Quantization of Hamiltonian (co)actions via twist

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Overview

1. Preliminaries
2. Hamiltonian actions compatible with twists
3. Quantum Hamiltonian (co)actions

Based on joint work with Bieliavsky and Nest [arXiv:1804.06160]

Preliminaries

Drinfel'd twist: philosophy

Let us consider a symmetry

$$G \times \mathbb{A} \to \mathbb{A}$$

Idea New structure F (twist) on G that can be used to produce deformations $\mathbb{A}_F$ via action of $\mathbb{A}$. Does $\mathbb{A}_F$ admit a natural symmetry? Yes, but not G.

Motivation To solve equations of motion. How? Find first integrals of motion (symmetries). Ok, done. And now, how can we build new ones? Perturbing them means changing $\mathbb{A}$, which implies new symmetries. Then hard to solve... but using Drinfel'd approach we know them.

Drinfel'd twist

Let $\mathfrak{g}$ be a Lie algebra and consider the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, equipped with the standard Hopf algebra structures Δ and ϵ.

Definition (Twist)

An element $\mathcal{F} \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$ is a twist if

- $\mathcal{F} = 1 \otimes 1 + \sum_{k=1}^{\infty} t^k \mathcal{F}_k$
- $(\mathcal{F} \otimes 1)(\Delta \otimes 1)(\mathcal{F}) = (1 \otimes \mathcal{F})(1 \otimes \Delta)(\mathcal{F})$
- $(\epsilon \otimes 1)\mathcal{F} = (1 \otimes \epsilon)\mathcal{F} = 1$

Equivalent data: 2-cocycle on $\mathscr{C}^\infty(G)$.

Drinfel'd twist properties

- Given a twist F, the antisymmetric part of its first order is an r-matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$.
- The converse is the hardest part, proved by Drinfel'd: let $r \in \mathfrak{g} \wedge \mathfrak{g}$ be a r-matrix. Then there exist a twist $\mathcal{F} \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$ whose semiclassical limit is r.
- $\Delta_{\mathcal{F}} = \mathcal{F}\Delta\mathcal{F}^{-1}$ defines a deformed Hopf algebra $\mathcal{U}_{\mathcal{F}}(\mathfrak{g})$

E., Schnitzer, Waldmann: *A Universal Construction of Universal Deformation Formulas, Drinfel'd Twists and their Positivity*. Pacific J. Math. 291 (2017)

Universal deformation formula

Given a Lie algebra action of $\varphi : \mathfrak{g} \longrightarrow \Gamma^\infty(TM)$, the fundamental vector fields $\varphi(\xi)$ determine a representation of $\mathfrak{g}$ on $\mathscr{C}^\infty(M)$ by derivations which extend to a Hopf algebra action

$$\triangleright : \mathscr{U}(\mathfrak{g}) \otimes \mathscr{C}^\infty(M) \longrightarrow \mathscr{C}^\infty(M)$$

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Thus

$$f \star g := m(F \triangleright (f \otimes g))$$

Question

Is UDF compatible with Hamiltonian actions?

Hamiltonian actions compatible with twists

Dressing action

Let us consider a Poisson Lie group G and denote by G^* and D the corresponding dual and double, respectively. Consider $g \in G$, $u \in G^*$ and let $ug \in D$ be their product.

Then, there exist elements ${}^u g \in G$ and $u^g \in G^*$ such that

$$ug = {}^u g u^g.$$

Hence, the action of $g \in G$ on $u \in G^*$ is given by

$$(u, g) \mapsto (ug)^*_G$$

where $(ug)^*_G$ denotes the G^*-factor of $ug \in D$. This defines a left action of G on G^*, called dressing action.

Definition: Dressing generators

The map $\alpha : \mathfrak{g} \to \Omega^1(G^*) : X \mapsto \alpha_X$ is said dressing generator with respect to the Poisson structure π on G^* if the fundamental vector field ℓ_X of the dressing action can be written as

$$\ell_X = \pi^\sharp(\alpha_X)$$

and satisfies

$$\alpha_{[X,Y]} = [\alpha_X, \alpha_Y]_{\pi_\ell} \quad \text{and} \quad \mathrm{d}\alpha_X = \alpha \wedge \alpha \circ \delta(X).$$

- If π is the standard Poisson dual structure α_X are just left-invariant one forms!

Definition: Hamiltonian action

Let $\Phi : G \times M \to M$ be an action of (G, π_G) on (M, π) and α_X the dressing generator wrt a Poisson structure π_{G^*} on G^*.

- A momentum map for Φ is a map $J : M \to G^*$ such that

$$\varphi(X) = \pi^\sharp(J^*(\alpha_X)),$$

  where $\varphi(X)$ is the fundamental vector field of Φ.
- A map $J : M \to G^*$ is said to be ℓ-equivariant if it intertwines the fundamental vector field $\varphi(X)$ and the dressing action ℓ_X for any X.

Quantum Hamiltonian (co)actions

We aim to obtain the analogue of the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & \Gamma^\infty(TM) \\
\Gamma^\infty(T^*G^*) \arrow[r, "J^*"'] & \Gamma^\infty(T^*M) \arrow[u, "\pi_M^\sharp"']
\end{tikzcd}$$

in terms of Hopf algebra actions (or equivalently, coactions).

Theorem

Let us consider a Hamiltonian action φ. Then

- The pullback $J^* : \mathscr{C}^\infty(G^*) \to \mathscr{C}^\infty(M)$ defines a $\mathscr{U}(\mathfrak{g})$-module algebra morphism.
- The pullback $J^* : \mathscr{C}^\infty(G^*) \to \mathscr{C}^\infty(M)$ of the momentum map J defines a $\mathscr{C}^\infty(G)$-comodule algebra morphism.

Proof (Sketch)

- The dressing action $\ell : \mathfrak{g} \to \Gamma^\infty(TG^*)$ is equivalent to a Hopf algebra action $\Lambda : \mathscr{U}(\mathfrak{g}) \times \mathscr{C}^\infty(G^*) \to \mathscr{C}^\infty(G^*)$ by
$$\Lambda(X, f) := \mathscr{L}_{\ell_X} f.$$
- Given a dressing generator $\alpha : \mathfrak{g} \to \Omega^1(G^*)$, the corresponding map given by $\mathscr{U}(\mathfrak{g}) \times \mathscr{C}^\infty(G^*) \to \mathscr{C}^\infty(G^*) : (X, f) \mapsto \mathscr{L}_{\alpha_X} f$ is a Hopf algebra action and
$$\Lambda(X, f) = \mathscr{L}_{\alpha_X} f.$$
- We can extend J to a map J^* acting on $\mathscr{L}_\alpha$ by $J^*\mathscr{L}_\alpha := \mathscr{L}_{J^*\alpha} \circ J^*$ and we get
$$\Phi(X, J^* f) = J^*(\Lambda(X, f)).$$

Definition: Hamiltonian (co)action

- A Hopf algebra action $\Phi : \mathcal{U}(\mathfrak{g}) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ is Hamiltonian if there exist an algebra morphism, called momentum map, $\mathbf{J} : \mathscr{C}^\infty(G^*) \to \mathscr{C}^\infty(M)$ which satisfies the condition
$$\Phi(X, \mathbf{J}f) = \mathbf{J}(\Lambda(X, f)).$$
- A Hopf algebra coaction $\delta_\Phi : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M) \otimes \mathscr{C}^\infty(G)$ is said Hamiltonian if there exist $\mathscr{C}^\infty(G)$-comodule algebra morphism $\mathbf{J}$ which intertwines it with the Hopf algebra coaction δ_Λ corresponding to the dressing action.

Quantized structures

Let us consider a twist $\mathcal{F}$ on $\mathcal{U}(\mathfrak{g})$ (or a 2-cocycle γ on $\mathscr{C}^\infty(G)$). Then we have:

- $\mathscr{C}^\infty_\hbar(G^*)$ is the deformed algebra given by the pair $(\mathscr{C}^\infty(G^*)[[\hbar]], \star_\ell)$, where $\star_\ell$ is the star product induced by ℓ via UDF.
- $\mathscr{C}^\infty_\hbar(M)$ is the deformed algebra given by the pair $(\mathscr{C}^\infty(M)[[\hbar]], \star_\varphi)$, where $\star_\varphi$ is the star product induced by ℓ via UDF.

Theorem: quantization

Let $\varphi : \mathfrak{g} \to \Gamma^\infty(TM)$ be an Hamiltonian action with momentum map $J : M \to G^*$. Then the corresponding quantum group action $\mathscr{U}_{\mathcal{F}}(\mathfrak{g}) \times \mathscr{C}^\infty_\hbar(M) \to \mathscr{C}^\infty_\hbar(M)$ is Hamiltonian with quantum momentum map $J^* : \mathscr{C}^\infty_\hbar(G^*) \to \mathscr{C}^\infty_\hbar(M)$. In other words, we have:

$$\Phi(X, J^* f) = J^*(\Lambda(X, f)).$$

and

$$J^*(f \star_\ell g) = J^* f \star_\varphi J^* g.$$

Outlook

- UDF and Hamiltonian are compatible, we obtained a quantum momentum map.
- What if the star product is not given by UDF? What is a quantum momentum map?
- Reduction theory
- strict DQ

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