Dual basis of $c_q[SL_2]$

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Motivation

- Famous quantum group $u_q(sl_2)$ at root of unity, n odd:

$$q^n = 1, \quad E^n = F^n = 0, \quad K^n = 1$$

$$KEK^{-1} = q^{-2}E, \quad KFK^{-1} = q^2 F, \quad [E, F] = K - K^{-1}$$

$$\Delta K = K \otimes K, \quad \Delta E = 1 \otimes E + E \otimes K, \quad \Delta F = K \otimes F + F \otimes 1$$

- **Quasitriangular** : $\mathcal{R} = \frac{1}{n} \sum_{r,a,b=0}^{n-1} \frac{(-1)^r q^{-2ab}}{[r]_{q^{-2}}!} F^r K^a \otimes E^r K^b$
- **PBW Basis** : $\{F^i K^j E^k\}_{0 \leq i,j,k < n}$

Motivation

- Dual to $u_q(sl_2)$, $c_q[SL_2]$ at odd root of unity :

$$q^n = 1, \quad a^n = d^n = 1, \quad b^n = c^n = 0$$

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd, \quad cb = bc$$

$$da - ad = (q - q^{-1})bc, \quad ad - q^{-1}bc = 1$$

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- Has basis $\{b^i c^j d^k\}_{0 \leq i,j,k < n}$

- Duality (up to normalisation)

$$\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, K \rangle = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}, \quad \langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, E \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix}, F \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- $\langle b^2, F^2 \rangle = \langle \Delta b^2, F \otimes F \rangle =$ $\langle a^2, F \rangle \langle b^2, F^2 \rangle + (1+q^2)\langle ab, F \rangle \langle bd, F \rangle + \langle b^2, F \rangle \langle d^2, F \rangle = \cdots$
- $\langle b^i c^j d^k, F^{i'} K^{j'} E^{k'} \rangle =???$

Problem

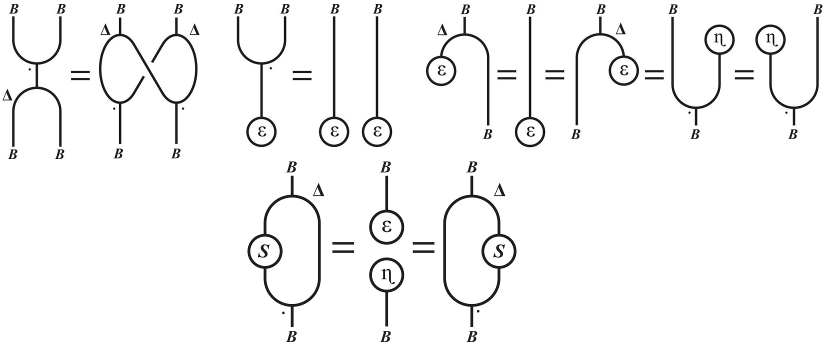
Find basis of $c_q[SL_2]$ dual to PBW basis of $u_q(sl_2)$.

Approach

(Co)-double Bosonisation

(Braided) Hopf algebra

- **Braided tensor category** $\mathcal{C}, \otimes, \Psi : V \otimes W \to W \otimes V$, obeys some axioms
- **Braided Hopf algebra**(SM'90s) $B \in \mathcal{C}$ is algebra, $\underline{\Delta} : B \to B\underline{\otimes}B$, $\underline{\epsilon} : B \to \underline{1}$, $\underline{S} : B \to B$



Double Bosonisation

- (SM'95) H quasitri.Hopf alg. $\Longrightarrow \mathcal{C} = \mathcal{M}_H \overset{B \in \mathcal{M}_H}{\Longrightarrow}$ ${}_{B'}\mathcal{M}_B \overset{\text{Tannaka-Krein}}{\Longrightarrow} Dbos(H) = B^{*\underline{\mathrm{cop}}} \bowtie H \bowtie B.$
- Cross relation between B and $B^{*\underline{\mathrm{cop}}}$:

$$bc = (\mathcal{R}_1^{(2)} \triangleright c_{\overline{(2)}})\mathcal{R}_2^{(2)}\mathcal{R}_1^{-(1)}(b_{\underline{(2)}} \triangleleft \mathcal{R}_2^{-(1)})\langle \mathcal{R}_1^{(1)} \triangleright c_{\overline{(1)}}, b_{\underline{(1)}} \triangleleft \mathcal{R}_2^{(1)}\rangle$$
$$\langle \mathcal{R}_1^{-(2)} \triangleright \bar{S}c_{\overline{(3)}}, b_{\underline{(3)}} \triangleleft \mathcal{R}_2^{-(2)}\rangle$$

- Quasitriangularity :

$$\mathcal{R}_{new} = \mathcal{R}_H.\overline{\exp},$$

$$\overline{\exp} = \sum_a f^a \otimes \underline{S}e_a$$

- **Example** (RA/SM) $H = \mathbb{C}_q\mathbb{Z}_n = \mathbb{C}[K]/(K^n - 1)$.
  $B = \mathbb{C}[E]/(E^n), E \triangleright K = qE$. $B^* = \mathbb{C}[F]/(F^n)$, $K \triangleleft F = qF$.
- **Double bosonisation** $Dbos(H) = \mathfrak{u}_q(sl_2)$

$$E^n = F^n = 0, \quad K^n = 1$$

$$[E, F] = K - K^{-1}, \quad KEK^{-1} = q^{-1}E, \quad KFK^{-1} = qF$$

$$\Delta K = K \otimes K, \quad \Delta E = 1 \otimes E + K \otimes K, \quad \Delta F = K^{-1} \otimes F + F \otimes 1$$

$$\mathcal{R}_{\mathfrak{u}_q(sl_2)} = \frac{1}{n} \sum_{r,a,b=0}^{n-1} \frac{(-1)^r q^{-ab}}{[r]_{q^{-1}}!} F^r K^a \otimes E^r K^b$$

- If $n = 2m + 1$, we have $\mathfrak{u}_q(sl_2) \cong u_{q^{-m}}(sl_2)$.
- If $n = 2m$, we have new quasitriangular Hopf algebra.
- Example : for $n = 2$ i.e. $q = -1$, $\mathfrak{u}_{-1}(sl_2)$ is 8-dimensional self-dual strictly quasitriangular

$$\mathcal{R}_{\mathfrak{u}_{-1}(sl_2)} = (1 \otimes 1 - F \otimes E)\mathcal{R}_K \neq 1 \otimes 1$$

$$E^2 = F^2 = 0, \quad K^2 = 1, \quad [E, F] = 0, \quad KE = -EK, FK = -KF$$

Co-double Bosonisation

- A coquasitri. Hopf Alg. $\Longrightarrow {}^{A}\mathcal{M} \overset{B\in {}^{A}\mathcal{M}}{\Longrightarrow} {}_{B'}\mathcal{M}_{B} \Longrightarrow$
  $coDbos(A) = \underline{B^{\mathrm{op}}} \rtimes\!\!\!\lhd A \rhd\!\!\!\ltimes B^{*}$
- $\underline{B^{\mathrm{op}}}$ and B^{*} commute.
- **Example** : Let $A = \mathbb{C}_q[t, t^{-1}]/(t^n - 1)$.
  $B = \mathbb{C}[X]/(X^n) \in {}^{A}\mathcal{M}$. $B^{*} = \mathbb{C}[Y]/(Y^n) \in \mathcal{M}^{A}$

$$\Delta t = t \otimes t, \quad \mathcal{R}(t^r, t^s) = q^{rs} \quad , \Delta_L X = t \otimes X, \quad \Delta_R Y = Y \otimes t$$

$$\underline{\Delta} X = 1 \otimes X + X \otimes 1, \Psi_L(X^r \otimes X^s) = q^{rs} X^s \otimes X^r.$$

$$\underline{\Delta} Y = 1 \otimes Y + Y \otimes 1, \Psi_R(Y^r \otimes Y^s) = q^{rs} Y^s \otimes Y^r.$$

Then its co-double bosonisation is $coDbos(A) = \mathfrak{c}_q[SL_2]$ such that

$$X^n = Y^n = 0, t^n = 1$$

$$XY = YX, \quad Xt = qtX, \quad Yt = qtY.$$

$$\Delta t = q^{-1} t \left(\frac{1}{(q^{-1} - (1 - q^{-1})Y \otimes X)(1 - (1 - q^{-1})Y \otimes X} \right) t$$

$$\Delta X = X \otimes 1 + t \left(\frac{1}{1 - (q - 1)Y \otimes X} \right) X$$

$$\Delta Y = 1 \otimes Y + Y \left(\frac{1}{1 - (1 - q^{-1})Y \otimes X} \right) t.$$

- If $n = 2m+1$, $\mathfrak{c}_q[SL_2] \cong c_{q^{-m}}[SL_2]$ with

$$X \mapsto bd^{-1}, \quad t \mapsto d^{-2}, \quad Y \mapsto \frac{d^{-1}c}{q^m - q^{-m}}.$$

- If n even, we have new coquasitriangular Hopf algebra, e.g. $\mathfrak{c}_{-1}[SL_2] \cong \mathfrak{u}_{-1}(sl_2)$.

- Theorem

$A = H^* \Longrightarrow (Dbos(H))^* = coDbos(A)$

(i.e. $B^{\underline{\mathrm{op}}} \bowtie A \bowtie B^* = (B^{*\underline{\mathrm{cop}}} \bowtie H \bowtie B)^*$).

- Corollary

$\mathfrak{c}_q[SL_2] \cong \mathfrak{u}_q(sl_2)$ *and* $\{X^i t^j Y^k\}_{0 \le i,j,k < n}$ *is the "dual PBW-basis"*

$$\langle X^i t^j Y^k, F^{i'} K^{j'} E^{k'} \rangle = \delta_{i,i'} \delta_{k,k'} q^{jj'} [i]_{q^{-1}}! [k]_q!$$

Inductive construction

May construct tree of quantum groups via double bosonisation (including Dynkin classification)

$$
\begin{tikzcd}
& & & & \cdots \\
& & & \mathfrak{u}_q(sl_4) \arrow[ur] \arrow[dr] & \\
& & \mathfrak{u}_q(sl_3) \arrow[ur] \arrow[dr] & & \cdots \\
\mathbb{C}\mathbb{Z}_n \arrow[r] & \mathfrak{u}_q(sl_2) \arrow[ur] \arrow[dr] & & \cdots & \\
& & \begin{array}{c}\text{new quantum group} \\ \text{(use other braided Hopf algebra)}\end{array} \arrow[r] & \cdots &
\end{tikzcd}
$$

Similarly for co-double bosonisation.

- Let $H = \widetilde{u_{q^{-m}}}(sl_2) = u_{q^{-m}}(sl_2) \otimes \mathbb{C}_{q^\alpha}[g]/(g^n - 1)$
- $B = \mathfrak{c}_q^2$ a reduced quantum plane

$$e_1^n = e_2^n = 0, \quad e_2 e_1 = q^{-m} e_1 e_2, \quad e_i \triangleright g = q^\alpha e_i$$

$$\underline{\Delta} e_i = 1 \otimes e_i + e_i \otimes 1, \quad \underline{\epsilon} e_i = 0, \quad \underline{S} e_i = -e_i$$

$$\Psi(e_i \otimes e_i) = q e_i \otimes e_i, \quad \Psi(e_1 \otimes e_2) = -q^{-m} e_2 \otimes e_1$$

$$\Psi(e_2 \otimes e_1) = -q^{-m} e_1 \otimes e_2 - (q-1) e_2 \otimes e_1.$$

- $\mathfrak{c}_q^2 \in \mathcal{M}_H$ iff $\alpha^2 = m(m-1) \mod n$.
- Then $Dbos(H) = \mathfrak{u}_q(sl_3)$ and

$$\mathfrak{u}_q(sl_3) = \begin{cases} u_{q^{-m}}(sl_3) & m > 1, \\ (u_{q^{-m}}(sl_3)/\langle K_1 - K_2 \rangle) \otimes (\mathbb{C}[g]/(g^n - 1)) & m = 1. \end{cases}$$

- Let $A = \widetilde{c_{q^{-m}}}(sl_2) = c_{q^{-m}}(sl_2) \otimes \mathbb{C}_{q^\alpha}[\varsigma]/(\varsigma^n - 1)$
- $B = \mathfrak{c}_q^2$ a reduced quantum plane.

$$X_1^n = X_2^n = 0, \quad X_2 X_1 = q^{-m} X_1 X_2, \quad \Delta_L X_i = {t^i}_j \varsigma \otimes X_j$$

$$\underline{\Delta} X_i = 1 \otimes X_i + X_i \otimes 1, \quad \underline{\epsilon} X_i = 0, \quad \underline{S} X_i = -X_i$$

- $\mathfrak{c}_q^2 \in {}^A\mathcal{M}$ iff $\alpha^2 = m(m-1) \mod n$.
- Then $coDbos(A) = \mathfrak{c}_q[SL_3]$ and

$$\mathfrak{c}_q[SL_3] = \begin{cases} c_{q^{-m}}[SL_3] & m > 1, \\ c'_{q^{-1}}[SL_3] = \langle {t^i}_j ({t^3}_3)^2 \rangle \hookrightarrow c_{q^{-1}}[SL_3] & m = 1 \end{cases}$$

where $(i, j) \neq (3, 3)$.

Fermionic quantum matrix group

- Let $A = \mathbb{C}_q[GL_2] = \langle \widetilde{s^i{}_j} \rangle$ and let B be fermionic quantum-plane $\in {}^A\mathcal{M}$ such that

$$e_i^2 = 0, \quad e_2 e_1 = -q^{-1} e_1 e_2, \Delta_L e_i = \widetilde{s^i{}_j} \otimes e_j$$

$$\underline{\Delta} e_i = e_i \otimes 1 + 1 \otimes e_i, \underline{\epsilon} e_i = 0, \quad \underline{S} e_i = -e_i$$

$$\Psi(e_i \otimes e_i) = -e_i \otimes e_i, \quad \Psi(e_1 \otimes e_2) = -q^{-1} e_2 \otimes e_1$$

$$\Psi(e_2 \otimes e_1) = -q^{-1} e_1 \otimes e_2 - (1 - q^{-2}) e_2 \otimes e_1$$

- Then $coDbos(A) = \mathbb{C}_q^{fer}[SL_3]$ is generated by

$$\mathbf{t} = \begin{pmatrix} t^1{}_1 & t^1{}_2 & t^1{}_3 \\ t^2{}_1 & t^2{}_2 & t^2{}_3 \\ t^3{}_1 & t^3{}_2 & t^3{}_3 \end{pmatrix} = \begin{pmatrix} X & t_1 & t_2 \\ s_1 & \tilde{a} & \tilde{b} \\ s_2 & \tilde{c} & \tilde{d} \end{pmatrix}$$

  where

$$s_1 = (q - q^{-1})(\tilde{b} f_1 - q \tilde{a} f_2), \quad s_2 = (q - q^{-1})(\tilde{d} f_1 - q \tilde{c} f_2)$$

$$t_1 = e_1 \tilde{c} - q^{-1} e_2 \tilde{a}, \quad t_2 = e_1 \tilde{d} - q^{-1} e_2 \tilde{b}$$

$$X = D + (t_1 D^{-1}(d s_1 - q b s_2) - q^{-1} t_2 D^{-1}(c s_1 - q a s_2))$$

- $\mathbb{C}_q^{fer}[SL_3]$ has standard matrix comultiplication, and some relations are 'fermionic'.
- It has non-standar R-matrix with some entries are 'fermionic'.

Hopf algebra Fourier transform (Not presented)

- Right integral $\int : H \to k$, $\int^* : A = H^* \to k$, $\mu = \int(\int^*)$.
- Hopf algebra Fourier transform

$$\mathcal{F}(h) = \sum_a (\int e_a h) f^a, \quad \mathcal{F}^*(\phi) = \sum_a e_a (\int^* \phi f^a), \quad \mathcal{F}^* \circ \mathcal{F} = \mu S$$

e_a basis of H, f^a dual basis of B^*

- $\mathcal{F} : \mathfrak{c}_q[SL_2] \to \mathfrak{u}_q(sl_2)$ is given by

$$\mathcal{F}(X^a t^b Y^c) = \sum_{l=0}^{n-1} \frac{q^{-(l+a)(1-b)+b(n-1-c)}}{n[n-1-a]_{q^{-1}}![n-1-c]_q!} F^{n-1-a} K^l E^{n-1-c}$$

$$\mu = \frac{q^{-1}}{n[n-1]_{q^{-1}}![n-1]_q!}$$

- $\mu \neq 0 \implies \mathcal{F}$ is invertible with $\mathcal{F}^{-1} = \mu^{-1} S^{-1} \mathcal{F}^* \implies \mathfrak{c}_q[SL_2] \cong \mathfrak{u}_q(sl_2)$ by $\mathcal{F}$.

3D Calculus (Not presented)

- 1^{st} order diff. calculus (Ω^1, d) over H is an H bimodule, s.t. $\mathrm{d}(ab) = (\mathrm{d}a)b + a(\mathrm{d}b)$.
- Ω^1 is free-modules over Λ^1, the basis of left-invariant one-form.
- $\Omega^1(\mathfrak{c}_q[SL_2])$ with n odd, has left-invariant basic 1-forms $e_\pm, e_0$ (3D-calculus) s.t. $e_\pm h = p^{|h|} h e_\pm$ and $e_0 h = p^{2|h|} h e_0$ where $p = q^{-m}$.

$$\mathrm{d}X = q^{-m} t e_-, \quad \mathrm{d}t = (1+q)\big(q(q^{-m} - q^m) t Y e_- + t e_0\big)$$

$$\mathrm{d}Y = (q^{-1} - 1)^{-1} e_+ + (1+q) Y e_0 + q(q^{-m} - q^m) Y^2 e_-$$

- These are well-behaves with Hopf algebra Fourier transform by

$$\mathcal{F}(\partial^a h) = \mathcal{F}(h)\chi_a$$

where $\chi_a(h) = \langle f^a, \varpi \pi_\epsilon S^{-1} h\rangle$ for all $h \in H$ defines $\chi_a \in H^{*+}$.

3D-calculus (Not presented)

The elements χ_a are

$$\chi_+ = \sum_{j=0}^{n-1} \delta_j(K)E = \sum_{i,j=0}^{n-1} \frac{(q^{-1}-1)^{-1}q^{-ij}}{n} K^i E = \frac{E}{(q^{-1}-1)}$$

$$\chi_0 = \sum_{j=0}^{n-1} (1+q)[j]_{q^2} \delta_j(K) = \sum_{i,j=0}^{n-1} \frac{q^{-ij}(1-q^{2j})}{n(1-q)} K^i = \frac{1-K^2}{1-q}$$

$$\chi_- = \sum_{j=0}^{n-1} q^{-m+j} F \delta_j(K) = \sum_{i,j=0}^{n-1} \frac{q^{-m}q^{j-ij}}{n} FK^i = q^{-m}FK.$$

Thank you for your kind attention