilde{D} = \begin{pmatrix} 0 & \partial_- \\ \partial_+ & 0 \end{pmatrix}, \quad \tilde{\pi}_0(f) = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

where $\partial_+ = \partial^*$. 
The minimal twist: type \((1, \rho)\)-twisted spectral triple

- **The algebra:** \(A = C^\infty(M) \oplus C^\infty(M) \cong C^\infty(M) \otimes \mathbb{C}^2\)
- Use the grading \(\gamma\) to extend \(\tilde{\pi}\) to the rep. of \(A\) on \(H\):

\[
\tilde{\pi}(f, g) = \frac{1}{2} \tilde{\pi}_0(f)(1 + \gamma) - \frac{1}{2} \tilde{\pi}_0(g)(1 - \gamma) = \text{diag}(f, g).
\]

- **The twist by the algebra automorphism:**

\[
\rho : A \to A, \quad (f, g) \mapsto (g, f),
\]

- **Bounded twisted commutators:**

\[
[\tilde{\mathcal{D}}, (f, g)]_{\rho} = \begin{pmatrix}
0 & [\partial_-, f] \\
[\partial_+, g] & 0
\end{pmatrix},
\]

- **Real structure** \(J = \text{(anti-)diag}(J_+, J_-)\).
Untwisting of the minimal twist

\( \rho \) is implemented by the operator:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

Choose a square root of this operator, e.g.:

\[
\nu = \frac{1}{2} \begin{pmatrix}
(1 + i) & (1 - i) \\
(1 - i) & (1 + i)
\end{pmatrix}.
\]

Note: \( \nu \) is unitary but not self-adjoint.

Representation comes out as:

\[
\pi_\nu(f, g) = \frac{1}{2} \begin{pmatrix}
(f + g) & i(g - f) \\
-i(f - g) & (f + g)
\end{pmatrix}.
\]
Untwisting of the minimal twist

- Untwisted Dirac operator:
  \[
  D = \nu \tilde{D}_\nu = \frac{1}{2} \left( \begin{array}{cc}
  (\partial_+ + \partial_-) & -i(\partial_+ - \partial_-) \\
  i(\partial_+ - \partial_-) & (\partial_+ + \partial_-)
  \end{array} \right).
  \]

- The commutator becomes:
  \[
  [D, \pi_\nu(f, g)] = \frac{1}{2} \left( \begin{array}{cc}
  [\partial_+ + \partial_-] & i([\partial_-, g] - [\partial_+, f]) \\
  i([\partial_+, f] - [\partial_-, g]) & [\partial_+, f] - [\partial_-, g]
  \end{array} \right).
  \]

- Constraint for $J$:
  \[
  J_+ = J_-.
  \]
Doubling of the untwisted minimal twist

- The operator $D$ is not self-adjoint (it is normal, though).
- We need to use the **doubling procedure**: The Hilbert space is doubled to: $H \otimes \mathbb{C}^2$.
- Representation, Dirac operator, grading, reality and the twist are:

$$
\pi(f, g) = \begin{pmatrix}
f & 0 & 0 & 0 \\
0 & g & 0 & 0 \\
0 & 0 & f & 0 \\
0 & 0 & 0 & g
\end{pmatrix}, \quad D = \begin{pmatrix}
0 & 0 & \partial_+ & 0 \\
0 & 0 & 0 & \partial_- \\
\partial_- & 0 & 0 & 0 \\
0 & \partial_+ & 0 & 0
\end{pmatrix},
$$

$$
J = \begin{pmatrix}
J & 0 & 0 & 0 \\
0 & J & 0 & 0 \\
0 & 0 & J & 0 \\
0 & 0 & 0 & J
\end{pmatrix}, \quad \gamma = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
$$
Conclusions

- A proposal for a unified approach to twisting real spectral triples was presented.
- The proposal involves twisting by an operator and an algebra automorphism.
- Two existing approaches to twisting of real structure, i.e. those of Landi & Martinetti, and Brzeziński, Ciccoli, Dąbrowski & Sitarz, are related by the untwisting procedure, provided the algebra automorphism is implemented by the square of an operator.
- The duality relates the second approach also to twisted spectral triples with untwisted 1-st order condition.
This is the way the talk ends
This is the way the talk ends
This is the way the talk ends
Not with a bang but a whimper.

T.S. Eliot, *The Hollow Men* (paraphrased)