László Fehér, Szeged and Budapest, Hungary

Global action-angle map and duality for a Poisson-Lie deformation of the BC_n Sutherland system

The talk is based on joint works with T.F. Görbe and I. Marshall.

Consider two *Liouville integrable* Hamiltonian systems (M, ω, H) and $(\widehat{M}, \hat{\omega}, \hat{H})$. These systems are said to be **in action-angle duality** if there exist Darboux coordinates q_i, p_i on (dense open subset of) M and Darboux coordinates $\hat{p}_k, \hat{q}_k$ on (dense open subset of) $\widehat{M}$ and a global symplectomorphism $\mathcal{R} : M \to \widehat{M}$ such that

$H \circ \mathcal{R}^{-1}$ depends only on $\hat{p}$ (action variables for H) and

$\widehat{H} \circ \mathcal{R}$ depends only on q (action variables for $\widehat{H}$).

This is non-trivial if both systems are themselves interesting ones.

A special case of duality is self-duality, where the leading Hamiltonians of the two systems have the same form.

It is a particularly interesting relation if both are many-body systems (interacting points) in such a way that

the q_i describe particle positions for $H(q,p)$ and

the $\hat{p}_i$ describe particle positions for $\widehat{H}(\hat{p},\hat{q})$.

• We study integrable many-body systems of Toda and Calogero (Sutherland, Moser, Olshanetsky-Perelomov, Ruijsenaars-Schneider) type, which describe "particles" moving on the line or on the circle.

• It was discovered by Ruijsenaars (1988-95) in his direct construction of action-angle variables for Calogero and Toda type systems that these systems enjoy duality relations.

• My research goal is to understand all known dualities in group theoretic terms and derive new ones.

• **In this talk** I shall focus on some many-body system associated with the BC_n root system.

Duality from Hamiltonian reduction: the basic idea

Goes back to Kazhdan-Kostant-Sternberg [1978], Ruijsenaars [1988]
Fock-Gorsky-Nekrasov-Roubtsov [2000] . . .

Start with a 'big phase space' $\mathcal{M}$ equipped with *two* Abelian Poisson algebras $\mathfrak{H}^1$ and $\mathfrak{H}^2$ generated by two families of 'free' Hamiltonians.

Apply some suitable reduction to the big phase space and construct *two* 'natural' models, M and $\widehat{M}$, of the *single* reduced phase space.

The two families of 'free' Hamiltonians turn into commuting **many-body Hamiltonians** and **particle positions** in terms of *both* models. Their rôle is *interchanged* in the two models.

The natural symplectomorphism between the two models of the reduced phase space yields the 'duality symplectomorphism'.

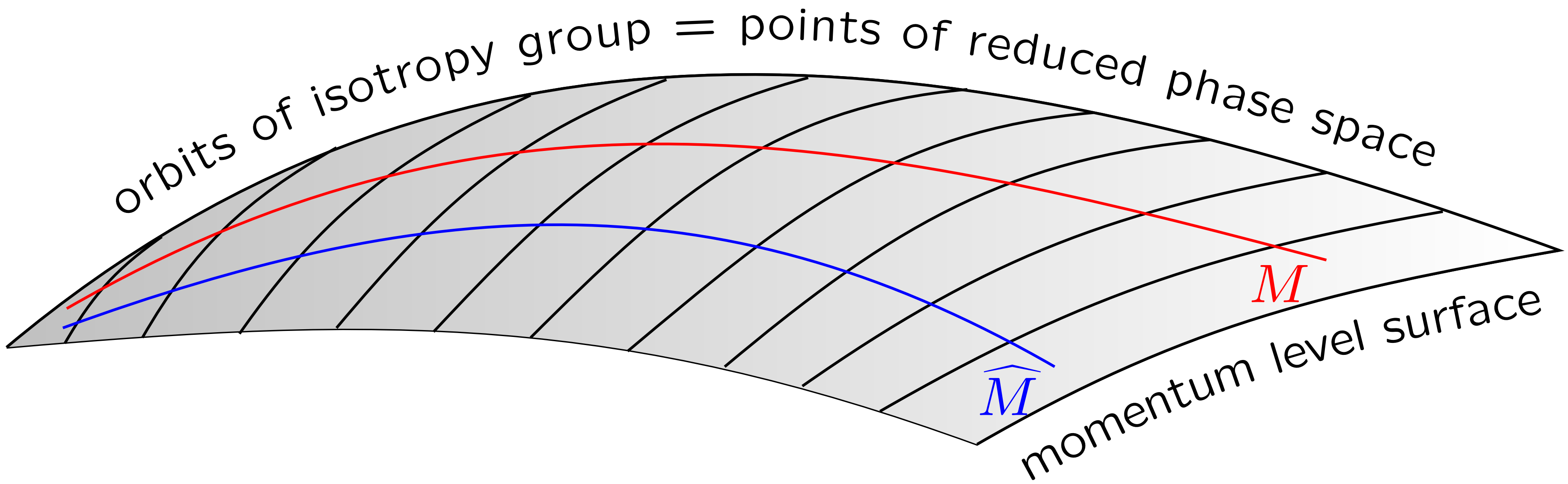

Figure. The geometry of reduction and action-angle duality.

Key question: What to reduce and how? The big phase space, the free Hamiltonians, the symmetry and constraints defining the reduction must be found by 'inspiration'.Then the work may start.

$$\begin{tikzcd}
& \mathcal{M}_0 \arrow[r, hook, "\iota_0"] \arrow[dl, "\psi"'] \arrow[d, "\pi_0"] \arrow[dr, "\hat{\psi}"] & \mathcal{M} \\
M \arrow[d, "q"] \arrow[rr, bend right, "\mathcal{R}"'] & \mathcal{M}_{\mathrm{red}} \arrow[l, "\Psi"'] \arrow[r, "\hat{\Psi}"] & \hat{M} \arrow[d, "\hat{p}"] \\
\mathbb{R}^n & & \mathbb{R}^n
\end{tikzcd}$$

$$\begin{tikzcd}
& \iota_0^*(\mathfrak{H}^1) \times \iota_0^*(\mathfrak{H}^2) & \\
\mathfrak{H} \times \mathfrak{Q} \arrow[ur, "\psi^*"] \arrow[r, "\Psi^*"] & \mathfrak{H}^1_{\mathrm{red}} \times \mathfrak{H}^2_{\mathrm{red}} \arrow[u, "\pi_0^*"'] & \hat{\mathfrak{P}} \times \hat{\mathfrak{H}} \arrow[l, "\hat{\Psi}^*"'] \arrow[ul, "\hat{\psi}^*"'] \arrow[ll, bend left, "\mathcal{R}^*"]
\end{tikzcd}$$

Starting with the Abelian algebras $\mathfrak{H}^1$ and $\mathfrak{H}^2$ on the master phase space $\mathcal{M}$, the reduced Abelian algebras are defined by $\mathfrak{H}^i_{\mathrm{red}} \circ \pi_0 = \mathfrak{H}^i \circ \iota_0$ for $i = 1, 2$. They turn into the Abelian algebras of the models M and $\hat{M}$ according to $\mathfrak{H} \circ \Psi = \mathfrak{H}^1_{\mathrm{red}} = \hat{\mathfrak{P}} \circ \hat{\Psi}$ and $\mathfrak{Q} \circ \Psi = \mathfrak{H}^2_{\mathrm{red}} = \hat{\mathfrak{H}} \circ \hat{\Psi}$.

The simplest self-dual system: $\quad H_{\mathrm{Cal}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k}\frac{\mu^2}{(q_k-q_j)^2}$

Symplectic reduction: Consider phase space $T^*\mathrm{i}u(n) \simeq \mathrm{i}u(n)\times \mathrm{i}u(n) := \{(Q,P)\}$ with two families of 'free' Hamiltonians $\{\mathrm{tr}\,(Q^k)\}$ and $\{\mathrm{tr}\,(P^k)\}$. Reduce by the adjoint action of $U(n)$ using the moment map constraint

$$[Q,P] = C(\mu) := \mathrm{i}\mu\sum_{j\neq k} E_{j,k}$$

This yields the rational Calogero system (OP [76], KKS [78]):

gauge slice (i): $Q = q := \mathrm{diag}(q_1,\dots,q_n)$, $\quad q_1 > \dots > q_n$, with $p := \mathrm{diag}(p_1,\dots,p_n)$

$$P = p + \mathrm{i}\mu\sum_{j\neq k}\frac{E_{jk}}{q_j-q_k} \equiv L_{\mathrm{Cal}}(q,p) \quad \text{Lax matrix,} \quad \mathrm{tr}\,(dP\wedge dQ) = \sum_{k=1}^{n} dp_k\wedge dq_k$$

gauge slice (ii): $P = \hat{p} := \mathrm{diag}(\hat{p}_1,\dots,\hat{p}_n)$, $\quad \hat{p}_1 > \dots > \hat{p}_n$, with $\hat{q} := \mathrm{diag}(\hat{q}_1,\dots,\hat{q}_n)$

$$Q = -L_{\mathrm{Cal}}(\hat{p},\hat{q}) \quad \text{dual Lax matrix,} \quad \mathrm{tr}\,(dP\wedge dQ) = \sum_{k=1}^{n} d\hat{q}_k\wedge d\hat{p}_k.$$

The alternative gauge slices give two models of the reduced phase space. Their natural symplectomorphism is the self-duality map.

Some examples of dual pairs due to Ruijsenaars [1995]

The trigonometric Sutherland system and its Ruijsenaars dual:

$$H_{\text{trigo}-\text{Suth}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{\mu^2}{2}\sum_{j\neq k}\frac{1}{\sin^2(q_k - q_j)}$$

$$\widehat{H}_{\text{rat}-\text{RS}} = \sum_{k=1}^{n}(\cos\hat{q}_k)\prod_{j\neq k}\left[1 - \frac{\mu^2}{(\hat{p}_k - \hat{p}_j)^2}\right]^{\frac{1}{2}}$$

Derived by reduction of $T^*U(n)$ [Kazhdan-Kostant-Sternberg 1978, Ayadi-LF 2010].

'Relativistic' deformation of the above dual pair:

$$H_{\text{trigo}-\text{RS}} = \sum_{k=1}^{n}(\cosh p_k)\prod_{j\neq k}\left[1 + \frac{\sinh^2\mu}{\sin^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

$$\widehat{H}_{\text{trigo}-\text{RS}} = \sum_{k=1}^{n}(\cos\hat{q}_k)\prod_{j\neq k}\left[1 - \frac{\sinh^2\mu}{\sinh^2(\hat{p}_k - \hat{p}_j)}\right]^{\frac{1}{2}}$$

Derived by reduction of the Heisenberg double of Poisson $U(n)$ [LF-Klimcik 2011].

These many-body systems are associated with the A_{n-1} root system.

A dual pair associated with the BC_n root system

The trigonometric BC_n Sutherland system

$$H(q,p) = \frac{1}{2}\sum_{j=1}^n p_j^2 + \sum_{1\leq j<k\leq n}\left(\frac{\gamma}{\sin^2(q_j-q_k)} + \frac{\gamma}{\sin^2(q_j+q_k)}\right) + \sum_{j=1}^n \frac{\gamma_1}{\sin^2(q_j)} + \sum_{j=1}^n \frac{\gamma_2}{\sin^2(2q_j)}$$

is dual to the (completed) rational Ruijsenaars-Schneider-van Diejen system

$$\widehat{H}(\lambda,\theta) = \sum_{j=1}^n \cos(\theta_j)\left[1-\frac{\nu^2}{\lambda_j^2}\right]^{\frac{1}{2}}\left[1-\frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}} \prod_{\substack{k=1\\(k\neq j)}}^n \left[1-\frac{\mu^2}{(\lambda_j-\lambda_k)^2}\right]^{\frac{1}{2}}\left[1-\frac{\mu^2}{(\lambda_j+\lambda_k)^2}\right]^{\frac{1}{2}}$$

$$-\frac{\nu\kappa}{\mu^2}\prod_{j=1}^n\left[1-\frac{\mu^2}{\lambda_j^2}\right] + \frac{\nu\kappa}{\mu^2}.$$

Coupling constants are subject to $\gamma>0$, $\gamma_2>0$, $4\gamma_1+\gamma_2>0$, and $\mu>0$, $|\nu|\neq|\kappa|\neq 0$. Duality holds under the relation $\gamma=\mu^2$, $\gamma_1=\frac{\nu\kappa}{2}$, $\gamma_2=\frac{(\nu-\kappa)^2}{2}$. (Sorry: $(\lambda,\theta)\equiv(\hat{p},\hat{q})$.)

The Sutherland positions q live in the open polytope (Weyl alcove)

$$\mathcal{D}_1 = \{q\in\mathbb{R}^n \mid \frac{\pi}{2} > q_1 > \cdots > q_n > 0\}$$

and the Sutherland actions λ (dual positions) fill the polyhedron

$$\mathcal{D}_2 = \{\lambda\in\mathbb{R}^n \mid \lambda_a-\lambda_{a+1}\geq\mu\ (a=1,\dots,n-1),\ \lambda_n\geq\max(|\nu|,|\kappa|)\}.$$

The Liouville tori collapse at the boundary of $\mathcal{D}_2$. The above description of dual system is valid on a dense open submanifold parametrized by $\mathcal{D}_2^o\times\mathbb{T}^n$.

Where does it come from and what is it good for?

Reduce the master phase space $T^*SU(2n) = \{(k,J) \mid k \in SU(2n),\, J \in su(2n)\}$ using the symmetry group $K_+ \times K_+$, where $K_+ = S(U(n) \times U(n))$ is the block-diagonal subgroup of $K = SU(2n)$.

Two Abelian algebras of free Hamiltonians are generated by

$$\mathcal{H}_a(k,J) = \frac{1}{2}\mathrm{tr}\,(\mathrm{i}J)^{2a} \quad \text{and} \quad \widehat{\mathcal{H}}_a(k,J) := \frac{1}{2}\mathrm{tr}\,(k^\dagger I k I)^a \quad \text{with} \quad I := \mathrm{diag}(\mathbf{1}_n, -\mathbf{1}_n).$$

The moment map values for the actions of K_+ generated by left- and respectively by right-multiplications are fixed to be

$$\mathrm{diag}(C(\mu), \mathbf{0}_n) + \mathrm{i}(\mu - \nu)I \quad \text{and} \quad -\mathrm{i}\kappa I.$$

Here, the matrix $C(\mu) \in u(n)$ is given by $C(\mu)_{lm} = \mathrm{i}\mu(\delta_{lm} - 1)$.

The free Hamiltonians $\mathcal{H}_a$ reduce to the Sutherland Hamiltonians, which in their action-angle variables λ, θ become

$$\mathcal{H}_a^{\mathrm{red}}(\lambda) = \sum_{j=1}^{n} (\lambda_j)^{2a}, \qquad a = 1, \dots, n.$$

The free Hamiltonians $\widehat{\mathcal{H}}_a$ reduce to the RSvD Hamiltonians, which in their action-angle variables q, p become

$$\widehat{\mathcal{H}}_a^{\mathrm{red}}(q) = \sum_{j=1}^{n} \cos(2aq_j), \quad a = 1, \dots, n.$$

• Each Sutherland Hamiltonian $\mathcal{H}_a^{\rm red}$ has a unique global minimum, corresponding to the 'special vertex' of the action polyhedron $\mathcal{D}_2$. Each of them is non-degenerate (the commutant is Abelian).

• Each RSvD Hamiltonian $\widehat{\mathcal{H}}_a^{\rm red}$ is maximally superintegrable, since it admits the following conserved quantities:

$$f_i(q,p) := \sum_{j=1}^{n} p_j Y_{j,i}(q), \qquad i \in \{1,\dots,n\} \setminus \{a\},$$

where $Y(q)$ is the inverse of the non-degenerate matrix $X_{i,j}(q) := \frac{\partial \widehat{\mathcal{H}}_i^{\rm red}(q)}{\partial q_j}$, $\forall q \in \mathcal{D}_1$.

This example was analyzed in the following paper:

L.F. and T.F. Görbe, Duality between the trigonometric BC(n) Sutherland system and a completed rational Ruijsenaars-Schneider-van Diejen system, Journ. Math. Phys. 55, 102704 (2014)

Next, I describe a generalization based on replacing the master system defined on the cotangent bundle $T^*SU(2n)$ by its natural Poisson-Lie analogue.

A new dual pair from a reduction of a Heisenberg double

For the big phase space, take the standard Poisson-Lie analogue of $T^*SU(2n)$. This is the (symplectic) Heisenberg double of Poisson $SU(2n)$, which as a manifold is the real Lie group $\mathcal{M} := SL(2n,\mathbb{C})$.

• Every $g \in \mathcal{M}$ admits the alternative Iwasawa decompositions

$$g = k_L b_R = b_L k_R, \qquad k_L, k_R \in K, \quad b_L, b_R \in B,$$

where $K := SU(2n)$ and $B := B(2n)$ consists of upper triangular matrices with positive diagonal. Using these, $\mathcal{M}$ is equipped with the Alekseev-Malkin symplectic form

$$\omega_{\mathcal{M}} = \frac{1}{2}\Im\mathrm{tr}\,(db_L b_L^{-1} \wedge dk_L k_L^{-1}) + \frac{1}{2}\Im\mathrm{tr}\,(b_R^{-1}db_R \wedge k_R^{-1}dk_R).$$

• The smooth functions depending only on b_L, or only on b_R, form two mutually commuting Poisson algebras, and similarly for k_L and k_R. These are (up to signs) the Poisson algebras of the standard Poisson groups K and B in duality.

The 'master system' $(\mathcal{M}, \omega_{\mathcal{M}}, \{\mathcal{H}_a\}, \{\widehat{\mathcal{H}}_a\})$ is now defined as follows. Using the Iwasawa decomposition of $g \in \mathcal{M} = SL(2n,\mathbb{C})$, written as $g = kb$, we introduce the 'unreduced Lax matrices'

$$\Omega(g) := bb^\dagger \quad \text{and} \quad L(g) := k^\dagger I k I, \quad \text{with} \quad I := \mathrm{diag}(\mathbf{1}_n, -\mathbf{1}_n).$$

Two Abelian Poisson algebras are generated by the Hamiltonians

$$\mathcal{H}_a(g) := \frac{1}{2}\mathrm{tr}\,\Omega(g)^a \quad \text{and} \quad \widehat{\mathcal{H}}_a(g) := \frac{1}{2}\mathrm{tr}\,L(g)^a, \qquad a = 1,2,\dots$$

They are invariant with respect to the action of the symmetry group $K_+ \times K_+$ defined by

$$K_+ \times K_+ \times \mathcal{M} \ni (\eta_L, \eta_R, g) \mapsto \eta_L g \eta_R^{-1} \in \mathcal{M}, \qquad K_+ = \{k \in K \mid IkI = k\}.$$

$K_+ < K$ is a Poisson subgroup and the action of $K_+ \times K_+$ is a Poisson action generated by the following moment map (in Lu's sense)

$$\mathcal{M} \in g \mapsto (\pi_N(b_L), \pi_N(b_R^{-1})) \in B/N \times B/N,$$

where $\pi_N : B \to B/N$ is the projection associated with the normal subgroup $N < B = B(2n)$ of elements having $\mathbf{1}_n$ as diagonal blocks.

The constraints and the key spectral invariants

Inspired by experience, we consider the moment map 'constraint surface'

$$\mathcal{M}_0 := \left\{ g \in \mathcal{M} \;\middle|\; b_R := b = \begin{pmatrix} e^{-v}\mathbf{1}_n & * \\ 0 & e^{v}\mathbf{1}_n \end{pmatrix},\; b_L = \begin{pmatrix} e^{u}\sigma & * \\ 0 & e^{-u}\mathbf{1}_n \end{pmatrix} \right\},$$

where $\sigma := \sigma(\mu) \in B(n)$ satisfies $\sigma\sigma^\dagger = e^{-2\mu}\mathbf{1}_n + \hat{v}\hat{v}^\dagger$ with fixed $\hat{v} \in \mathbb{C}^n$ verifying $|\hat{v}|^2 = e^{-2\mu}(e^{2n\mu} - 1)$. Here, u, v and $\mu > 0$ are real constants ($|u| \neq |v|$).

Our task is to construct two suitable models of the reduced phase space

$$\mathcal{M}_{\text{red}} = \mathcal{M}_0/(K_+(\sigma) \times K_+) \quad \text{where} \quad K_+(\sigma) = \{\eta \in K_+ \mid \eta\sigma\sigma^\dagger\eta^{-1} = \sigma\sigma^\dagger\}.$$

$\mathcal{M}_0$ is a principal bundle over $\mathcal{M}_{\text{red}}$. It inherits a symplectic structure and two reduced Abelian Poisson algebras.

For any $g = kb \in \mathcal{M}_0$, $L(g) = k^\dagger I k I$ and $\Omega(g) = bb^\dagger$ are conjugate to unique diagonal matrices of the following form:

$$L(g) \sim \text{diag}(e^{2\mathrm{i}q_1}, \ldots, e^{2\mathrm{i}q_n}, e^{-2\mathrm{i}q_1}, \ldots, e^{-2\mathrm{i}q_n}) \quad \text{with} \quad \frac{\pi}{2} \geq q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$$

and

$$\Omega(g) \sim \text{diag}(e^{2\lambda_1}, \ldots, e^{2\lambda_n}, e^{-2\lambda_1}, \ldots, e^{-2\lambda_n}) \quad \text{with} \quad \lambda_1 \geq \lambda_2 \geq \ldots \lambda_n \geq |v|.$$

The respective 'spectral invariants' q_i and λ_i descend to functions on $\mathcal{M}_{\text{red}}$. Naturally, they (or their suitable functions) give rise to action variables. A crucial problem is to find their range of values.

Darboux coordinates and reduced Hamiltonians: Act I

We proved that the domain of the λ-variables is

$$\mathcal{D}_\lambda = \left\{ \lambda \in \mathbb{R}^n \mid \lambda_i - \lambda_{i+1} \geq \mu \ (i = 1, \ldots, n-1),\ \lambda_n \geq \max(|v|, |u|) \right\}.$$

We can parametrize a dense open subset $\mathcal{M}^0_{\mathrm{red}} \subset \mathcal{M}_{\mathrm{red}}$ by Darboux coordinates λ_i, θ_j varying in $\mathcal{D}^0_\lambda \times \mathbb{T}^n = \{(\lambda, e^{\mathrm{i}\theta}\}$, where $\mathcal{D}^0_\lambda \subset \mathcal{D}_\lambda$ is the interior. In these coordinates $\widehat{\mathcal{H}}^{\mathrm{red}}_1$ becomes the RSvD type Hamiltonian

$$e^{u-v}\, \widehat{\mathcal{H}}^{\mathrm{red}}_1(\lambda, \theta) = V(\lambda) + \sum_{k=1}^n \frac{\cos\theta_k}{\cosh^2\lambda_k} \left[1 - \frac{\sinh^2 v}{\sinh^2\lambda_k}\right]^{1/2} \left[1 - \frac{\sinh^2 u}{\sinh^2\lambda_k}\right]^{1/2}$$

$$\times \prod_{\substack{l=1 \\ (l\neq k)}}^n \left[1 - \frac{\sinh^2\mu}{\sinh^2(\lambda_k - \lambda_l)}\right]^{1/2} \left[1 - \frac{\sinh^2\mu}{\sinh^2(\lambda_k + \lambda_l)}\right]^{1/2} \quad \text{with}$$

$$V(\lambda) = \frac{\sinh(v)\sinh(u)}{\sinh^2\mu} \prod_{k=1}^n \left[1 - \frac{\sinh^2\mu}{\sinh^2\lambda_k}\right] - \frac{\cosh(v)\cosh(u)}{\sinh^2\mu} \prod_{k=1}^n \left[1 + \frac{\sinh^2\mu}{\cosh^2\lambda_k}\right] + C.$$

We also have the reduced Lax matrix $L_{\mathrm{red}}(\lambda, \theta)$ generating $\widehat{\mathcal{H}}^{\mathrm{red}}_j$ for $j = 1, \ldots, n$.

Even globally on $\mathcal{M}_{\mathrm{red}}$, the other family $\mathcal{H}^{\mathrm{red}}_j$ of reduced Hamiltonians reads

$$\mathcal{H}^{\mathrm{red}}_j = \sum_{i=1}^n \cosh(2j\lambda_i).$$

As was promised, the λ_i play the double role of positions and actions.

Darboux coordinates and reduced Hamiltonians: Act II

The action variables (2π-periodic flows) corresponding to eigenvalues of $L(g)$ are

$$x_i := \log \sin q_i \qquad \text{and their domain is proved to be}$$

$$\mathcal{D}_x = \big\{ x \in \mathbb{R}^n \mid x_1 \leq s := \min(0, v-u),\ \ x_j - x_{j+1} \geq \mu \ (j = 1, \ldots, n-1) \big\}.$$

The pair $(x, e^{\mathrm{i}y}) \in \mathcal{D}_x^0 \times \mathbb{T}^n$ gives Darboux coordinates on a dense open subset $\mathcal{M}'_{\text{red}} \subset \mathcal{M}_{\text{red}}$. In these coordinates $\mathcal{H}_1^{\text{red}}$ becomes the RSvD type Hamiltonian

$$\mathcal{H}_1^{\text{red}}(x,y) = U(x) - \sum_{j=1}^{n} \cos(y_j) U_1(x_j)^{\frac{1}{2}} \prod_{\substack{k=1 \\ (k \neq j)}}^{n} \left[1 - \frac{\sinh^2(\mu)}{\sinh^2(x_j - x_k)} \right]^{\frac{1}{2}}$$

$$U(x) = \frac{e^{-2u} + e^{2v}}{2} \sum_{j=1}^{n} e^{-2x_j}, \quad U_1(x_j) = \big[1 - (1 + e^{2(v-u)}) e^{-2x_j} + e^{2(v-u)} e^{-4x_j} \big].$$

We also have the reduced Lax matrix $\Omega_{\text{red}}(x,y)$ generating $\mathcal{H}_j^{\text{red}}$ for $j = 1, \ldots, n$.

The other family $\widehat{\mathcal{H}}_j^{\text{red}}$ of reduced Hamiltonians takes the following form:

$$\widehat{\mathcal{H}}_j^{\text{red}} = \sum_{i=1}^{n} \cos(2jq_i), \qquad (\cos(2jq_i) \text{ is a polynomial in } \sin q_i = e^{x_i}).$$

Therefore the x_i also play the double role of positions and actions.

Consequence: Each reduced Hamiltonian $\widehat{\mathcal{H}}_j^{\text{red}}$ and $\mathcal{H}_j^{\text{red}}$ is non-degenerate and possesses a unique equilibrium point (which is shared by its family).

Sketch of the derivation of $\mathcal{D}_\lambda$ and the model $\mathcal{D}_\lambda^0 \times \mathbb{T}^n$ of $\mathcal{M}_{\rm red}^0$

By standard algebra, each element of $\mathcal{M}_0$ can be transformed to $g = kb$ for which

$$b = \begin{pmatrix} e^{-v}\mathbf{1}_n & \beta \\ 0 & e^{v}\mathbf{1}_n \end{pmatrix} \quad \text{with } \beta = \mathrm{diag}(\beta_1, \dots, \beta_n), \quad \beta_1 \geq \dots \geq \beta_n \geq 0.$$

We can easily diagonalize $bb^\dagger$ and write it as

$$bb^\dagger = \rho(\lambda)\mathrm{diag}(e^{2\lambda_1}, \dots, e^{2\lambda_n}, e^{-2\lambda_1}, \dots, e^{-2\lambda_n})\rho(\lambda)^\dagger,$$

where, on a dense open subset, $\lambda_1 > \dots > \lambda_n > |v|$.

The $\mathbb{C}^{2n}$ vector $\tilde{w} := \rho(\lambda)^\dagger k^\dagger \hat{w}$, with $\hat{w} = (\hat{v}, 0, \dots, 0)^T$, turns out to transform by the residual gauge transformations as

$$\tilde{w} \mapsto \mathrm{diag}(\tau_1, \dots, \tau_n, \tau_1, \dots, \tau_n)\tilde{w}, \qquad \tau_i \in U(1), \quad i = 1, \dots, n.$$

Therefore, $|\tilde{w}_a|$ ($a = 1, \dots, 2n$) is gauge invariant (a function on $\mathcal{M}_{\rm red}^0$), and we showed that it depends only on the spectral invariants λ. Moreover, we were able to compute the functions

$$\mathcal{F}_a(\lambda) = |\tilde{w}_a|^2.$$

These functions must be positive on a dense domain, and there we obtain the complementary invariants

$$\tilde{w}_j^* \tilde{w}_{j+n} = |\tilde{w}_j^* \tilde{w}_{j+n}| e^{\mathrm{i}\theta_j}, \quad j = 1, \dots, n.$$

The interior of $\mathcal{D}_\lambda$ was found as the domain where $\mathcal{F}_a(\lambda) > 0$ for all a.

The full domain $\mathcal{D}_\lambda$ was then determined by invoking a density argument. The invariants λ_j, $e^{\mathrm{i}\theta_j}$ parametrize a dense open submanifold $\mathcal{M}_{\rm red}^0 \subset \mathcal{M}_{\rm red}$.

Concretely, the open domain was found by requiring positivity of the functions

$$\mathcal{F}_j(\lambda) = e^{-\mu}\left(e^{2\lambda_j} - e^{-2u}\right)\frac{\sinh(\mu)}{\sinh(2\lambda_j)}\prod_{\substack{i=1\\(i\neq j)}}^{n}\left(\frac{\sinh(\lambda_j+\lambda_i+\mu)\sinh(\lambda_j-\lambda_i+\mu)}{\sinh(\lambda_j-\lambda_i)\sinh(\lambda_j+\lambda_i)}\right)$$

and

$$\mathcal{F}_{n+j}(\lambda) = e^{-\mu}\left(e^{-2u} - e^{-2\lambda_j}\right)\frac{\sinh(\mu)}{\sinh(2\lambda_j)}\prod_{\substack{i=1\\(i\neq j)}}^{n}\left(\frac{\sinh(\lambda_j+\lambda_i-\mu)\sinh(\lambda_j-\lambda_i-\mu)}{\sinh(\lambda_j-\lambda_i)\sinh(\lambda_j+\lambda_i)}\right)$$

for all $j = 1, \ldots, n$.

Two global models of $\mathcal{M}_{\rm red}$

The action-angle variables λ_i, θ_j are not good coordinates on $\mathcal{M}_{\rm red}$ where λ reaches the boundary of the polyhedron $\mathcal{D}_\lambda$. To describe the global structure of $\mathcal{M}_{\rm red}$, we introduce the complex variables

$$\zeta_j = \sqrt{\lambda_j - \lambda_{j+1} - \mu} \prod_{k=1}^{j} e^{-\mathrm{i}\theta_k}, \quad j = 1, \dots, n-1, \quad \zeta_n = \sqrt{\lambda_n - |u|} \prod_{k=1}^{n} e^{-\mathrm{i}\theta_k}.$$

The boundary of $\mathcal{D}_\lambda$ is characterized by the vanishing of some ζ_k, and for the dense open part $\mathcal{M}^0_{\rm red}$ we have

$$\mathcal{D}^0_\lambda \times \mathbb{T}^n \Longleftrightarrow (\mathbb{C}^*)^n, \quad \text{with} \quad \mathbb{C}^* = \mathbb{C} \setminus \{0\}.$$

The complex variables remain valid when we 'add the zeros', and the standard symplectic vector space $(\widehat{M}, \hat{\omega}) = (\mathbb{C}^n, \mathrm{i} \sum_{j=1}^n d\zeta_j \wedge d\zeta_j^*)$ gives a global model of $\mathcal{M}_{\rm red}$. The point $\zeta = 0$ corresponds to the common equilibrium of the reduced Hamiltonians $\mathcal{H}_j^{\rm red}$.

Analogously, we combine the variables x_i, y_j into complex coordinates

$$\mathcal{Z}_j = \sqrt{x_j - x_{j+1} - \mu} \prod_{k=j+1}^{n} e^{\mathrm{i} y_k}, \quad j = 1, \dots, n-1, \quad \mathcal{Z}_n = \sqrt{s - x_1} \prod_{k=1}^{n} e^{\mathrm{i} y_k}.$$

Using these, the symplectic manifold $(M, \omega) = (\mathbb{C}^n, \mathrm{i} \sum_{j=1}^n d\mathcal{Z}_j \wedge d\mathcal{Z}_j^*)$ represents an alternative model of $\mathcal{M}_{\rm red}$. The point $\mathcal{Z} = 0$ corresponds to common equilibrium of the dual Hamiltonians $\widehat{\mathcal{H}}_j^{\rm red}$.

We have the reduced Lax matrices, generating the reduced Hamiltonians, in terms of both global models of $\mathcal{M}_{\rm red}$ explicitly.

Remark: In $\mathcal{Z}_n$ we have $s = \min(0, v - u)$, and in the formula of ζ_n we assumed that $|u| > |v|$.

Consequence: The identity map of the reduced phase space $\mathcal{M}_{\text{red}}$ translates into a (very non-trivial) symplectomorphism from $\mathbb{C}^n$ to $\mathbb{C}^n$, which parametrizes both models M and $\widehat{M}$ of $\mathcal{M}_{\text{red}}$. This is the duality map $\mathcal{R}$ that produces action-angle variables for our pair of integrable systems obtained by Hamiltonian reduction. The Hamiltonian flows of the action variables are equivalent to the standard torus action on the symplectic vector space $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

In summary, we have generalized the reduction treatment of the duality between the trigonometric BC_n Sutherland system

$$H(q,p) = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{1\leq j<k\leq n}\left(\frac{\gamma}{\sin^2(q_j - q_k)} + \frac{\gamma}{\sin^2(q_j + q_k)}\right) + \sum_{j=1}^{n}\frac{\gamma_1}{\sin^2(q_j)} + \sum_{j=1}^{n}\frac{\gamma_2}{\sin^2(2q_j)}$$

and the rational Ruijsenaars-Schneider-van Diejen system

$$\widehat{H}(\lambda,\theta) = \sum_{j=1}^{n}\cos(\theta_j)\left[1 - \frac{\nu^2}{\lambda_j^2}\right]^{\frac{1}{2}}\left[1 - \frac{\kappa^2}{\lambda_j^2}\right]^{\frac{1}{2}}\prod_{\substack{k=1\\(k\neq j)}}^{n}\left[1 - \frac{\mu^2}{(\lambda_j - \lambda_k)^2}\right]^{\frac{1}{2}}\left[1 - \frac{\mu^2}{(\lambda_j + \lambda_k)^2}\right]^{\frac{1}{2}}$$
$$-\frac{\nu\kappa}{\mu^2}\prod_{j=1}^{n}\left[1 - \frac{\mu^2}{\lambda_j^2}\right] + \frac{\nu\kappa}{\mu^2}.$$

It is well-known that the cotangent bundle $T^*SU(2n)$ can be recovered as a limit of the Heisenberg double $SL(2n,\mathbb{C})$. Next, I outline how the limit appears for the corresponding dual pairs of integrable Hamiltonians.

The cotangent bundle limit

• Limit of $\widehat{\mathcal{H}}_1^{\text{red}}$ to dual BC_n Sutherland: Take any positive parameter r and define

$\widehat{H}_r(\lambda,\theta;u,v,\mu) := \widehat{\mathcal{H}}_1^{\text{red}}(r\lambda,\theta;ru,rv,r\mu)$ on the domain $\mathcal{D}_\lambda^0 \times \mathbb{T}^n$. Then we obtain

$$\lim_{r\to 0} \widehat{H}_r(\lambda,\theta;u,v,\mu) = \widehat{H}(\lambda,\theta;u,v,\mu),$$

where $\widehat{H}$ is the dual of the BC_n Sutherland Hamiltonian. The symplectic form is also rescaled during the limit and the parameters match as $(u,v,\mu) \leftrightarrow (-\nu,\kappa,\mu)$.

• Limit of $\mathcal{H}_1^{\text{red}}$ to BC_n Sutherland: Define new Darboux coordinates q_i, p_i by

$$\exp(x_i) = \sin(q_i) \quad \text{and} \quad y_i = p_i \tan(q_i),$$

and then make the substitution

$u \to \beta u, \quad v \to \beta v, \quad \mu \to \beta\mu, \quad p \to \beta p, \quad \omega_{\text{red}} \to \beta\omega_{\text{red}},$ using a parameter $\beta > 0$.

The 'deformed Hamiltonians' $H_\beta(q,p;u,v,\mu) := \mathcal{H}_1^{\text{red}}(\log\sin q, \beta p\tan q; \beta u, \beta v, \beta\mu)$ are found to satisfy

$$\lim_{\beta\to 0} \frac{H_\beta(q,p;u,v,\mu) - n}{\beta^2} = H_{\text{Suth}}(q,p;\gamma_1,\gamma_2,\gamma),$$

with $\gamma = \mu^2$ etc. The domain of x and correspondingly that of q, p depends on β, and in the $\beta \to 0$ limit we recover the usual BC_n domain (Weyl alcove) for q.

The second limit is rather singular: $e^{\mathrm{i}y_j} \in U(1)$ and p_j runs over $\mathbb{R}$ in the limit.

Relation to van Diejen's many-body models

We recall that van Diejen's trigonometric integrable many-body model (RSvD model or BC_n RS model) has the Hamiltonian

$$H_{\mathrm{RSvD}}(q,p) = \sum_{j=1}^{n} \Big(\cosh(\beta p_j)\sqrt{\mathsf{V}_j(-q)\mathsf{V}_j(q)} - [\mathsf{V}_j(q) + \mathsf{V}_j(-q)]/2 \Big),$$

with V_j defined by

$$\mathsf{V}_j(q) = \mathsf{w}(q_j) \prod_{\substack{k=1 \\ (k\neq j)}}^{n} \mathsf{v}(q_j + q_k)\mathsf{v}(q_j - q_k),$$

where v, w denote the functions

$$\mathsf{v}(z) = \frac{\sin(z + \mathrm{i}g)}{\sin(z)} \quad \text{and} \quad \mathsf{w}(z) = \frac{\sin(z + \mathrm{i}g_0)}{\sin(z)} \frac{\cos(z + \mathrm{i}g_1)}{\cos(z)} \frac{\sin(z + \mathrm{i}g_0')}{\sin(z)} \frac{\cos(z + \mathrm{i}g_1')}{\cos(z)},$$

and g, g_0, g_1, g_0', g_1' are arbitrary coupling parameters.

In the trigonometric case $\pi/2 > q_1 > q_2 > \cdots > q_n > 0$, $p \in \mathbb{R}^{2n}$ and the constants are positive. The model admits a plethora of analytic continuations and limits.

We can show that by applying suitable analytic continuations, specializations and limits H_{RSvD} reproduces our reduced Hamiltonians $\mathcal{H}_1^{\mathrm{red}}$ and $\widehat{\mathcal{H}}_1^{\mathrm{red}}$.

The limit yielding $\mathcal{H}_1^{\mathrm{red}}$ is rather singular, it involves a suitable infinite shift of the positions like in the well-known 'Toda limit' of the hyperbolic Sutherland model. For the details, see our papers (which use slightly different notations).

CONCLUSION

I have illustrated how Hamiltonian reduction leads to integrable many-body systems enjoying action-angle duality.

The advantage of this approach is that once the correct starting point is 'guessed', the global phase space (with complete flows) and the duality symplectomorphism result automatically.

Open problems:

Can our 3-parametric reduction be extended so as to accommodate 5-parameters, and yield 5-parametric systems of van Diejen?

What is the reduction origin of the hyperbolic RS system?

Finally, what about quantum Hamiltonian reduction?

References

This talk was mainly based on the following papers:

L.F. and T.F. Görbe, Duality between the trigonometric BC(n) Sutherland system and a completed rational Ruijsenaars-Schneider-van Diejen system, Journ. Math. Phys. 55, 102704 (2014)

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L.F. and I. Marshall, Global description of action-angle duality for a Poisson-Lie deformation of the trigonometric BC_n Sutherland system, arXiv:1710.08760

Predecessors of the joint works with Görbe and Marshall are:

L.F. and B.G. Pusztai, A class of Calogero type reductions of free motion on a simple Lie group, Lett. Math. Phys. 79, 263-277 (2007)

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I. Marshall, A new model in the Calogero-Ruijsenaars family, Commun. Math. Phys. 338, 563-587 (2015)

References to the fundamental papers of Ruijsenaars, van Diejen, Fock-Gorsky-Nekrasov-Roubtsov, and others, as well as to the reductions techniques used, can be found in the above sources.