NONTRIVIAL PROPERTIES OF MOMENTUM SPACE AND RELATIVE LOCALITY IN K-POINCARE AND K-DE SITTER ALGEBRAS

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Non-quantum limit of quantum gravity and Deformed Special Relativity

- We want to look into the $\hbar \to 0$ regime of Quantum Gravity

$$E_P = \sqrt{\frac{\hbar c^5}{G}} \qquad\qquad \ell_P = \sqrt{\frac{\hbar G}{c^3}}$$

- Planck scale effects are still present and governed by the Planck Energy scale, if the limit is taken so that also $G \to 0$, keeping $\frac{\hbar}{G} = const$

- Because we can define an energy scale but not a length scale, it is natural to look at physics from the point of view of momentum space rather than spacetime (focus on relativistic symmetries)

- Given an energy scale, we can construct a deformation of the Poincaré algebra, such that the energy scale becomes a second relativistic invariant besides the speed of light

•*Amelino-Camelia, IJMPD 2002, PLB 2001*
•*Kowalski-Glikman, IJMPA 2001*
•*Magueijo, Smolin, PRL 2002, PRD 2003*

- Indications that the effective action of matter coupled to 2+1 quantum gravity describes matter fields subject to deformed Poincaré symmetries

•*Freidel, Kowalski-Glikman, Smolin, PRD 2004*
•*Freidel, Livine PRL 2006*
•*Cianfrani, Kowaslki-Glikman, Pranzetti, Rosati, PRD 2016*

- Indications that the spacetime symmetries emerging in the Minkowski regime of LQG are described by a deformed Poincaré group

•*Bojowald, Paily, PRD 2013*
•*Amelino-Camelia, da SIlva, Ronco, Cesarini, Lecian PRD2017*
•*Brahma, Ronco, Amelino-Camelia, Marciano, PRD2017*
•*Brahma, Ronco, PLB 2018*

Poisson-Hopf algebra description of relativistic symmetries

✦ Hopf algebras provide a consistent mathematical framework to deform special-relativistic symmetries and introduce an invariant energy scale

•J. Lukierski, A. Nowicki, H. Ruegg, Phys. Lett. B 1992
•S. Majid, H. Ruegg, Phys. Lett. B 1994
•J. Kowalski-Glikman, S. Nowak 2002-2003

✦ in the semiclassical approximation, the symmetries of phase space are described by Poisson brackets satisfying the same relations as the commutators of the Hopf algebra

✦ k-Poincaré is the most used Hopf algebra to develop phenomenology associated to deformed Poincaré symmetry, in particular focussing on energy-dependent time of travel of relativistic particles

•Amelino-Camelia, Kowalski-Glikman, Mandanici, Procaccini, Int. J. Mod. Phys. A20 (2005)

✦ opportunities for phenomenology arise for example in the study of the propagation of very high energy particles (photons, neutrinos) from astrophysical sources

•Amelino-Camelia, Ellis, Mavromatos, Nanopoulos, Sarkar, Nature 393 (1998)
•M. Ackermann et al. (Fermi GBM/LAT), Nature 462(2009)
•Xu, Ma, Astropart.Phys. 82 (2016)
•Amelino-Camelia, D'Amico, Rosati, Loret, Nat.Astron. 1 (2017)

k-Poincaré Poisson-Hopf algebra

- algebra of symmetries in bicrossproduct coordinates (1+1 dimensions)

$$\{\mathcal{P}_1, \mathcal{P}_0\} = 0$$
$$\{\mathcal{N}, \mathcal{P}_0\} = \mathcal{P}_1$$
$$\{\mathcal{N}, \mathcal{P}_1\} = \frac{1 - e^{-2\ell \mathcal{P}_0}}{2\ell} - \frac{\ell}{2}\mathcal{P}_1^2$$

$$\left[\ell = \frac{1}{\kappa} \sim \frac{1}{E_p}\right]$$

- first Casimir

$$\mathcal{C}_\ell = \left(\frac{2}{\ell}\sinh\left(\frac{\ell \mathcal{P}_0}{2}\right)\right)^2 - e^{\ell \mathcal{P}_0}\mathcal{P}_1^2$$

- coproducts and antipodes

$$\Delta(\mathcal{P}_0) = \mathcal{P}_0 \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{P}_0 \qquad S(\mathcal{P}_0) = -\mathcal{P}_0$$
$$\Delta(\mathcal{P}_1) = \mathcal{P}_1 \otimes \mathbb{I} + e^{-\ell \mathcal{P}_0} \otimes \mathcal{P}_1 \qquad S(\mathcal{P}_1) = -e^{\ell \mathcal{P}_0}\mathcal{P}_1$$
$$\Delta(\mathcal{N}) = \mathcal{N} \otimes \mathbb{I} + e^{-\ell \mathcal{P}_0} \otimes \mathcal{N} \qquad S(\mathcal{N}) = -e^{\ell \mathcal{P}_0}\mathcal{N}$$

•*Lukierski, Nowicki, Ruegg, Phys. Lett. B 293 (1992)*
•*Lukierski, Ruegg, Nowicki, Tolstoi, Phys. Lett. B 264 (1991)*
•*Majid, Ruegg, Phys.Lett. B334 (1994)*

k-Poincaré representation on momentum space

- because spacetime translations close a subalgebra, they can be represented as an algebra of functions over momentum space •*Kowalski-Glikman, Nowak, CQG 2003*
- correspondence between structures of the Hopf sub-algebra and structures of the momentum space:

translations $P_\mu(p)$	coordinates over manifold p_μ
change of basis of the algebra	diffeomorphism
coproduct map $\Delta P_\mu(p,q)$	composition law of momenta $(p \oplus q)_\mu$
antipode $S(P_\mu)(p)$	inversion $(\ominus p)_\mu$
coassociativity $(\Delta \otimes \mathtt{Id}) \circ \Delta = (\mathtt{Id} \otimes \Delta) \circ \Delta$	associativity of composition rule $(p \oplus q) \oplus k = p \oplus (q \oplus k)$

•*Gubitosi, Mercati, CQG 2013*

Geometric properties of the k-Poincaré momentum space manifold

- k-Poincaré momenta live on a (portion of) de Sitter manifold

•*Kowalski-Glikman, Nowak, CQG 2003*
•*Kowalski-Glikman PLB 2002*
•*Gubitosi, Mercati, CQG 2013*
•*Amelino-Camelia, Arzano, Kowalski-Glikman, Rosati, Trevisan, CQG 2012*

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change to a basis where the algebra is trivial (coproducts still nontrivial)

$$P_0(p_0,p_1) = \frac{\sinh(\ell p_0)}{\ell} + \frac{\ell p_1^2}{2} e^{\ell p_0}$$
$$P_1(p_0,p_1) = p_1 e^{\ell p_0}$$
$$P_4(p_0,p_1) = \frac{\cosh(\ell p_0)}{\ell} - \frac{\ell p_1^2}{2} e^{\ell p_0}$$

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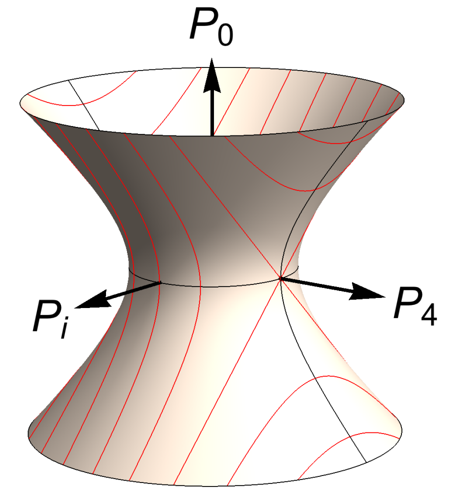
these new generators turn out to satisfy the relation

$$P_0^2 - P_1^2 - P_4^2 = -\frac{1}{\ell^2}$$

this is the defining relation of a 1+1 dimensional de Sitter manifold embedded in a 2+1 Minkowski manifold
the energy scale is playing a crucial role in the geometry of momentum space, since it defines its radius of curvature



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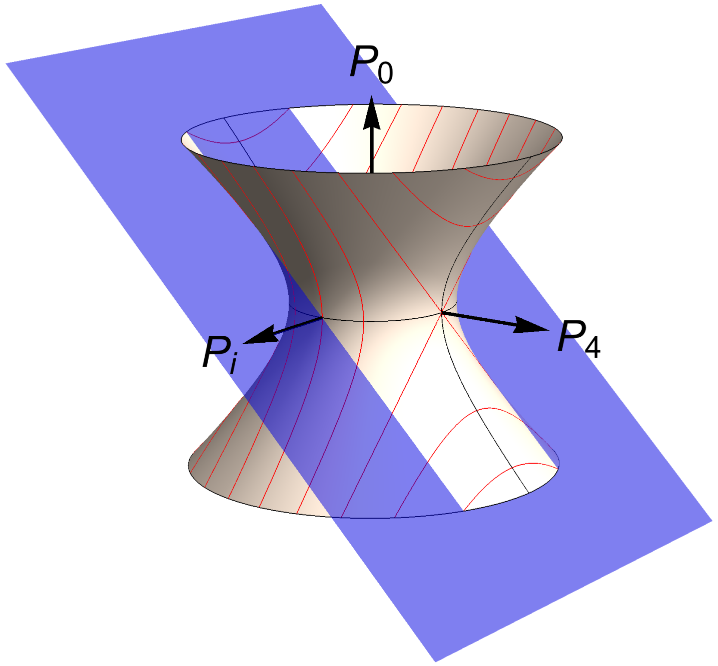

- the bicrossproduct coordinates only cover half of the manifold:

$$P_0 + P_4 > 0$$

Curved momentum space and kinematics of free particles

- the first Casimir of the algebra gives the mass-shell condition

$$\mathcal{C}_\ell = \left(\frac{2}{\ell}\sinh\left(\frac{\ell \mathcal{P}_0}{2}\right)\right)^2 - e^{\ell \mathcal{P}_0}\mathcal{P}_1^2 \quad \Longrightarrow \quad m^2 = \left(\frac{2}{\ell}\sinh\left(\frac{\ell p_0}{2}\right)\right)^2 - e^{\ell p_0} p_1^2$$

in the massless case : $p_1(p_0) = \dfrac{1 - e^{-\ell p_0}}{\ell}$

- the dispersion relation of free particles is invariant under boosts

$$[\mathcal{C}_\ell, \mathcal{N}] = 0 \quad \Longrightarrow \quad \left(\frac{2}{\ell}\sinh\left(\frac{\ell p_0'}{2}\right)\right)^2 - e^{\ell p_0'} p_1'^2 = m^2$$

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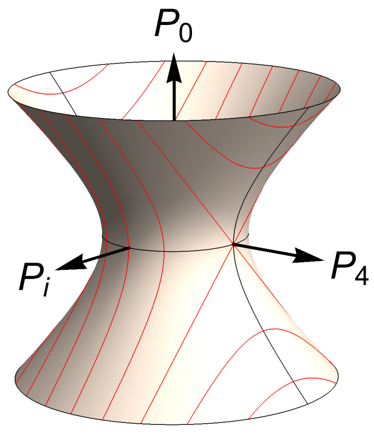
- from the point of view of the momentum space:

the dispersion relation is given by the curves of constant geodesic distance from the origin of momentum space



invariance of the dispersion relation is due to the invariance of the line element $ds_p^2 = dp_0^2 - e^{2\ell p_0}dp_1^2$

$$\begin{aligned} p_0' &= p_0 + \xi p_1 \\ p_1' &= p_1 + \xi\left(\frac{1-e^{-2\ell p_0}}{2\ell} - \frac{\ell}{2}p_1^2\right) \end{aligned} \longrightarrow (ds_p^2)' \equiv dp_0'^2 - e^{2\ell p_0'}dp_1'^2 = ds_p^2$$

so boosts are isometries of the k-Poincaré momentum space

Definition of a semi-classical spacetime

✦ we want to study the propagation of free relativistic particles, without purely quantum effects (quantum correlations, fuzziness etc..)

✦ to this aim spacetime is defined via a classical phase-space construction: coordinates are the objects that define a trivial symplectic structure together with momenta:

$$\{p_1, p_0\} = 0$$
$$\{x^1, x^0\} = 0$$
$$\{x^\mu, p_\nu\} = \delta^\mu_\nu$$

✦ at the quantum level the k-Poincaré group is dual to the k-Minkowski noncommutative spacetime, whose coordinates can be related to the (quantum version of these) commutative coordinates by a momentum-dependent redefinition

$$[\hat{x}_0, \hat{x}_j] = i\lambda x_j$$
$$[\hat{x}_i, \hat{x}_j] = 0$$

k-Poincaré particle kinematics

- representation of the algebra of symmetries on phase space

$$\{p_1, p_0\} = 0$$
$$\{x^1, x^0\} = 0$$
$$\{x^\mu, p_\nu\} = \delta^\mu_\nu$$

$$\mathcal{P}_0 = p_0$$
$$\mathcal{P}_1 = p_1$$
$$\mathcal{N} = p_1 x^0 + \left(\frac{1 - e^{-2\ell p_0}}{2\ell} - \frac{\ell}{2} p_1^2 \right) x^1$$

•Amelino-Camelia, Barcaroli, Gubitosi, Loret, Class.Quant.Grav. 30 (2013)
•Gubitosi, Barcaroli, PRD 93 (2016)

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✦ evolution of phase space coordinates is given by the Hamilton equations with the k-Poincaré Casimir as Hamiltonian

$$\dot{x}^0 = \{\mathcal{C}_\ell, x^0\} = \frac{1}{\ell}\left(e^{\ell p_0} - e^{-\ell p_0}\right) - \ell p_1^2 e^{\ell p_0}$$
$$\dot{x}^1 = \{\mathcal{C}_\ell, x^1\} = 2\, p_1\, e^{\ell p 0}.$$

✦ massless coordinate velocity depends on the energy of the particle $v \equiv \dfrac{\dot{x}^1}{\dot{x}^0} = -e^{\ell p_0}$

→ massless particle worldline: $x^1 - \bar{x}^1 = -e^{\ell p 0}(x^0 - \bar{x}^0)$

•*Amelino-Camelia, Barcaroli, Gubitosi, Loret, Class.Quant.Grav. 30 (2013)*
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$\longrightarrow$

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$\longrightarrow$ massless particle worldline: $x^1 - \bar{x}^1 = -e^{\ell p 0}(x^0 - \bar{x}^0)$

- one would obtain the same result using the 'k-Minkowski coordinates' and appropriately accounting for the deformed action of translations upon them

$$\chi^1 = x^1$$
$$\chi^0 = x^0 - \ell x^1 p_1$$

$$\{\chi^0, \chi^1\} = \ell \chi^1$$
$$\{\chi^0, p_1\} = -\ell p_1$$
$$\{\chi^0, p_0\} = 0$$

•*Amelino-Camelia, Barcaroli, Gubitosi, Loret, Class.Quant.Grav. 30 (2013)*
•*Gubitosi, Barcaroli, PRD 93 (2016)*

de Sitter spacetime - symmetries and phase space

- 1+1 dimensional de Sitter manifold can be described as the 2-dim hypersurface embedded in a 3-dim Minkowski manifold

$$(z^0)^2 - (z^1)^2 - (z^2)^2 = -H^{-2}$$

- line element in comoving coordinates

$$ds^2 = (dx^0)^2 - e^{2Hx^0}(dx^1)^2$$

- algebra of symmetries (co-algebra sector is trivial)

$$\begin{aligned} \{\mathcal{P}_0, \mathcal{P}_1\} &= H\,\mathcal{P}_1 \\ \{\mathcal{P}_0, \mathcal{N}\} &= \mathcal{P}_1 - H\,\mathcal{N} \\ \{\mathcal{P}_1, \mathcal{N}\} &= \mathcal{P}_0 \end{aligned}$$

- mass Casimir

$$\mathcal{C}_{dS} = \mathcal{P}_0^2 - \mathcal{P}_1^2 + 2H\mathcal{N}\mathcal{P}_1$$

- representation of symmetry generators:

$$\begin{aligned} \{x^\mu, x^\nu\} &= 0\,, \\ \{x^\mu, p_\nu\} &= -\delta^\mu_\nu\,, \\ \{p_\mu, p_\nu\} &= 0\,. \end{aligned} \quad \longrightarrow \quad \begin{aligned} \mathcal{P}_0 &= p_0 - Hx^1 p_1 \\ \mathcal{P}_1 &= p_1 \\ \mathcal{N} &= x^1 p_0 + p_1\left(\frac{1-e^{-2Hx^0}}{2H} - \frac{H}{2}(x^1)^2\right) \end{aligned}$$

de Sitter particle kinematics

✦ evolution of worldline coordinates is given by Hamilton equations with the Hamiltonian given by the Casimir

$$\begin{aligned}\dot{x}^1 &= \{\mathcal{C}_{dS}, x^1\} = -2e^{-2Hx^0} p_1 \\ \dot{x}^0 &= \{\mathcal{C}_{dS}, x^0\} = 2p_0\end{aligned}$$

✦ the massless condition $\mathcal{C}_{dS} = 0$ relates energy and spatial momentum:

$$p_0 = |p_1| e^{-Hx^0}$$

(this encodes redshift of energy)

✦ coordinate velocity: $v \equiv \frac{\dot{x}^1}{\dot{x}^0} = e^{-Hx^0}$

→ massless particle worldline

$$x^1(x^0) - \bar{x}^1 \equiv \int_{\bar{x}^0}^{x^0} \frac{\dot{x}^1}{\dot{x}^0} dx^0 = \left(\frac{e^{-H\bar{x}^0} - e^{-Hx^0}}{H} \right)$$

Duality between de Sitter spacetime and de Sitter momentum space

de Sitter spacetime	de Sitter momentum space
spacetime metric $ds^2 = (dx^0)^2 - e^{2Hx^0}(dx^1)^2$	momentum space metric $ds_p^2 = dp_0^2 - e^{2\ell p_0} dp_1^2$
worldline $x^1 = \frac{1 - e^{-Hx^0}}{H}$	dispersion relation $p_1 = \frac{1 - e^{-\ell p_0}}{\ell}$
dispersion relation $p_1 = -e^{Hx^0} p_0$	worldline $x^1 = -e^{\ell p_0} x^0$
generators of translations $\mathcal{P}_1 = p_1$, $\mathcal{P}_0 = p_0 - Hx^1 p_1$	‘k-Minkowski coordinates’ $\chi^1 = x^1$, $\chi^0 = x^0 - \ell x^1 p_1$

•*Amelino-Camelia, Barcaroli, Gubitosi, Loret, Class.Quant.Grav. 30 (2013)*

✦ related to the fact that in Hopf algebras noncommutativity induces curvature in the dual space, and viceversa

•*Majid arXiv: hep-th/0604130*

Putting spacetime and momentum space curvature together

✦ in physically relevant scenarios we want to implement deformed (local) relativistic transformations over a curved spacetime

✦ this is motivated by the fact that opportunities for phenomenology arise in contexts where spacetime curvature is actually non-negligible (early universe, propagation of photons from Gamma-ray Bursts etc…)

✦ extension of results fond in kP to curved spacetime is non-trivial, as one would in general expect some sort of interplay between effects of curvature in spacetime and in momentum space

- *Amelino-Camelia, Smolin, Starodubtsev, Class. Quant. Grav. 21(2004)*
- *Marciano, Amelino-Camelia, Bruno, Gubitosi, Mandanici, Melchiorri, JCAP 1006 (2010)*

✦ in the context of Hopf algebras, one can study a k-deformation of the de Sitter algebra

- *Lukierski, Ruegg, Nowicki and Tolstoi, Phys. Lett. B 264 (1991)*
- *Ballesteros, Herranz, del Olmo, Santander, J. Phys. A: Math. Gen. 26 (1993)*
- *Ballesteros, Herranz, del Olmo, Santander, J. Phys. A: Math. Gen. 27 (1994)*

Preliminaries: revisiting the k-Poincaré momentum space construction

- algebra in bicrossproduct coordinates (2+1 dimensions) $\left[z = \ell = \frac{1}{\kappa}\right]$

$$
\begin{array}{lll}
\{J, P_1\} = P_2, & \{J, P_2\} = -P_1, & \{J, P_0\} = 0, \\
\{J, K_1\} = K_2, & \{J, K_2\} = -K_1, & \{K_1, K_2\} = -J, \\
\{P_0, P_a\} = 0, & \{P_a, P_b\} = 0, & \{K_a, P_0\} = P_a
\end{array}
$$

$$
\{K_a, P_b\} = \delta_{ab}\left(\frac{1}{2z}\left(1 - e^{-2zP_0}\right) + \frac{z}{2}\,P^2\right) - zP_aP_b\,,
$$

- coproducts

$$
\begin{aligned}
\Delta_z(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \\
\Delta_z(P_a) &= P_a \otimes 1 + e^{-zP_0} \otimes P_a, \\
\Delta_z(J) &= J \otimes 1 + 1 \otimes J, \\
\Delta_z(K_a) &= K_a \otimes 1 + e^{-zP_0} \otimes K_a + z\,\epsilon_{abc} P_b \otimes J_c\,.
\end{aligned}
$$

- Poisson algebra dual to translations

$$
\left[X^0, X^i\right] = -z\,X^i, \qquad \left[X^i, X^j\right] = 0
$$

obtained by dualizing the cocommutator map (its form can be read off from the first-order deformation of the coproducts of P) and gives the k-Minkowski spacetime algebra

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&\{P_0,P_a\}=0, \qquad \{P_a,P_b\}=0, \qquad \{K_a,P_0\}=P_a\\
&\{K_a,P_b\}=\delta_{ab}\left(\frac{1}{2z}\left(1-e^{-2zP_0}\right)+\frac{z}{2}\,P^2\right)-zP_aP_b\,,
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\Delta_z(K_a) &= K_a\otimes 1+e^{-zP_0}\otimes K_a+z\,\epsilon_{abc}P_b\otimes J_c\,.
\end{aligned}
$$

- Poisson algebra dual to translations

$$
\left[X^0,X^i\right]=-z\,X^i, \qquad \left[X^i,X^j\right]=0
$$

obtained by dualizing the cocommutator map (its form can be read off from the first-order deformation of the coproducts of P) and gives the k-Minkowski spacetime algebra

this construction is possible because the translations close a Hopf sub-algebra

Preliminaries: revisiting the k-Poincaré momentum space construction

- the generic element of the dual Poisson-Lie group is constructed via exponentiation

$$G^*(p_0,p_1,p_2) = \exp\left(p_1\rho(X^1)\right)\exp\left(p_2\rho(X^2)\right)\exp\left(p_0\rho(X^0)\right)\,,$$

the coproducts of P_μ can be re-obtained from the group law of G* upon identifying $P_\mu \equiv p_\mu$
a different choice of ordering of the exponentials would result in a different choice of basis of the translation generators

- 4d representation of X^μ

$$\rho(X^0) = z\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\,, \quad \rho(X^1) = z\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\,, \quad \rho(X^2) = z\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\,.$$

- then the group element reads

$$G^*(p) = \begin{pmatrix} \cosh(zp_0) + \frac{1}{2}\,e^{z\,p_0}\,z^2\,\bar{p}^2 & zp_1 & zp_2 & \sinh(zp_0) + \frac{1}{2}\,e^{z\,p_0}\,z^2\,\bar{p}^2 \\ e^{z\,p_0}\,zp_1 & 1 & 0 & e^{z\,p_0}\,zp_1 \\ e^{z\,p_0}\,zp_2 & 0 & 1 & e^{z\,p_0}\,zp_2 \\ \sinh(zp_0) - \frac{1}{2}\,e^{z\,p_0}\,z^2\,\bar{p}^2 & -zp_1 & -zp_2 & \cosh(zp_0) - \frac{1}{2}\,e^{z\,p_0}\,z^2\,\bar{p}^2 \end{pmatrix}$$

Preliminaries: revisiting the k-Poincaré momentum space construction

✦ the k-Poincaré momentum space is generated by the orbits of the dual Poisson-Lie group G^* acting on the ambient Minkowski space that pass through the point (0,0,0,1):

$$G^* \cdot (0,0,0,1)^T = (S_0, S_1, S_2, S_4)^T \, .$$

✦ where we recover the coordinates defined earlier:

$$\begin{aligned} S_0 &= \sinh(zp_0) + \tfrac{1}{2}\, e^{z\,p_0}\, z^2\, \bar{p}^2, \\ S_1 &= e^{z\,p_0}\, z\, p_1, \\ S_2 &= e^{z\,p_0}\, z\, p_2, \\ S_4 &= \cosh(zp_0) - \tfrac{1}{2}\, e^{z\,p_0}\, z^2\, \bar{p}^2 \, . \end{aligned}$$

such that $-S_0^2 + S_1^2 + S_2^2 + S_4^2 = 1$ and $S_0 + S_4 = e^{z\,p_0} > 0$

✦ this defines half of a 2+1 dimensional de Sitter manifold

• *J. Kowalski-Glikman, Int. J. Mod. Phys. A 28 (2013)*
• *Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PLB773 (2017)*
• *Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, arXiv:1711.05050*

k-(anti) de Sitter algebra

- in the k-(anti) de Sitter algebra there is a nontrivial interplay between the 'quantum' defamation parameter z and the cosmological constant Λ, that is a classical deformation parameters ($\Lambda > 0$ de Sitter, $\Lambda < 0$ anti de Sitter)

- algebra in 2+1 dimensions (bicrossproduct basis)

• *Ballesteros, Herranz, del Olmo, Santander, J. Phys. A (1994)*

$$\{J,P_0\}=0,\qquad \{J,P_1\}=P_2,\qquad \{J,P_2\}=-P_1,$$
$$\{J,K_1\}=K_2,\qquad \{J,K_2\}=-K_1,\qquad \{K_1,K_2\}=-\frac{\sin(2z\sqrt{\Lambda}J)}{2z\sqrt{\Lambda}},$$
$$\{P_0,P_1\}=-\Lambda\,K_1,\qquad \{P_0,P_2\}=-\Lambda\,K_2,\qquad \{P_1,P_2\}=\Lambda\,\frac{\sin(2z\sqrt{\Lambda}J)}{2z\sqrt{\Lambda}},$$
$$\{K_1,P_0\}=P_1,\qquad \{K_2,P_0\}=P_2,$$
$$\{P_2,K_1\}=z\,(P_1P_2-\Lambda K_1K_2)\qquad \{P_1,K_2\}=z\,(P_1P_2-\Lambda K_1K_2)\,,$$
$$\{K_1,P_1\}=\frac{1}{2z}\left(\cos(2z\sqrt{\Lambda}J)-e^{-2zP_0}\right)+\frac{z}{2}\,(P_2^2-P_1^2)-\frac{z\Lambda}{2}\,(K_2^2-K_1^2)\,,$$
$$\{K_2,P_2\}=\frac{1}{2z}\left(\cos(2z\sqrt{\Lambda}J)-e^{-2zP_0}\right)+\frac{z}{2}\,(P_1^2-P_2^2)-\frac{z\Lambda}{2}\,(K_1^2-K_2^2)\,,$$

- coproducts

$$\Delta(P_0)\;=\;P_0\otimes 1+1\otimes P_0,\qquad \Delta(J)=J\otimes 1+1\otimes J,$$
$$\Delta(P_1)\;=\;P_1\otimes\cos(z\sqrt{\Lambda}J)+e^{-zP_0}\otimes P_1+\Lambda\,K_2\otimes\frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}},$$
$$\Delta(P_2)\;=\;P_2\otimes\cos(z\sqrt{\Lambda}J)+e^{-zP_0}\otimes P_2-\Lambda\,K_1\otimes\frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}},$$
$$\Delta(K_1)\;=\;K_1\otimes\cos(z\sqrt{\Lambda}J)+e^{-zP_0}\otimes K_1+P_2\otimes\frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}},$$
$$\Delta(K_2)\;=\;K_2\otimes\cos(z\sqrt{\Lambda}J)+e^{-zP_0}\otimes K_2-P_1\otimes\frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}},$$

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\begin{aligned}
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&\{P_0,P_1\}=-\Lambda\,K_1, \qquad \{P_0,P_2\}=-\Lambda\,K_2, \qquad \{P_1,P_2\}=\Lambda\,\frac{\sin(2z\sqrt{\Lambda}J)}{2z\sqrt{\Lambda}},\\
&\{K_1,P_0\}=P_1, \qquad \{K_2,P_0\}=P_2,\\
&\{P_2,K_1\}=z\,(P_1P_2-\Lambda K_1K_2) \qquad \{P_1,K_2\}=z\,(P_1P_2-\Lambda K_1K_2)\,,\\
&\{K_1,P_1\}=\frac{1}{2z}\left(\cos(2z\sqrt{\Lambda}J)-e^{-2zP_0}\right)+\frac{z}{2}\left(P_2^2-P_1^2\right)-\frac{z\Lambda}{2}\left(K_2^2-K_1^2\right),\\
&\{K_2,P_2\}=\frac{1}{2z}\left(\cos(2z\sqrt{\Lambda}J)-e^{-2zP_0}\right)+\frac{z}{2}\left(P_1^2-P_2^2\right)-\frac{z\Lambda}{2}\left(K_1^2-K_2^2\right),
\end{aligned}
$$

- coproducts

$$
\begin{aligned}
\Delta(P_0) &= P_0\otimes 1+1\otimes P_0, \qquad \Delta(J)=J\otimes 1+1\otimes J,\\
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\Delta(K_2) &= K_2\otimes\cos(z\sqrt{\Lambda}J)+e^{-zP_0}\otimes K_2-P_1\otimes\frac{\sin(z\sqrt{\Lambda}J)}{\sqrt{\Lambda}},
\end{aligned}
$$

- spacetime translations do not close a sub algebra any more

k-(anti) de Sitter algebra - dual Lie algebra and group

✦ because in particular the coalgebra sector of translations does not close, the dual Lie algebra needs to be constructed with respect to the whole set of k-de Sitter generators

$$
\begin{array}{lll}
[X^0, X^1] = -z\,X^1, & [X^0, X^2] = -z\,X^2, & [X^1, X^2] = 0, \\
[X^0, L^1] = -z\,L^1, & [X^0, L^2] = -z\,L^2, & [L^1, L^2] = 0, \\
[R, X^2] = -z\,L^1, & [R, L^1] = z\,\Lambda\,X^2, & [L^1, X^2] = 0, \\
[R, X^1] = z\,L^2, & [R, L^2] = -z\,\Lambda\,X^1, & [L^2, X^1] = 0, \\
[R, X^0] = 0, & [L^1, X^1] = 0, & [L^2, X^2] = 0.
\end{array}
$$

X are dual to translations P, R is dual to the rotation J and L are dual to boosts K

✦ the generic element of the corresponding dual Poisson-Lie group G* is again defined via exponentiation

$$G^*_\Lambda = \exp\left(\theta\rho(R)\right)\exp\left(p_1\rho(X^1)\right)\exp\left(p_2\rho(X^2)\right)\exp\left(\chi_1\rho(L^1)\right)\exp\left(\chi_2\rho(L^2)\right)\exp\left(p_0\rho(X^0)\right)$$

however now the local group coordinates include 'generalized momenta', associated to boosts and rotations besides spacetime translations

as seen before, the coproducts and algebra of the q-de Sitter algebra can be recovered from the group law of G* upon identifying $P_\mu \equiv p_\mu, \quad \chi_i \equiv K_i, \quad \theta = J$

•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PLB773 (2017)*
•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, ArXiv:1711.05050*

k-de Sitter algebra - construction of the generalised momentum space

✦ the dual Lie algebra generators admit a 6-dimensional representation

$$\rho(X^0) = z\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix} \quad \rho(X^1) = z\begin{pmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&-1&0&0&0&0 \end{pmatrix}$$

$$\rho(X^2) = z\begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&0 \end{pmatrix} \quad \rho(L^1) = z\sqrt{\Lambda}\begin{pmatrix} 0&0&0&1&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&1\\ 0&0&0&0&0&0\\ 0&0&0&-1&0&0 \end{pmatrix}$$

$$\rho(L^2) = z\sqrt{\Lambda}\begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&1\\ 0&0&0&0&-1&0 \end{pmatrix} \quad \rho(R) = z\sqrt{\Lambda}\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&-1&0\\ 0&0&0&1&0&0\\ 0&0&-1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix}.$$

✦ then the orbits of the dual Poisson-Lie group G* acting on the ambient Minkowski space that pass through the point (0,0,0,0,0,1) are given by

$$G^* \cdot (0,0,0,0,0,1)^T = (S_0, S_1, S_2, S_3, S_4, S_5)^T .$$

and generate the generalised momentum space of k-de Sitter

•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PLB773 (2017)*
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k-de Sitter algebra - properties of the generalised momentum space

- the orbits are parameterised by

$$
\begin{aligned}
S_0 &= \sinh(zp_0) + \frac{1}{2} e^{z\,p_0} z^2 \left(p_1^2 + p_2^2 + \Lambda\left(\chi_1^2 + \chi_2^2\right)\right) \\
S_1 &= e^{z\,p_0} z \left(\cos(z\sqrt{\Lambda}\,\theta)\,p_1 - \sqrt{\Lambda}\,\sin(z\sqrt{\Lambda}\,\theta)\,\chi_2\right), \\
S_2 &= e^{z\,p_0} z \left(\cos(z\sqrt{\Lambda}\,\theta)\,p_2 + \sqrt{\Lambda}\,\sin(z\sqrt{\Lambda}\,\theta)\,\chi_1\right), \\
S_3 &= e^{z\,p_0} z \left(-\sin(z\sqrt{\Lambda}\,\theta)\,p_2 + \sqrt{\Lambda}\,\cos(z\sqrt{\Lambda}\,\theta)\,\chi_1\right), \\
S_4 &= e^{z\,p_0} z \left(\sin(z\sqrt{\Lambda}\,\theta)\,p_1 + \sqrt{\Lambda}\,\cos(z\sqrt{\Lambda}\,\theta)\,\chi_2\right), \\
S_5 &= \cosh(zp_0) - \frac{1}{2} e^{z\,p_0} z^2 \left(p_1^2 + p_2^2 + \Lambda\left(\chi_1^2 + \chi_2^2\right)\right)
\end{aligned}
$$

these coordinates satisfy the conditions $-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 = 1$ and $S_0 + S_5 = e^{z\,p_0} > 0$

this is the embedding of (half of) a 4+1 dimensional de Sitter space into the ambient 5+1 dimensional Minkowski space, M_{dS_5}

•Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PLB773 (2017)
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S_3 &= e^{z\,p_0} z \left(-\sin(z\sqrt{\Lambda}\,\theta)\, p_2 + \sqrt{\Lambda}\,\cos(z\sqrt{\Lambda}\,\theta)\,\chi_1\right),\\
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S_5 &= \cosh(zp_0) - \frac{1}{2} e^{z\,p_0} z^2 \left(p_1^2 + p_2^2 + \Lambda\left(\chi_1^2 + \chi_2^2\right)\right)
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✦ the momentum space has a lower dimensionality compared to the number of generators because rotations generate the isotropy group of the point (0,0,0,0,0,1).
Taking this into account the full momentum space is $M_{dS_5} \times S^1$

•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, PLB773 (2017)*
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S_3 &= e^{z\,p_0} z \left(-\sin(z\sqrt{\Lambda}\,\theta)\, p_2 + \sqrt{\Lambda}\, \cos(z\sqrt{\Lambda}\,\theta)\, \chi_1\right), \\
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these coordinates satisfy the conditions $-S_0^2 + S_1^2 + S_2^2 + S_3^2 + S_4^2 + S_5^2 = 1$ and $S_0 + S_5 = e^{z\,p_0} > 0$

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✦ the momentum space has a lower dimensionality compared to the number of generators because rotations generate the isotropy group of the point (0,0,0,0,0,1).
Taking this into account the full momentum space is $M_{dS_5} \times S^1$

✦ this allows to write the dispersion relation associated to the Casimir in a simplified way

$$
\mathcal{C}_z = \frac{2}{z^2}\left[\cosh(zp_0)\cos(z\sqrt{\Lambda}\theta) - 1\right] - e^{zp_0}\left(p_1^2 + p_2^2 - \Lambda(\chi_1^2 + \chi_2^2)\right)\cos(z\sqrt{\Lambda}\theta) - 2\,\Lambda\, e^{zp_0}\,\frac{\sin(z\sqrt{\Lambda}\theta)}{\sqrt{\Lambda}} R_3,
$$

$[R_3 = \epsilon_{3bc}\chi_b p_c]$

$$
\theta = 0 \quad \longrightarrow \quad \mathcal{C}_z = \frac{2}{z^2}\left[\cosh(zp_0) - 1\right] - e^{zp_0}\left(p_1^2 + p_2^2 - \Lambda(\chi_1^2 + \chi_2^2)\right)
$$

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Generalised momentum space of k-(anti) de Sitter algebra - further results

✦ generalisation to k-anti de Sitter algebra:

this is obtained for $\Lambda < 0$ (upon appropriately modifying the dual Lie algebra representation so to keep it real)

the resulting orbit of the dual Lie group generates a manifold defined by the quadratic constraint

$$-S_0^2 + S_1^2 + S_2^2 - S_3^2 - S_4^2 + S_5^2 = 1$$

✦ generalisation to higher dimensions:

the main nontrivial additional ingredient are rotations, which close a deformed sub algebra with a privileged direction

$$\begin{aligned} \Delta_z(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, \\ \Delta_z(J_1) &= J_1 \otimes e^{z\sqrt{\omega}J_3} + 1 \otimes J_1, \\ \Delta_z(J_2) &= J_2 \otimes e^{z\sqrt{\omega}J_3} + 1 \otimes J_2, \end{aligned}$$

still one can show that the generalised momentum space is half of a 6+1 dimensional de Sitter manifold and the rotation sector only has the role of generating the isotropy subgroup of its origin (0,0,0,0,0,0,0,1)

•*Ballesteros, Gubitosi, Gutierrez-Sagredo, Herranz, ArXiv:1711.05050*