The Algebroid Structure of Double Field Theory

Larisa Jonke

Division of Theoretical Physics
Rudjer Bošković Institute, Zagreb

Based on:

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General motivation

- String geometry departs from Riemannian geometry, notably in presence of fluxes
  - open strings $\rightsquigarrow$ noncommutativity - Poisson structure - $\star$-product - Kontsevich '97 DQ
    Chu, Ho '99; Seiberg, Witten '99
  - closed strings $\rightsquigarrow$ noncommutativity/nonassociativity - (twisted) Poisson - $\star$-product
    Halmagyi '09; Lüst '10; Blumenhagen, Plauschinn '10; Mylonas, Schupp, Szabo '12; & c.
- Dualities relate different geometries/topologies $\rightsquigarrow$ "non-geometric backgrounds"
- Manifestly duality-invariant theories - double and exceptional field theories
  Hull, Hohm, Zwiebach; Hohm, Samtleben; & c.
- Evidence that the correct language is algebroid/generalized geometry
  Courant; Liu, Weinstein, Xu, Ševera; Roytenberg; Hitchin; Gualtieri; Cavalcanti; Bouwknegt, Hannabuss, Mathai; & c.

- Courant Algebroids and Generalized Geometry double the bundle, e.g. $TM \oplus T^*M$
- DFT doubles the base, $\mathcal{M} = M \times \widetilde{M}$ — comes with constraints

- Solving the strong constraint, reduces DFT data to the data of the standard CA
- What is the geometric origin of the DFT data and the strong constraint?
  cf. also Deser, Stasheff '14; Deser, Saemann '16

- CAs provide membrane sigma models $\rightsquigarrow$ describe non-geometric backgrounds
  Roytenberg '06
  Mylonas, Schupp, Szabo '12; ACh, Jonke, Lechtenfeld '15; Bessho, Heller, Ikeda, Watamura '15
- Is there a "DFT algebroid" that could provide a DFT membrane sigma model?

Plan

- Basic DFT data
- Courant Algebroids
- The Burger Proposal
- Membrane Sigma Model for DFT
- Algebroid structure of DFT
- Closing Remarks

Basic DFT Data

Double Field Theory (Generalised Metric Formulation)

A field theory invariant under $O(d,d)$; T-duality becomes manifest.

It uses doubled coordinates $(x^I) = (x^i, \tilde{x}_i)$, and all fields depend on both.

The $O(d,d)$ structure is associated to a (constant) $O(d,d)$-invariant metric

$$\eta = (\eta_{IJ}) = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \ , \quad h^t \eta h = \eta \ , \quad h \in O(d,d) \ ,$$

used to raise and lower $I = 1, \ldots, 2d$ indices.

Derivatives are also doubled accordingly: $(\partial_I) = (\partial_i, \tilde{\partial}^i)$.

The fields g and B are packed together in a generalised metric

$$\mathcal{H}_{IJ} = \begin{pmatrix} g_{ij} - B_{ik} g^{kl} B_{lj} & B_{ik} g^{kj} \\ -g^{ik} B_{kj} & g^{ij} \end{pmatrix} \ ,$$

satisfying

$$\mathcal{H}^t = \mathcal{H} \quad \text{and} \quad \mathcal{H}\eta\mathcal{H} = \eta^{-1} \ , \quad h^K{}_I h^L{}_J \mathcal{H}'_{KL}(x') = \mathcal{H}_{IJ}(x)$$

DFT action, its symmetries and constraints

The $O(d,d)$-invariant action for $\mathcal{H}$ and the invariant dilaton d ($e^{-2d} = \sqrt{-g}e^{-2\phi}$) is

$$S = \int \mathrm{d}x\mathrm{d}\tilde{x} e^{-2d} \left(\tfrac{1}{8}\mathcal{H}^{IJ}\partial_I\mathcal{H}^{KL}\partial_J\mathcal{H}_{KL} - \tfrac{1}{2}\mathcal{H}^{IJ}\partial_I\mathcal{H}^{KL}\partial_L\mathcal{H}_{KJ} - 2\partial_I d\partial_J\mathcal{H}^{IJ} + 4\mathcal{H}^{IJ}\partial_I d\partial_J d \right) \ .$$

Gauge transformations are also included with a parameter $\epsilon^I = (\epsilon^i, \tilde{\epsilon}_I)$:

$$\begin{aligned} \delta_\epsilon \mathcal{H}^{IJ} &= \epsilon^K\partial_K H^{IJ} + (\partial^I\epsilon_K - \partial_K\epsilon^I)\mathcal{H}^{KJ} + (\partial^J\epsilon_K - \partial_K\epsilon^J)\mathcal{H}^{IK} := L_\epsilon\mathcal{H}^{IJ} \ , \\ \delta_\epsilon d &= -\tfrac{1}{2}\partial_K\epsilon^K + \epsilon^K\partial_K d \ , \end{aligned}$$

and $L_\epsilon\cdot$ is called the generalised Lie derivative. But S is not automatically invariant.

The theory is constrained.

- Weak constraint: $\Delta\cdot := \partial^I\partial_I\cdot = 0$; stems from the level matching condition.
- Strong constraint: $\partial^I\partial_I(\dots) = 0$ on products on fields.

Only when the strong constraint is satisfied, the action S is gauge invariant.

The C-bracket of DFT

The generalised Lie derivative, *after imposing the strong constraint*, satisfies

$$L_{\epsilon_1} L_{\epsilon_2} - L_{\epsilon_2} L_{\epsilon_1} = L_{[\![\epsilon_1,\epsilon_2]\!]} \ ,$$

with the bracket operation, called the C-bracket, being

$$[\![\epsilon_1, \epsilon_2]\!]^J = \epsilon_1^K \partial_K \epsilon_2^J - \tfrac{1}{2}\epsilon_1^K \partial^J \epsilon_{2K} - (\epsilon_1 \leftrightarrow \epsilon_2) \ .$$

Solving the strong constraint amounts to eliminating half of the coordinates.

Then, DFT reduces to ordinary sugra in different duality frames.

$\rightsquigarrow$ the geometric structure reduces to a Courant Algebroid (C-bracket $\xrightarrow{s.c.}$ Courant bracket &c.)

But, what is the underlying geometric structure of DFT, and where does it come from?

Can the strong constraint be relaxed?

Courant Algebroids

Definition of a Courant Algebroid

Courant '90; Liu, Weinstein, Xu '97

$(E \xrightarrow{\pi} M, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \rho : E \to TM)$, such that for $A, B, C \in \Gamma(E)$ and $f, g \in C^\infty(M)$:

1. Modified Jacobi identity where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is derivative defined by $\langle \mathcal{D}f, A\rangle = \frac{1}{2}\rho(A)f$.

$$[[A,B],C] + \text{c.p.} = \mathcal{D}\mathcal{N}(A,B,C) \ , \quad \text{where} \quad \mathcal{N}(A,B,C) = \tfrac{1}{3}\langle [A,B], C\rangle + \text{c.p.} \ ,$$

2. Modified Leibniz rule

$$[A, fB] = f[A,B] + (\rho(A)f)B - \langle A, B\rangle \mathcal{D}f \ ,$$

3. Compatibility condition

$$\rho(C)\langle A,B\rangle = \langle [C,A] + \mathcal{D}\langle C,A\rangle, B\rangle + \langle [C,B] + \mathcal{D}\langle C,B\rangle, A\rangle \ ,$$

The structures also satisfy the following properties (they follow...):

4. Homomorphism

$$\rho[A,B] = [\rho(A), \rho(B)] \ ,$$

5. "Strong constraint"

$$\rho \circ \mathcal{D} = 0 \quad \Leftrightarrow \quad \langle \mathcal{D}f, \mathcal{D}g\rangle = 0 \ .$$

Alternative definition of a Courant Algebroid

Ševera '98

Definition in terms of a bilinear, non-skew operation (Dorfman derivative)

$$[A, B] = A \circ B - B \circ A \ ,$$

notably satisfying instead of 1, the Jacobi identity (in Loday-Leibniz form):

$$A \circ (B \circ C) = (A \circ B) \circ C + B \circ (A \circ C) \ .$$

Axioms 2 and 3 do not contain $\mathcal{D}$-terms any longer,

$$\begin{aligned} A \circ fB &= f(A \circ B) + (\rho(A)f)B \ , \\ \rho(C)\langle A, B\rangle &= \langle C \circ A, B\rangle + \langle C \circ B, A\rangle \ . \end{aligned}$$

with two additional properties:

$$\begin{aligned} \rho(A \circ B) &= [\rho(A), \rho(B)] \ , \\ A \circ A &= \mathcal{D}\langle A, A\rangle \ . \end{aligned}$$

The two definitions are equivalent, as proven by Roytenberg '99

In a local basis (e^I) of $\Gamma(E)$, $I = 1, \ldots, 2d$, we can write the local form of the operations:

$$
\begin{aligned}
[e^I, e^J] &= \eta^{IK}\eta^{JL} T_{KLM} e^M \ , \\
\langle e^I, e^J \rangle &= \tfrac{1}{2}\eta^{IJ} \ , \\
\rho(e^I) f &= \eta^{IJ} \rho^i{}_J \partial_i f \ , \\
\mathcal{D} f &= \mathcal{D}_I f \ e^I = \rho^i{}_I \partial_i f \ e^I \ ,
\end{aligned}
$$

with $(\rho^i{}_J)$ the anchor components. The axioms and properties of a CA take the form:

$$
\begin{aligned}
&\eta^{IJ} \rho^i{}_I \rho^j{}_J = 0 \ , \\
&\rho^i{}_I \partial_i \rho^j{}_J - \rho^i{}_J \partial_i \rho^j{}_I - \eta^{KL} \rho^i{}_K T_{LIJ} = 0 \ , \\
&4 \rho^i{}_{[L} \partial_i T_{IJK]} - 3 \eta^{MN} T_{M[IJ} T_{KL]N} = 0 \ .
\end{aligned}
$$

Brackets for standard and non-standard CAs

The *standard* CA: $E = TM \oplus T^*M$, $\rho = (\mathrm{id}, 0)$, and $T_{IJK} = (H_{ijk}, 0, 0, 0)$ with $\mathrm{d}H = 0$. For this case, the (Courant) bracket is

$$\begin{aligned} [A, B]_s &= [A_V, B_V] + \mathcal{L}_{A_V} B_F - \mathcal{L}_{B_V} A_F - \tfrac{1}{2}\mathrm{d}(\iota_{A_V} B_F - \iota_{B_V} A_F) + H(A_V, B_V) \\ &= A^l \partial_l B^k \partial_k + (A^l \partial_l B_k + \tfrac{1}{2} A^l \partial_k B_l)\mathrm{d}x^k - (A \leftrightarrow B) + A^l B^m H_{lmk} \mathrm{d}x^k \ , \end{aligned}$$

where $A = A_V + A_F \in \Gamma(E)$, with $A_V \in \Gamma(TM)$ and $A_F \in \Gamma(T^*M)$.

Another simple example: $\rho = (0, \Pi^\sharp)$, $T_{IJK} = (0, 0, 0, R^{ijk})$ with $[\Pi, \Pi]_S = [\Pi, R]_S = 0$. Here, Π is a Poisson 2-vector. The bracket is not any longer the standard Courant.

In general, the Courant bracket is given by an expression of the form Liu, Weinstein, Xu

$$\begin{aligned} [A, B] &= [A_V, B_V] + \mathcal{L}_{A_F} B_V - \mathcal{L}_{B_F} A_V + \tfrac{1}{2}\mathrm{d}_*(\iota_{A_V} B_F - \iota_{B_V} A_F) \\ &+ [A_F, B_F] + \mathcal{L}_{A_V} B_F - \mathcal{L}_{B_V} A_F - \tfrac{1}{2}\mathrm{d}(\iota_{A_V} B_F - \iota_{B_V} A_F) + T(A, B) \ , \\ &= \left(\rho^i{}_J (A^J \partial_i B_K - B^J \partial_i A_K) - \tfrac{1}{2}\rho^i{}_K (A^J \partial_i B_J - B^J \partial_i A_J)\right) e^K + A^L B^M T_{LMK} e^K \ . \end{aligned}$$

AKSZ sigma models

AKSZ sigma model - topological sigma models satisfying the classical master equation Alexandrov, Kontsevich, A. Schwarz, Zaboronsky '97.

In 2d Poisson sigma model is most general TFT Cattaneo, Felder '01. Quantization of this model lead to Kontsevich deformation quantization formula.

In 3d the AKSZ sigma model requires a dg-manifolds for source and target, symplectic form on a target, and a self-commuting hamiltonian of degree 3. Given the data of a CA, $(E, [\cdot,\cdot]_E, \langle\cdot,\cdot\rangle_E, \rho)$, one can uniquely construct a membrane sigma model Roytenberg '06:

$$S[X,A,F] = \int_{\Sigma_3} \left(\langle F, \mathrm{d}X \rangle + \langle A, \mathrm{d}A \rangle_E - \langle F, \rho(A) \rangle + \tfrac{1}{3} \langle A, [A,A]_E \rangle_E \right)$$

For a manifold with boundaries on can add both topological and non-topological terms Cattaneo, Felder '01; Park '00

$$S_b[X,A] = \int_{\partial\Sigma_3} \tfrac{1}{2} g_{IJ} A^I \wedge *A^J + \tfrac{1}{2} \mathcal{B}_{IJ} A^I \wedge A^J \ .$$

In local coordinates

$$S[X,A,F]=\int_{\Sigma_3} F_i\wedge \mathrm{d}X^i+\tfrac{1}{2}\eta_{IJ}A^I\wedge \mathrm{d}A^J-\rho^i{}_I(X)A^I\wedge F_i+\tfrac{1}{6}T_{IJK}(X)A^I\wedge A^J\wedge A^K\ .$$

$i=1,\dots,d$ (target space index) and $I=1,\dots,2d$ (CA index).

Maps $X=(X^i):\Sigma_3\to M$, 1-forms $A\in\Omega^1(\Sigma_3,X^*E)$, and auxiliary 2-form $F\in\Omega^2(\Sigma_3,X^*T^*M)$.
Symmetric bilinear form of the CA $\rightsquigarrow$ $O(d,d)$ invariant metric

$$\eta=(\eta_{IJ})=\begin{pmatrix}0 & 1_d\\ 1_d & 0\end{pmatrix}\ ,$$

ρ and T are the anchor and twist of the CA, the latter generating a generalized Wess-Zumino term.

Gauge invariance of the Courant sigma model $\Rightarrow$ CA axioms and properties.

The Burger Proposal

The relation of DFT and CAs

Solving the s.c. by elimination of $\widetilde{x}$, i.e. $\widetilde{\partial}^i = 0$, takes us from DFT to the standard CA.

Indeed, the C-bracket/gen. Lie derivative reduces then to the Courant/Dorfman bracket.

However, CAs double the bundle, DFT doubles the space.

What if we take a CA on doubled space?

- Geometric origin of the DFT operations and the strong constraint?
- Definition of a DFT algebroid?

Our proposal is instead that the DFT geometry should lie "in between" two CAs.

$\longleftarrow$ Large CA over $M \times \widetilde{M}$

$\longleftarrow$ Projection

$\longleftarrow$ DFT structure

$\longleftarrow$ Strong Constraint

$\longleftarrow$ Canonical CA over M

Doubling and rewriting

In order to relate to DFT, we consider a Courant algebroid over the doubled space.

At least locally, we can work with a 2^{nd} order bundle $\mathbb{E} = (T \oplus T^*)\mathcal{M}$, over $\mathcal{M} = T^*M$.

In order to reveal the DFT structure, for simplicity start with the standard CA over $\mathcal{M}$. A section $\mathbb{A} \in \mathbb{E}$ is

$$\mathbb{A} := \mathbb{A}_V + \mathbb{A}_F = \mathbb{A}^I \partial_I + \widetilde{\mathbb{A}}_I \mathrm{d}\mathbb{X}^I \ .$$

Now introduce the following combinations: (N.B. η_{IJ} is not the metric of the CA over $\mathcal{M}$)

$$\mathbb{A}^I_{\pm} = \tfrac{1}{2}(\mathbb{A}^I \pm \eta^{IJ} \widetilde{\mathbb{A}}_J) \ .$$

Strategy: rewrite all structural data of $\mathbb{E}$ in terms of $\mathbb{A}_{\pm}$.

Projected sections and bilinear

Starting with sections of the large CA:

$$\mathbb{A} = \mathbb{A}_+^I e_I^+ + \mathbb{A}_-^I e_I^- \ , \quad \text{where} \quad e_I^\pm = \partial_I \pm \eta_{IJ} \mathrm{d}\mathbb{X}^J \ ,$$

a projection to the subbundle L_+ spanned by local sections (e_I^+)

$$\begin{aligned} p : \mathbb{E} \quad &\to \quad L_+ \\ (\mathbb{A}_V, \mathbb{A}_F) \quad &\mapsto \quad \mathbb{A}_+ := A \ , \end{aligned}$$

leads exactly to the form of a DFT $O(d,d)$ vector

$$A = A_i(\mathrm{d}X^i + \tilde{\partial}^i) + A^i(\mathrm{d}P_i + \partial_i) \ .$$

Projection of the symmetric bilinear of $\mathbb{E}$, leads to the O(d,d) invariant DFT metric:

$$\langle \mathbb{A}, \mathbb{B} \rangle_{\mathbb{E}} = \tfrac{1}{2} \eta_{\hat{I}\hat{J}} \mathbb{A}^{\hat{I}} \mathbb{B}^{\hat{J}} = \eta_{IJ}(\mathbb{A}_+^I \mathbb{B}_+^J - \mathbb{A}_-^I \mathbb{B}_-^J) \quad \mapsto \quad \eta_{IJ} A^I B^J = \langle A, B \rangle_{L_+} \ ,$$

where $\hat{I} = 1, \ldots, 4d$, while $I = 1, \ldots, 2d$.

Projected brackets

Rewriting the Courant bracket on $\mathbb{E}$ in terms of the $\pm$ components:

$$\begin{aligned} [\mathbb{A},\mathbb{B}]_E &= \eta_{IK}(\mathbb{A}_+^K\partial^I\mathbb{B}_+^L - \mathbb{A}_-^K\partial^I\mathbb{B}_+^L - \tfrac{1}{2}(\mathbb{A}_+^K\partial^L\mathbb{B}_+^I - \mathbb{A}_-^K\partial^L\mathbb{B}_-^I) - \{\mathbb{A}\leftrightarrow\mathbb{B}\})e_L^+ + \\ &+ \eta_{IK}(\mathbb{A}_+^K\partial^I\mathbb{B}_-^L - \mathbb{A}_-^K\partial^I\mathbb{B}_-^L + \tfrac{1}{2}(\mathbb{A}_+^K\partial^L\mathbb{B}_+^I - \mathbb{A}_-^K\partial^L\mathbb{B}_-^I) - \{\mathbb{A}\leftrightarrow\mathbb{B}\})e_L^- \ . \end{aligned}$$

The C-bracket of DFT is obtained from the large standard Courant bracket as:

$$[\![A,B]\!] = p[p(\mathbb{A}),p(\mathbb{B})]_{\mathbb{E}} \ .$$

(L_+ is not an involutive subbundle, thus neither a Dirac structure of $\mathbb{E}$.)

Projection of the Dorfman derivative on $\mathbb{E}$ to the generalised Lie derivative of DFT:

$$L_A B = p\left(p(\mathbb{A})\circ p(\mathbb{B})\right) \ .$$

Projection of the anchor map $\rho^I{}_J = (\rho^I{}_J, \widetilde{\rho}^{IJ})$

$$(\rho_\pm)^I{}_J = \rho^I{}_J \pm \eta_{JK}\widetilde{\rho}^{IK}$$

Thus, the map p sends all CA structures to the corresponding DFT structures.

Membrane Sigma Model for DFT

The DFT Membrane Sigma Model

Using the data of DFT algebroid we propose cf. Chatzistavrakidis, Jonke, Lechtenfeld '15

$$S[\mathbb{X}, A, F] = \int \Big(F_I \wedge \mathrm{d}\mathbb{X}^I + \eta_{IJ} A^I \wedge \mathrm{d}A^J - (\rho_+)^I{}_J A^J \wedge F_I + \tfrac{1}{3} \widehat{T}_{IJK} A^I \wedge A^J \wedge A^K \Big) \ ,$$

where $\rho_+ : L_+ \to T\mathcal{M}$ is a map to the tangent bundle and $\widehat{T}$ corresponds to DFT fluxes.

Maps $\mathbb{X} = (\mathbb{X}^I) : \Sigma_3 \to \mathcal{M}$, 1-forms $A \in \Omega^1(\Sigma_3, \mathbb{X}^* L_+)$, and auxiliary 2-form $F \in \Omega^2(\Sigma_3, \mathbb{X}^* T^* \mathcal{M})$.

Symmetric bilinear form η is $O(d,d)$ invariant metric.

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Symmetric bilinear form η is $O(d, d)$ invariant metric.

We use the DFT MSM to

- describe the flux backgrounds
- find the relation with flux formulation of DFT

Consider a doubled torus as target of the DFT MSM and DFT structural data as $(\rho_+)^I{}_J = (\rho^i{}_j, \rho^{ij}, \rho_i{}^j, \rho_{ij})$, $T_{IJK} = (H_{ijk}, f_{ij}{}^k, Q_i{}^{jk}, R^{ijk})$,

Add symmetric term on boundary $g_{IJ} = (g_{ij}, g_i{}^j, g^i{}_j, g^{ij})$,

Expand coordinate and vector components $\mathbb{X}^I = (X^i, \widetilde{X}_i)$, $A^I = (q^i, p_i)$,

Goal here is to describe the standard T-duality chain relating geometric and non-geometric fluxes schematically through Shelton, Taylor, Wecht '05

$$H_{ijk} \xleftrightarrow{\mathrm{T}_k} f_{ij}{}^k \xleftrightarrow{\mathrm{T}_j} Q_i{}^{jk} \xleftrightarrow{\mathrm{T}_i} R^{ijk} \ ,$$

using DFT membrane action.

H flux background

Choose

$$(\rho_+)^I{}_J = (\delta^i{}_j, 0, 0, 0) \ , \quad T_{IJK} = (H_{ijk}, 0, 0, 0) \ , \quad g_{IJ} = (0, 0, 0, g^{ij}) \ .$$

Then the membrane action becomes

$$\begin{aligned} S_{\mathrm{DFT}} \quad &= \quad \int_{\Sigma_3} \big(F_I \wedge \mathrm{d}\mathbb{X}^I + q^i \wedge \mathrm{d}p_i + p_i \wedge \mathrm{d}q^i - q^i \wedge F_i + \tfrac{1}{6}\, H_{ijk}\, q^i \wedge q^j \wedge q^k \big) \\ &\quad + \int_{\partial\Sigma_3} \tfrac{1}{2}\, g^{ij}\, p_i \wedge *p_j \ . \end{aligned}$$

The on-shell membrane theory $\rightsquigarrow$ integrate F_I

$$q^i = \mathrm{d}X^i \qquad \text{and} \qquad \mathrm{d}\widetilde{X}_i = 0 \ .$$

The action now takes the form

$$\int_{\partial\Sigma_3} \big(p_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\, g^{ij}\, p_i \wedge *p_j \big) + \int_{\Sigma_3} \tfrac{1}{6}\, H_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \ ,$$

which, after integrating out p_i using $*^2 = 1$, takes precisely the desired form

$$S_H[X] := \int_{\partial\Sigma_3} \tfrac{1}{2}\, g_{ij}\, \mathrm{d}X^i \wedge *\mathrm{d}X^j + \int_{\Sigma_3} \tfrac{1}{6}\, H_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k$$

We choose

$$(\rho_+)^I{}_J = \left(\delta^i{}_j, R^{ijk}\,\widetilde{X}_k, -\delta_i{}^j, 0\right) \qquad T_{IJK} = (0,0,0,R^{ijk}) \qquad g_{IJ} = (0,0,0,g^{ij}) \ .$$

The topological part of the membrane action becomes

$$\begin{aligned} S \quad &= \quad \int_{\Sigma_3} \big(F_I \wedge \mathrm{d}\mathbb{X}^I + q^i \wedge \mathrm{d}p_i + p_i \wedge \mathrm{d}q^i - q^i \wedge F_i + p_i \wedge F^i \\ &\qquad\qquad - R^{ijk}\,\widetilde{X}_k\, p_j \wedge F_i + \tfrac{1}{6}\, R^{ijk}\, p_i \wedge p_j \wedge p_k \big) \ . \end{aligned}$$

Integrating out the auxiliary fields F_I gives

$$q^i = \mathrm{d}X^i - R^{ijk}\,\widetilde{X}_k\, p_j \qquad \text{and} \qquad p_i = -\mathrm{d}\widetilde{X}_i \ ,$$

leading to

$$S_R[X,\widetilde{X}\,] = \int_{\partial\Sigma_3} \left(\mathrm{d}\widetilde{X}_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\, R^{ijk}\,\widetilde{X}_k\, \mathrm{d}\widetilde{X}_i \wedge \mathrm{d}\widetilde{X}_j + \tfrac{1}{2}\, g^{ij}\, \mathrm{d}\widetilde{X}_i \wedge *\mathrm{d}\widetilde{X}_j \right) \ ,$$

The resulting action was proposed by Mylonas, Schupp, Szabo '12.

R flux and nonassociativity

They defined a bivector $\Theta = \frac{1}{2}\,\Theta^{IJ}\,\partial_I \wedge \partial_J$ on phase space T^*M given by

$$\Theta^{IJ} = \begin{pmatrix} R^{ijk}\,\widetilde{X}_k & \delta^i{}_j \\ -\delta_i{}^j & 0 \end{pmatrix} \ .$$

It induces a twisted Poisson bracket given by

$$\{\mathbb{X}^I, \mathbb{X}^J\}_\Theta = \Theta^{IJ} \ ,$$

with non-vanishing Jacobiator

$$\{X^i, X^j, X^k\}_\Theta := \tfrac{1}{3}\,\{\{X^i, X^j\}_\Theta, X^k\}_\Theta + \text{cyclic} = R^{ijk} \ .$$

Alternatively, modify the anchor to

$$(\rho_+)^I{}_J = (0, 0, \delta_i{}^j, 0) \ .$$

The resulting worldsheet action

$$S_R[\widetilde{X}\,] = \int_{\partial\Sigma_3} \tfrac{1}{2}\,g^{ij}\,\mathrm{d}\widetilde{X}_i \wedge *\mathrm{d}\widetilde{X}_j + \int_{\Sigma_3} \tfrac{1}{6}\,R^{ijk}\,\mathrm{d}\widetilde{X}_i \wedge \mathrm{d}\widetilde{X}_j \wedge \mathrm{d}\widetilde{X}_k \ ,$$

is the same as the sigma-model action with H-flux under the duality exchanges of all fields X^i with $\widetilde{X}_i$.

Flux backgrounds from DFT MSM - Summary

In terms of the doubled space of DFT, the four T-dual backgrounds with H-, f-, Q- and R-flux all correspond to the *standard Courant algebroid* over different submanifolds of the doubled space.

This does not include the nonassociative (nor noncommutative) models, which violate the strong constraint of DFT and therefore do not correspond to Courant sigma-models.

Global properties important additional input.

Taking a parametrization of the ρ_+ components to be $(\rho_+)^I{}_J = (\delta^i{}_j, \beta^{ij}, \delta_i{}^j + \beta^{jk}B_{ki}, B_{ij})$, we make contact with flux formulation of DFT Geissbuhler, Marques, Nunez, Penas '13. This gives

$$\eta^{IJ}\rho^K{}_I\rho^L{}_J = \eta^{KL}$$
$$2\rho^L{}_{[I}\partial_{\underline{L}}\rho^K{}_{J]} - \rho^K{}_J\eta^{JL}\hat{T}_{LIJ} = \rho_{L[I}\partial^K\rho^L{}_{J]} \ .$$

Considering gauge transformations of the form (here $\hat{T} = \frac{1}{2}T$.)

$$\delta_\epsilon \mathbb{X}^I = \rho^I{}_J(\mathbb{X})\epsilon^J \ ,$$
$$\delta_\epsilon A^I = \mathrm{d}\epsilon^I + \eta^{IJ}\hat{T}_{JKL}(\mathbb{X})A^K\epsilon^L \ ,$$
$$\delta_\epsilon F_K = -\epsilon^J(\partial_K\rho^I{}_J F_I - \partial_K\hat{T}_{ILJ}A^I\wedge A^L) \ ,$$

gives the following necessary conditions for gauge invariance of the DFT MSM action:

$$4\rho^M{}_{[L}\partial_{\underline{M}}\hat{T}_{IJK]} + 3\eta^{MN}\hat{T}_{M[IJ}\hat{T}_{KL]N} = \mathcal{Z}_{IJKL} \ .$$

Sufficiency also requires use of the strong constraint.

$\rightsquigarrow$ Strong constraint needed for gauge invariance of DFT sigma model and for (on-shell) closure of gauge algebra.

Algebroid structure of DFT

Towards a DFT Algebroid structure

Strategy: Replace $[\cdot,\cdot]_E \to [\![\cdot,\cdot]\!]$, $\langle\cdot,\cdot\rangle_E \to \langle\cdot,\cdot\rangle_{L_+}$ and $\rho \to \rho_+$, and also define $\mathcal{D}_+$ as

$$\langle A, \mathcal{D}_+ f\rangle_{L_+} = \tfrac{1}{2}\rho_+(A) f \ ,$$

and determine one by one the obstructions to the CA axioms and properties.

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1. Modified Jacobi identity ($\mathcal{N}(A,B,C) = \frac{1}{3}\langle [\![A,B]\!], C\rangle_{L_+} + \text{c.p.}$) $\rightsquigarrow$ obstructed

   $$[\![[\![A,B]\!],C]\!] + \text{c.p.} = \mathcal{D}_+\mathcal{N}(A,B,C) + \mathcal{Z}(A,B,C) + SC_1(A,B,C) \ ,$$

   where the last term (which vanishes on the strong constraint) is explicitly given by

   $$\begin{aligned} SC_1(A,B,C)^L = &-\tfrac{1}{2}\left(A^I\partial_J B_I \partial^J C^L - B^I \partial_J A_I \partial^J C^L\right) - \\ - \ & \rho_{I[J}\partial_M \rho^I{}_{N]}\left(A^J B^N \partial^M C^L - \tfrac{1}{2}C^J A^K \partial^M B_K \eta^{NL} + \tfrac{1}{2}C^J B^K \partial^M A_K \eta^{NL}\right) + \\ + \ & \text{c.p.}(A,B,C) \ . \end{aligned}$$

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2. Modified Leibniz rule $\rightsquigarrow$ unobstructed

   $$[\![A, fB]\!] = f[\![A,B]\!] + (\rho_+(A) f)\, B - \langle A, B\rangle_{L_+}\mathcal{D}_+ f \ .$$

3. Compatibility condition $\rightsquigarrow$ unobstructed

$$\langle [\![C,A]\!] + \mathcal{D}_+ \langle C,A\rangle_{L_+}, B\rangle_{L_+} + \langle [\![C,B]\!] + \mathcal{D}_+ \langle C,B\rangle_{L_+}, A\rangle_{L_+} = \rho_+(C)\langle A,B\rangle_{L_+} \ .$$

Towards a DFT Algebroid structure

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4. Homomorphism $\rightsquigarrow$ obstructed

$$\rho_+[\![A,B]\!] = [\rho_+(A), \rho_+(B)] + SC_2(A,B) \ ,$$

where the last term (which vanishes on the strong constraint) is explicitly given by

$$SC_2(A,B) = \left(\rho_{L[I}\partial^K \rho^L{}_{J]} A^I B^J + \tfrac{1}{2}\left(A^I \partial^K B_I - B^I \partial^K A_I\right)\right)\partial_K \ .$$

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5. "Strong constraint" $\rightsquigarrow$ obstructed

$$\langle \mathcal{D}_+ f, \mathcal{D}_+ g\rangle_{L_+} = \tfrac{1}{4}\langle \mathrm{d}f, \mathrm{d}g\rangle_{L_+} = \eta^{IJ}\rho^K{}_I\rho^L{}_J\partial_K f\partial_L g = \partial^L f\partial_L g \ .$$

A DFT Algebroid

Let M be d-dimensional manifold. A DFT algebroid on $\mathrm{T}^*\mathrm{M}$ is a quadruple $(L_+, [\![\cdot,\cdot]\!], \langle\cdot,\cdot\rangle_{L_+}, \rho_+)$, where L_+ is rank-$(2d)$ vector bundle over $\mathrm{T}^*\mathrm{M}$ equipped with a skew-symmetric bracket $[\![\cdot,\cdot]\!] : \Gamma(L_+)\otimes\Gamma(L_+)\to\Gamma(L_+)$, a non-degenerate symmetric form $\langle\cdot,\cdot\rangle_{L_+} : \Gamma(L_+)\otimes\Gamma(L_+)\to C^\infty(\mathrm{T}^*\mathrm{M})$ and a smooth bundle map $\rho_+ : L_+ \to \mathrm{T}(\mathrm{T}^*\mathrm{M})$, such that

1. $\langle\mathcal{D}_+ f, \mathcal{D}_+ g\rangle_{L_+} = \frac{1}{4}\langle \mathrm{d}f, \mathrm{d}g\rangle_{L_+}$,
2. $[\![A, fB]\!] = f[\![A,B]\!] + (\rho_+(A)f)\,B - \langle A,B\rangle_{L_+}\mathcal{D}_+ f$,
3. $\langle[\![C,A]\!] + \mathcal{D}_+\langle C,A\rangle_{L_+}, B\rangle_{L_+} + \langle[\![C,B]\!] + \mathcal{D}_+\langle C,B\rangle_{L_+}, A\rangle_{L_+} = \rho_+(C)\langle A,B\rangle_{L_+}$,

for all $A, B, C \in \Gamma(L_+)$ and $f \in C^\infty(\mathrm{T}^*\mathrm{M})$, where $\mathcal{D}_+ : C^\infty(\mathrm{T}^*\mathrm{M}) \to \Gamma(L_+)$ is the derivative defined through $\langle A, \mathcal{D}_+ f\rangle_{L_+} = \frac{1}{2}\rho_+(A)f$.

When the s.c. is imposed, it reduces to a CA and ρ_+ becomes a homomorphism.

A DFT Algebroid - discussion

A DFT algebroid is an example of the structure we call *pre-DFT algebroid*.

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Recall the definition of a Courant algebroid:
$(E \xrightarrow{\pi} M, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \rho : E \to TM)$, such that for $A, B, C \in \Gamma(E)$ and $f, g \in C^\infty(M)$:

1. Modified Jacobi identity where $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined by $\langle \mathcal{D}f, A\rangle = \frac{1}{2}\rho(A)f$.
$$[[A,B],C] + \text{c.p.} = \mathcal{D}\mathcal{N}(A,B,C) \ , \quad \text{where} \quad \mathcal{N}(A,B,C) = \tfrac{1}{3}\langle [A,B], C\rangle + \text{c.p.} \ ,$$
2. Modified Leibniz rule
$$[A, fB] = f[A,B] + (\rho(A)f)B - \langle A, B\rangle \mathcal{D}f \ ,$$
3. Compatibility condition
$$\rho(C)\langle A,B\rangle = \langle [C,A] + \mathcal{D}\langle C,A\rangle, B\rangle + \langle [C,B] + \mathcal{D}\langle C,B\rangle, A\rangle \ ,$$
4. Homomorphism $\qquad \rho[A,B] = [\rho(A), \rho(B)]$,
5. *Ker*(ρ) $\qquad \rho \circ \mathcal{D} = 0 \quad \Leftrightarrow \quad \langle \mathcal{D}f, \mathcal{D}g\rangle = 0$.

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A DFT Algebroid - discussion

In general, one can obstruct the properties 1, 4, 5 in an independent way!

$$\text{Pre-DFT algebroid} \xleftarrow{\not 5} \text{Ante-Courant algebroid} \xleftarrow{\not 4} \text{Pre-Courant algebroid} \xleftarrow{\not 1} \text{Courant algebroid}$$

DFT algebroid is a special case of pre-DFT algebroid where all three obstructions are controled by the strong constraint.

Relaxing the strong constraint?

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Pre-Courant algebroid was introduced by Vaisman Vaisman '05; cf. Hansen, Strobl '09; Bruce, Grabowski '16.

Pre-DFT algebroid corresponds to metric algebroid defined in terms of Dorman bracket Vaisman '12.

Closing remarks

- We showed that the geometric structure of DFT is in between two Courant algebroids. It is an example of pre-DFT algebroid related to the metric algebroid of Vaisman.
- We proposed membrane sigma model for DFT, gauge invariant under the strong constraint. It provides universal description of geometric and non-geometric fluxes.
- Gauge structure of DFT membrane sigma model needs further study. In particular, can we propose gauge invariant action violating the strong constriant and realize non-associative R-flux in that framework?
- The global properties might be understood in the framework of para-Hermitean geometry (Vaisman '12; Freidel, Rudolph, Svoboda '17; Svoboda '18), where symplectic structure plays the key role.