Quantum orbit method

a geometric quantization approach

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This paper gives wrong solutions to trivial problems. The basic error, however, is not new.
(Clifford Truesdell, Mathematical Reviews)

(replace **trivial** with **solved**)

Quantum orbit method

Poisson manifold (M, π) $\xrightarrow{\text{quantization}}$ NC associative $*$-algebra $\mathcal{F}_\hbar(M)$

Set of symplectic leaves $\mathcal{S}_M$ $\xleftrightarrow{\textit{quantum orbit relation}}$ Set of irreducible unitary representations of $\mathcal{F}_\hbar$

The example you may know

$$\begin{array}{ccc}
\boxed{(M,\pi) = (\mathfrak{g}^*, \pi_{\mathrm{lin}})} & \xrightarrow{\text{quantization}} & \boxed{\begin{array}{l} U_g(\mathfrak{g}) \text{ identified to} \\ \mathcal{F}_h(\mathfrak{g}^*) \\ \mathrm{gr}(U(\mathfrak{g})) \simeq \mathrm{Pol}(\mathfrak{g}^*) \end{array}} \\
\boxed{\mathcal{S}_{\mathfrak{g}^*} = \text{coadjoint orbits}} & \xleftrightarrow{\textit{quantum orbit relation}} & \boxed{\begin{array}{l} *\text{-irreps of } U(\mathfrak{g}) \leftrightarrow \\ \text{real irreps of } \mathfrak{g} \end{array}}
\end{array}$$

An even simpler case

$$(M, \pi) = (M, 0) \xrightarrow{\text{no quantization}} \mathcal{C}(M) \text{ identified to } \mathcal{F}_h(M)$$

$\mathcal{S}_M =$ points of M; p $\overset{p \mapsto \rho_p}{\longleftrightarrow}$ $*$-irreps of $\mathcal{C}(M)$
$\ker \rho_p =$
$\{f \in \mathcal{C}(M) : f(p) = 0\}$

The example you probably don't know

(K, π_K) standard compact Poisson-Lie group $\xrightarrow{\text{quantization}}$ NC associative $*$-algebra $\mathcal{F}_q(M)$ (quantum function algebra)

$\mathcal{S}_K$ = dressing orbits of K^* on K $\xleftrightarrow{\textit{quantum orbit relation}}$ Set of $*$-irreps of $\mathcal{F}_q(M)$

Combinatorial data attached to $\mathrm{Wl}(K)$ $\xleftrightarrow{\text{Soibelman 1991}}$ Combinatorial data attached to $\mathrm{Wl}(K)$

A case in which it does not work, does it?

$(\mathbb{T}^2, \theta)$ invariant symplectic torus $\xrightarrow{\text{whichever quantization}}$ $\mathcal{F}_\theta(\mathbb{T}^2)$ quantum torus C^*-irrational rotation algebra

$\mathcal{S}_M = \{*\}$ just one point $\xleftrightarrow{\text{HEY! They don't match...}}$ infinitely many $*$-irreps of quantum torus

Innocent looking questions

- Under which constraints on the Poisson manifolds does QOM work?
- Which quantization procedure is the *right* one? Which quantization outcome works better?
- Just *bijective* correspondence or maybe something more (topology?)?
- $\mathcal{S}_M$ is the roughest Poisson (Morita) invariant: to which other invariants does this apply?

Quantization via symplectic groupoid

- Introduced in the late 80ies (Karasev, Weinstein, Zakrzewski).
- Revived by Hawkins 2008.
- Outcome is a groupoid (twisted) C^*-algebra.
- Relies on geometric quantization: pros= geometric data involved, cons= choices.

Outline of QSG

Integration

Integrate (M, π) to a symplectic groupoid (Σ, ω_π).

Polarization

Choose a multiplicative Lagrangian fibration $\mathcal{L}$ of Σ: projection to leaves $\Sigma \to \Sigma_{\mathcal{L}}$ is a groupoid morphism.

Bohr-Sommerfeld

Consider Bohr-Sommerfeld conditions. Under a suitable geometrical constraint the set of BS leaves Σ_{BS} is a subgroupoid of $\Sigma_{\mathcal{L}}$.

And finally...

Construct the groupoid C^*-algebra: $\mathcal{F}_\hbar(M) = C^*(\Sigma_{BS})$.

The dust under the rug

Prequantization

Prequantization of symplectic groupoid was fully understood by Weinstein-Xu (1991). We will assume that our Poisson manifolds are prequantizable **and** that the prequantization data involves a **trivial** groupoid 2-cocycle, so that the resulting C^*-algebra is **not twisted**.This relates to properties of the Poisson cohomology class $[\pi]$.

Integration

Integrating a Poisson manifold (M, π) means finding a Lie groupoid $\Sigma \rightrightarrows M$ with space of units $M = \Sigma_0$, and a *multiplicative* symplectic form ω_Σ on Σ such that the source (resp. target) is a Poisson (resp. anti Poisson) map.

1. Theoretically *almost* always possible (R.L.Fernandes, M. Crainic);
2. Explicit integration often difficult;
3. Symplectic leaves of M correspond to orbits of Σ;
4. Trivial case (everybody knows what a symplectic groupoid is): $T^*M \to M$, (symplectic groupoid when $\pi_M = 0$, with $s = t = p$ and Liouville symplectic form).

Choice of polarization

Real Lagrangian Multiplicative Polarization - Hawkins 2008

A real LMP on a symplectic groupoid (Σ, ω) is an integrable Lagrangian distribution $\mathcal{L}$ on Σ such that $m^*\mathcal{L}^\perp = (pr_1^* + pr_2^*)\mathcal{L}^\perp$ and $\mathrm{inv}_*(\mathcal{L}) \subseteq \mathcal{L}$.

$$m : \Sigma_2 \to \Sigma; \, m(\gamma, \eta) = \gamma\eta \, , \qquad \mathrm{inv} : \Sigma \to \Sigma; \, \mathrm{inv}(\gamma) = \gamma^{-1}$$

In a different language a real LMP is a wide sub LA-groupoid of $T\Sigma$.

Multiplicativity guarantees that the set of Lagrangian leaves is a quotient groupoid $\Sigma_{\mathcal{L}}$. Real LMP does not always exist. We will allow singularities but such that $\Sigma \to \Sigma_{\mathcal{L}}$ is still a groupoid fibration (Bonechi, —, Qiu, Tarlini (2013)).

Bohr-Sommerfeld condition

When Lagrangian leaves are not simply connected existence of covariantly costant polarization along the leaves is not guaranteed. Connection holonomy along the leaf should be trivial.

Bohr-Sommerfeld

The trivial holonomy condition (under geometrical constraints... BCQT 2013) selects a subgroupoid Σ_{BS}, called the Bohr-Sommerfeld subgroupoid. We associate to it the quotient of Bohr-Sommerfeld Lagrangian leaves $\Sigma^{\mathcal{L}}_{BS}$.

We will take for granted that it is always possible to fix a left Haar system $\{\lambda\}$ on the groupoid $\Sigma^{\mathcal{L}}_{BS}$ and therefore construct a groupoid C^*-algebra $C^*(\Sigma^{\mathcal{L}}_{BS}; \lambda)$.

The quantization of (M, π) is the groupoid C^*-algebra $C^*(\Sigma^{\mathcal{L}}_{BS}; \lambda)$.

Why do I have to choose?

Real multiplicative Lagrangian polarizations are not unique. Different choices of polarization give rise to different subset of Bohr-Sommerfeld leaves. There are no general results granting indipendence from this choice.

In principle it is possible that different polarizations will behave diferently with respect to quantum orbit method. QMO can be seen also as selecting *well behaved* polarizations.

The trivial case

Let M be a (compact) manifold with zero Poisson structure.

- The symplectic groupoid of M is T^*M, with $s = t = p_{T^*M}$ and the Liouville symplectic form.
- The vertical polarization is a real multiplicative Lagrangian polarization of T^*M.
- All leaves are simply connected and therefore there are no BS conditions.
- The resulting C^*-algebra is the C^*-algebra $\mathcal{C}^0(M)$.
- Unitary irreducible representations of $\mathcal{C}^0(M)$ are characters and with Jacobson topology the unitary dual of $\mathcal{C}^0(M)$ is homemorphic to M.

The linear case

Let $\mathfrak{g}$ be a Lie algebra and let $M = \mathfrak{g}^*$ with the linear KKS Poisson bracket.

- The symplectic groupoid of $\mathfrak{g}^*$ is T^*G, with $s = L_*$, $t = R_*$ and the Liouville symplectic form.
- The vertical polarization is a real multiplicative Lagrangian polarization of T^*G.
- All leaves are simply connected and therefore there are no BS conditions.
- The resulting C^*-algebra is the group C^*-algebra $C^*(G)$.
- It can be shown that $C^*(G)$ is a completion of $U(\mathfrak{g})$ (identified with e-supported distributions).
- Under suitable hypothesis $\mathrm{Irrep}*(C^*(G)) \simeq \mathrm{Irrep}_{\mathbb{R}}{}^{\mathrm{alg}} U(\mathfrak{g})$.

The standard symplectic case

Let $M = \mathbb{R}^{2n}$ with the standard symplectic form ω: let $\pi = \omega^{-1}$

- The symplectic groupoid of $\mathbb{R}^{2n}$ is $\mathbb{R}^{4n}$, with pair groupoid structure and standard symplectic form.
- Many possible choices of real multiplicative Lagrangian polarization of $\mathbb{R}^{4n}$: choose a *horizontal* one;
- All leaves are simply connected and therefore there are no BS conditions.
- The resulting C^*-algebra is the C^*-algebra $\mathcal{K}(L^2(\mathbb{R}^{2n}))$ of compact operators.
- Naimark's theorem: the unitary dual of $\mathcal{K}(L^2(\mathbb{R}^{2n}))$ consists of only one point.

Groupoid C^*-algebra

Let Σ be a topological groupoid and let $u \in \Sigma_0$. Let A be an abelian subgroup of the isotropy group Σ_u^u.
Let ρ be a representation of A on $\mathcal{H}_\rho$.
Then there is a well defined induced representation $\mathrm{Ind}(u, A, \rho)$ of $C^*(\Sigma, \lambda)$ on a suitable Hilbert space completion of $\mathcal{C}_c(\Sigma_u^u) \otimes \mathcal{H}_\rho$.
From this you can try to build up a correspondence between irreps of the C^*-algebra and its orbits.

Abelian isotropy - Muhly, Renault, Williams 1996

Let Σ be a 2^{nd}-countable locally compact topological groupoid with abelian isotropy:

- For any $u \in \Sigma_0$ and $\chi \in \widehat{\Sigma_u^u}$ the representation $\mathrm{Ind}(u, \Sigma_u^u, \chi)$ is irreducible;
- If $\gamma \in \Sigma_u$ then there is a unitary equivalence

$$\mathrm{Ind}(u, \Sigma_u^u, \chi) \simeq \mathrm{Ind}(u\gamma, \Sigma_{s(\gamma)}^{s(\gamma)}, \chi \cdot \gamma)$$

- The corresponding map

$$\Psi; \Sigma_0/\Sigma \to \widehat{C^*(\Sigma, \lambda)}$$

  is injective.
- If Σ_0/Σ is T_2 then Ψ is continuous (overly restrictive but...).
- If Σ_0/Σ is T_2 is not even T_0 then the C^*-algebra is not postliminal and therefore

$$\widehat{C^*(\Sigma, \lambda)} \not\simeq \mathrm{Prim}(C^*(\Sigma, \lambda)) .$$

Topologically principal - Sims and Williams 2015

Let Σ be an amenable, etale, Hausdorff groupoid such that for any X closed invariant subspace of Σ_0 then $\Sigma\big|_X$ is topologically principal (i.e. it has trivial isotropy on a dense subset). Then $C^*(\Sigma, \lambda)$ is type I and the induced representation map Ψ establishes an homeomorphism:

$$Q(\Sigma) \to \widehat{C^*(\Sigma, \lambda)}$$

between the quasi-orbit space of Σ (slight topological regularization of the space of orbits) and the space of unitary irreps with its natural (Jacobson) topology.

Back to the torus

- Let $\mathbb{T}^2$ with a right invariant symplectic form θ. The corresponding symplectic groupoid is $T^*\mathbb{T}^2$ as a manifold.
- There is a natural Lagrangian multiplicative polarization by cilnders such that the groupoid of Lagrangian leaves is the action groupoid $\mathbb{R} \ltimes \mathbb{R}$.
- After selecting BS leaves one gets the action groupoid $\mathbb{Z} \ltimes_\theta \mathbb{R}$ where ($\theta \notin \mathbb{Q}$):

$$n \cdot_\theta x = e^{i\theta n} x \, .$$

- The space of orbits has infinitely many points but trivial topology, thus not even T_0.
- $\mathrm{Prim}(C^*(\mathbb{T}^2_\theta)) = \{P\}$ but $\widehat{C^*(\mathbb{T}^2_\theta)}$ has infinitely many elements.

My case study

I will consider the case of covariant Poisson $\mathbb{C}P^n_t$; when $t = 0, 1$ it is called standard while when $t \in]0, 1[$ non standard.

- Poisson quotient of standard Poisson-Lie $SU(n)$.
- In the standard case QOM is known to hold (Stokman 1995, Nevsheyev-Tuset 2013).
- Groupoid quantization can be explicitely determined (Bonechi, –, Qiu, Tarlini CMP 2013).

The standard case

- Symplectic foliation: one cell in each even dimension. Topology of the space of leaves:

$$\{\emptyset, S_1, \{S_1, S_2\}, \{S_1, S_2, S_3\}, \ldots, \{S_1, \ldots, S_{n+1}\}\}$$

- *Singular* multiplicative Lagrangian polarization + non trivial BS conditions (BCQT 2013).
- The groupoid satisfies Sims-Williams hypothesis; this implies QOM holds true (–, Rend. Sem. Mat. PoliTo 2016).

The non standard case

- Coisotropic quotient of standard Poisson-Lie $SU(n)$.
- Neither Stokman or Nevshveyev-Tuset results apply.
- Groupoid quantization and singular MLP in (BCQT 2013).
- BS groupoid is amenable, etale, topologically principal and with abelian isotropy (postliminal C^*-algebra).
- Have to take into account non trivial isotropy corresponding to $\mathbb{S}^1$-families of symplectic cells ($\mathbb{T}$–leaves).

Thank You