The gauge group of a Hopf-Galois extension and twist deformations

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principal bundles & Hopf-Galois extensions

- H Hopf algebra (structure group)
- A an H-comodule algebra (total space) with coaction
  $\delta : A \to A \otimes H$, $a \mapsto a_{(0)} \otimes a_{(1)}$, algebra map
- B algebra (base space), $B \simeq A^{co(H)} := \{b \in A | \delta(b) = b \otimes 1\!\!1_H\}$

$+$ principality condition: the algebra extension $B \subseteq A$ is Hopf-Galois:

$$\chi = (m_A \otimes id)(id \otimes_B \delta) : A \otimes_B A \to A \otimes H$$
$$a \otimes_B a' \mapsto aa'_{(0)} \otimes a'_{(1)}$$

(canonical map) is bijective.

$$\begin{tikzcd} P \times G \arrow[r, "\mu"] & P \arrow[d, "G"] \\ & X \simeq P/G \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} A = \mathcal{O}(P) \arrow[r, "\delta=\mu^*"] & A \otimes H \\ B = \mathcal{O}(X) \simeq \mathcal{O}(P/G) \arrow[u, hook, "H=\mathcal{O}(G)"] & \end{tikzcd}$$

The classical gauge group

HG-gauge

For a principal G-bundle $\pi : P \to X$, the group $\mathcal{G}_P$ of gauge transformations is

- the subgroup of principal bundle automorphisms which are vertical:

$$\mathcal{G}_P = \mathrm{Aut}_V(P) := \{\varphi : P \to P;\ \varphi(pg) = \varphi(p)g\ , \pi \circ \varphi = \pi\},$$

 with group law given by the composition of maps;

- the group of G-equivariant maps,

$$\mathcal{G}_P = \{\sigma : P \to G;\ \sigma(pg) = g^{-1}\sigma(p)g\}$$

 with pointwise product, $(\sigma \cdot \tau)(p) := \sigma(p)\tau(p) \in G$.
 (Locally, $x \in X \to g(x) \in G$)

The group of gauge transformations acts by pullback on the set $\mathcal{A}_P$ of connections of the bundle $\pi : P \to X$.
ω, η connection forms are gauge equivalent iff $\exists \varphi \in \mathcal{G}_P$ such that $\varphi^* \omega = \eta$.
Indeed gauge equivalence defines an equivalence relation on $\mathcal{A}_P$

$$\rightsquigarrow \mathcal{M} = A_P / \mathcal{G}_P \quad \text{moduli space of connections}$$

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Aim: extend the notion of gauge transformations to the algebraic framework of (NC) Hopf-Galois extensions.

- [Brzeziński (1996)]
- [Aschieri, Landi, P. (2018)] in the framework of coquasitriangular Hopf algebras

For the HG extension $\mathcal{O}(X) \subseteq \mathcal{O}(P)$ associated to a principal G-bundle $P \to X$, the canonical map

$$\chi = (m \otimes id) \circ (id \otimes_B \delta^A) : A \otimes_B A \to A \otimes H \qquad a \otimes_B a' \mapsto aa'_{(0)} \otimes a'_{(1)}$$

is an algebra map.
Moreover, $\forall \varphi \in \mathcal{G}_P$, φ^* is an algebra map.

Remark: In general the canonical map χ of a HG extension is NOT an algebra map, but a morphism of relative Hopf-modules, $\chi \in Hom({}_A\mathcal{M}_A^H)$.

$$\left(Hom({}_A\mathcal{M}_A^H) = \{\text{linear maps of } H\text{-comodules and of } A\text{-bimodules}\} \right)$$

Problem: if we defined

$$Aut_V := \{\mathsf{F} : A \to A \,|\mathsf{F} \text{ invertible left } B\text{-module morph. s.t. } \delta \mathsf{F} = (\mathsf{F} \otimes \mathrm{id})\delta\}$$

without the assumption that the maps F are algebra morphisms, then

- for a Hopf-Galois extension associated to a classical bundle, the group Aut_V of 'quantum' gauge transformations would be too big, much bigger than the 'classical' gauge group.
  Ex. For $P = \mathbb{Z}_2 \to \{*\}$ with $G = \mathbb{Z}_2$, the group Aut_V is the group
  $$Aut_V = \left\{ \begin{pmatrix} a & 1-a \\ 1-a & a \end{pmatrix}, \quad \text{with} \quad 2a \neq 1 \right\}.$$
  Whereas with the additional hypothesis that the maps F are algebra morphisms this group collapses to $\mathbb{Z}_2$.
- the condition of invertibility of the maps F would be a requirement (while classical gauge transformations are automatically invertible).

Coquasitriangular Hopf algebras

A Hopf algebra H is called coquasitriangular if it is endowed with

$$R : H \otimes H \to \mathbb{K} \qquad \text{(universal } r\text{-form)}$$

linear map such that

(i) R is invertible for the convolution product, with inverse denoted by $\bar{R}$;

(ii) $m_{op} = R * m * \bar{R}$, i.e. for all $h, k \in H$

$$kh = R(h_{(1)} \otimes k_{(1)})\, h_{(2)} k_{(2)} \bar{R}(h_{(3)} \otimes k_{(3)})$$

(iii) $R \circ (m \otimes \mathrm{id}) = R_{13} * R_{23} \quad$ and $\quad R \circ (\mathrm{id} \otimes m) = R_{13} * R_{12}$,
where $R_{12}(h \otimes k \otimes l) = R(h \otimes k)\,\varepsilon(l)$ and similarly for R_{13} and R_{23}.

• If (H, R) is coquasitriangular then so is $(H, \bar{R}_{21})$ where $\bar{R}_{21}(h \otimes k) := \bar{R}(k \otimes h)$
(H, R) is called cotriangular if $R = \bar{R}_{21}$.

Examples

- Any commutative H is cotriangular with trivial universal r-form $R = \varepsilon \otimes \varepsilon$.

- The noncommutative FRT bialgebras

$$\mathcal{O}_q(G) = \mathbb{C}\langle u_{ij}\rangle / \langle \mathcal{R}^{ji}_{kl} u_{km} u_{ln} = u_{ik} u_{jl} \mathcal{R}^{lk}_{mn}\rangle \ , \quad q \in \mathbb{R}$$

deforming the coordinate functions on Lie groups are coquasitriangular, $R(u_{ik} \otimes u_{jl}) \propto \mathcal{R}^{ij}_{kl}$

- If (H, R) is a coquasitriangular Hopf algebra and $\gamma : H \otimes H \to \mathbb{K}$ is a 2-cocycle on H, then the Hopf algebra H_γ with twisted product and antipode is also coquasitriangular with universal r-form

$$R_\gamma := \gamma_{21} * R * \bar{\gamma} : h \otimes k \longmapsto \gamma\,(k_{(1)} \otimes h_{(1)})\, R(h_{(2)} \otimes k_{(2)})\, \bar{\gamma}\,(h_{(3)} \otimes k_{(3)})$$

Some useful facts from the theory of cqt Hopf algebras:

- For (H, R) coquasitriangular, the monoidal category of right H-comodules $\mathcal{M}^H$ is braided monoidal with braiding given by the H-comodule isomorphisms

$$R_{V,W} : V \otimes W \longrightarrow W \otimes V \ , \quad v \otimes w \longmapsto w_{(0)} \otimes v_{(0)} \ R(v_{(1)} \otimes w_{(1)})$$

- The category $(\mathcal{A}^H, \boxtimes)$ of H-comodule algebras is monoidal:

Proposition

Let $(A, \delta^A), (C, \delta^C) \in \mathcal{A}^H$, then the H-comodule $A \otimes C$ (with tensor product coaction $\delta^{A\otimes C} : a \otimes c \mapsto a_{(0)} \otimes c_{(0)} \otimes a_{(1)} c_{(1)}$) is a right H-comodule algebra,

$$A \boxtimes C := (A \otimes C, \bullet) \quad \textit{(braided product algebra)}$$

when endowed with the product

$$(a \otimes c) \bullet (a' \otimes c') := a \ R_{C,A}(c \otimes a')c' = aa'_{(0)} \otimes c_{(0)}c' \ R(c_{(1)} \otimes a'_{(1)}) \ .$$

Moreover, if $\phi : A \to E$ and $\psi : C \to F$ are morhisms of H-comodule algebras, then so is $\phi \boxtimes \psi := \phi \otimes \psi : (A \otimes C, \bullet) \to (E \otimes F, \bullet)$ where $(\phi \otimes \psi)(a \otimes c) = \phi(a) \otimes \psi(c)$.

Proposition

Let (H, R) be a coquasitriangular Hopf algebra. The right H-comodule $\underline{H} = (H, \mathrm{Ad})$ becomes an H-comodule algebra $\underline{H} = (H, \star, \mathrm{Ad})$ when endowed with the product

$$h \star k := h_{(2)} k_{(2)} R(S(h_{(1)}) h_{(3)} \otimes S(k_{(1)}))$$

and unit $\eta : \mathbb{K} \to \underline{H}$ given as linear map by the unit of H.

- $(\underline{H}, \star, \eta, \Delta, \epsilon, \underline{S}, \mathrm{Ad})$ is a braided Hopf algebra (associated with H):
  - $\Delta : \underline{H} \to \underline{H} \boxtimes \underline{H}$ is an algebra map w.r.t. the braided product $m_{\underline{H} \boxtimes \underline{H}}$
  - the antipode $\underline{S} : \underline{H} \to \underline{H}$ defined by

    $$\underline{S}(h) := S(h_{(2)}) R\Big(S^2(h_{(3)}) S(h_{(1)}) \otimes h_{(4)}\Big) \ ,$$

    is an H-comodule map and turns out to be a braided anti-algebra map and a braided anti-coalgebra map

    $$\underline{S} \circ \star = \star \circ R_{\underline{H},\underline{H}} \circ (\underline{S} \otimes \underline{S}) \ , \quad \Delta \circ \underline{S} = (\underline{S} \otimes \underline{S}) \circ R_{\underline{H},\underline{H}} \circ \Delta \ .$$

Hopf-Galois extensions for coquasitriangular Hopf algebras and their gauge groups. [P. Aschieri, G. Landi, C.P. (2018)]

Theorem

Let (H, R) be a coquasitriangular Hopf algebra and $A \in \mathcal{A}_{qc}^{(H,R)}$ a quasi-commutative H-comodule algebra. Let $B \subseteq Z(A)$ be the corresponding subalgebra of coinvariants. Then the canonical map

$$\chi = (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes_B \delta^A) : A \boxtimes_B A \longrightarrow A \boxtimes \underline{H} \,, \qquad a' \boxtimes_B a \longmapsto a' \, a_{(0)} \boxtimes a_{(1)}$$

is an algebra map, thus a morphism in $\mathcal{A}^H$.

Definition

Let (H, R) be a coquasitriangular Hopf algebra. A right H-comodule algebra A is quasi-commutative (with respect to the universal r-form R), $A \in \mathcal{A}_{qc}^{(H,R)}$ if

$$m_A = m_A \circ R_{A,A} \,, \qquad ac = c_{(0)} a_{(0)} \; R(a_{(1)} \otimes c_{(1)}) \qquad a, c \in A$$

Examples

- Clearly, for $(H, \varepsilon \otimes \varepsilon)$, every commutative algebra $A \in \mathcal{A}^H$ is quasi-commutative.
- Twist deformations $A_\gamma \in \mathcal{A}^{H_\gamma}$ of quasi-commutative algebras A via a 2-cocycle on H are quasi-commutative algebras.
- A main example of quasi-commutative comodule algebra is the H-comodule algebra $(\underline{H}, \star, \mathrm{Ad})$ associated with a cotriangular Hopf algebra (H, R).
- $H = \mathcal{O}(GL_q(2))$ is coquasitriangular with (not cotriangular) universal r-form

$$R(u_{ij} \otimes u_{kl}) = q^{-1} \mathcal{R}^{ik}_{jl} \ , \quad R\Big(D^{-1} \otimes u_{ij}\Big) = R\Big(u_{ij} \otimes D^{-1}\Big) = q\, \delta_{ij} \ ,$$

The quantum plane $\mathcal{O}(\mathbb{C}^2_q) = \mathbb{C}[x, x_2]/\langle x_1 x_2 - q\ x_2 x_1 \rangle$ is a quasi-commutative $\mathcal{O}(GL_q(2))$-comodule algebra with coaction $\delta(x_i) = \sum_j x_j \otimes u_{ji}$.

The gauge group of a (coquasi△) Hopf-Galois extension.

classical

Theorem

Let (H, R) *be a coquasitriangular Hopf algebra,* $(\underline{H}, \star, \eta, \Delta, \epsilon, \underline{S}, \mathrm{Ad})$ *the associated braided Hopf algebra,* $A \in \mathcal{A}_{qc}^{(H,R)}$ *and* $B = A^{coH} \subseteq A$ *a Hopf-Galois extension. The* $\mathbb{K}$*-module*

$$\mathcal{G}_A := \mathrm{Hom}_{\mathcal{A}^H}(\underline{H}, A)$$

of H*-equivariant algebra maps* $\underline{H} \to A$ *is a group with respect to the convolution product, with inverse* $\bar{\mathsf{f}} := \mathsf{f} \circ \underline{S}$*, for* $\mathsf{f} \in \mathrm{Hom}_{\mathcal{A}^H}(\underline{H}, A)$*.*

- the convolution product of two such algebra maps is an algebra map is shown by using the fact that A is quasi-commutative.
- moreover the hypothesis A quasi commutative is a sufficient condition for the invertibility in $\mathcal{G}_A$ of each map $\mathsf{f} : \underline{H} \to A$, in particular for its inverse (with respect to the convolution product) to be again an algebra morphism.

Theorem

Let $B = A^{coH} \subseteq A$ be an H-Hopf-Galois extension with H coquasitriangular and $A \in \mathcal{A}_{qc}^{(H,R)}$. The $\mathbb{K}$-module

$$\mathrm{Aut}_V(A) := \mathrm{Hom}_{{}_B\mathcal{A}^H}(A,A) = \{\mathsf{F} \in \mathrm{Hom}_{\mathcal{A}^H}(A,A), \text{ such that } \mathsf{F}_{|_B} = \mathrm{id}\}$$

of left B-module, right H-comodule algebra morphisms is a group with respect to map composition

$$\mathsf{F}\cdot\mathsf{G} := \mathsf{G}\circ\mathsf{F}\,, \quad \mathsf{F},\mathsf{G} \in \mathrm{Aut}_V(A)$$

with inverse

$$\mathsf{F}^{-1} : a \longmapsto a_{(0)}\mathsf{F}({a_{(1)}}^{<1>}){a_{(1)}}^{<2>}$$

for $\mathsf{F} \in \mathrm{Hom}_{{}_B\mathcal{A}^H}(A,A)$.

- The translation map $\tau = \chi^{-1}_{|_{1\otimes \underline{H}}} : \underline{H} \longrightarrow A \boxtimes_B A\,,\ h \mapsto h^{<1>} \otimes h^{<2>}$ is an algebra map;
- the translation map satisfies

$$\tau \circ \underline{S} = R_{A,A} \circ \tau$$

(for $\tau = t^*$, the classical translation map of a principal bundle $\pi : P \to B$ corresponds by duality to the property $t(q,p) = t(p,q)^{-1}$, $p, q \in P$).

Proposition

The groups $(\mathcal{G}_A, *)$ *and* $(\mathrm{Aut}_V(A), \cdot)$ *are isomorphic via the map*

$$\begin{aligned} \theta_A : \mathcal{G}_A &\longrightarrow \mathrm{Aut}_V(A) \\ \mathsf{f} &\mapsto \mathsf{F}_{\mathsf{f}} := m_A \circ (\mathrm{id}_A \otimes_B \mathsf{f}) \circ \delta^A \quad : a \mapsto a_{(0)}\mathsf{f}(a_{(1)}) \end{aligned}$$

with inverse

$$\mathsf{F} \mapsto \mathsf{f}_{\mathsf{F}} := m_A \circ (\mathrm{id}_A \otimes \mathsf{F}) \circ \tau \quad : h \mapsto h^{<1>}\mathsf{F}(h^{<2>}).$$

Definition

We term $\mathcal{G}_A \simeq \mathrm{Aut}_V(A)$ *the gauge group of the Hopf-Galois extension* $B \subseteq A \in \mathcal{A}_{qc}^{(H,R)}$.

Example

Let us consider the principal G-bundle $G \to \{*\}$ over a point, or dually the Hopf-Galois extension $\mathbb{C} \subseteq \mathcal{O}(G)$, where $A := \mathcal{O}(G)$ is a comodule algebra with coaction of $H := \mathcal{O}(G)$ given by the coproduct, then

$$\mathcal{G}_A = \mathrm{Hom}_{{}_B\mathcal{A}^H}(A, A) \simeq G$$

Deformation of Hopf-Galois extensions and gauge groups via 2-cocycles.

Let $\gamma : H \otimes H \to \mathbb{K}$ be a convolution invertible unital 2-cocycle or Drinfel'd twist on H,

$$\gamma\left(g_{(1)} \otimes h_{(1)}\right)\gamma\left(g_{(2)}h_{(2)} \otimes k\right) = \gamma\left(h_{(1)} \otimes k_{(1)}\right)\gamma\left(g \otimes h_{(2)}k_{(2)}\right) .$$

- $H \rightsquigarrow$ twisted Hopf-algebra H_γ
  with twisted product $h \cdot_\gamma k := \gamma\left(h_{(1)} \otimes k_{(1)}\right) h_{(2)}k_{(2)}\bar{\gamma}\left(h_{(3)} \otimes k_{(3)}\right)$
  and twisted antipode $S_\gamma := u_\gamma * S * \bar{u}_\gamma$.
- $A \in \mathcal{A}^H \rightsquigarrow$ twisted comodule-algebra $A_\gamma \in \mathcal{A}^{H_\gamma}$
  with twisted product $a \bullet_\gamma a' := a_{(0)}a'_{(0)}\bar{\gamma}\left(a_{(1)} \otimes a'_{(1)}\right)$
  and coaction $\delta = \delta_\gamma : A_\gamma \to A_\gamma \otimes H_\gamma$.
- $\rightsquigarrow$ apply to HG extensions [Aschieri, Bieliavsky, P., Schenkel, CMP 2017]

$$\begin{array}{ccc} A & & A_\gamma \\ {\scriptstyle H}\big\uparrow & \rightsquigarrow\overset{twisting}{\underset{\gamma \;\; on \;\; H}{}}\rightsquigarrow & {\scriptstyle H_\gamma}\big\uparrow \\ B = A^{coH} & & B = A_\gamma^{coH_\gamma} \end{array}$$

Theorem (Aschieri, Bieliavsky, P., Schenkel, CMP 2017)

The following diagram in ${}_{A_\gamma}\mathcal{M}_{A_\gamma}^{H_\gamma}$ commutes:

$$\begin{tikzcd}
A_\gamma \otimes_B^\gamma A_\gamma \arrow[r, "\chi_\gamma"] \arrow[dd, "\varphi_{A,A}"'] & A_\gamma \otimes^\gamma \underline{(H_\gamma)} \arrow[d, "\mathrm{id}\otimes^\gamma \mathcal{Q}"] \\
& A_\gamma \otimes^\gamma (\underline{H})_\gamma \arrow[d, "\varphi_{A,H}"] \\
(A \otimes_B A)_\gamma \arrow[r, "\Gamma(\chi)=\chi"] & (A \otimes \underline{H})_\gamma
\end{tikzcd}$$

Corollary

The extension $B = A^{coH} \subset A$ is H-Galois $\iff$ the extension $B \simeq A_\gamma^{coH_\gamma} \subset A_\gamma$ is H_γ-Galois.

Case of H coquasi$\triangle$

Theorem

Let γ be a twist on (H, R). Let $A \in \mathcal{A}_{qc}^{(H,R)}$ with $B = A^{coH} \subseteq A$ and twist deformation $B = A_\gamma^{coH_\gamma} \subseteq A_\gamma \in \mathcal{A}_{qc}^{(H_\gamma, R_\gamma)}$. Then the following diagram in $\mathcal{A}^{H_\gamma}$ commutes.

$$\begin{tikzcd}
A_\gamma \boxtimes_B^\gamma A_\gamma \arrow[r, "\chi_\gamma"] \arrow[dd, "\varphi_{A,A}"'] & A_\gamma \boxtimes^\gamma \underline{H_\gamma} \arrow[d, "\mathrm{id} \otimes^\gamma \mathcal{Q}"] \\
& A_\gamma \boxtimes^\gamma \underline{H}_\gamma \arrow[d, "\varphi_{A,\underline{H}}"] \\
(A \boxtimes_B A)_\gamma \arrow[r, "\Gamma(\chi)"] & (A \boxtimes \underline{H})_\gamma
\end{tikzcd}$$

Twisting gauge groups

Theorem

The functor $\Gamma : \mathcal{A}^H \to \mathcal{A}^{H_\gamma}$ *composed with the pullback* $Q^* : \mathrm{Hom}_{\mathcal{A}^{H_\gamma}}\left(\underline{H_\gamma}, A_\gamma\right) \to \mathrm{Hom}_{\mathcal{A}^{H_\gamma}}\left(\underline{H_\gamma}, A_\gamma\right)$ *of the map* $\mathcal{Q} : \underline{H_\gamma} \longrightarrow \underline{H}_\gamma$ *gives the gauge group isomorphism*

$$\Gamma_Q := \mathcal{Q}^* \circ \Gamma : \mathrm{Hom}_{\mathcal{A}^H}\left(\underline{H}, A\right) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{A}^{H_\gamma}}\left(\underline{H_\gamma}, A_\gamma\right).$$

Twisting gauge groups

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$$\Gamma_{\mathcal{Q}} := \mathcal{Q}^* \circ \Gamma : \mathrm{Hom}_{\mathcal{A}^H}\left(\underline{H}, A\right) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{A}^{H_\gamma}}\left(\underline{H_\gamma}, A_\gamma\right).$$

Proposition

For θ_A *and* θ_{A_γ} *the isomorphisms described before the following diagram*

$$\begin{tikzcd}
\mathcal{G}_A \arrow[r, "\theta_A"] \arrow[d, "\Gamma_{\mathcal{Q}}"'] & \mathrm{Aut}_V(A) \arrow[d, "\Gamma"] \\
\mathcal{G}_{A_\gamma} \arrow[r, "\theta_{A_\gamma}"] & \mathrm{Aut}_V(A_\gamma)
\end{tikzcd}$$

commutes.

Open questions

- study of the gauge group of a twisted Hopf-Galois extension obtained by deformation with a cocycle σ on an external Hopf algebra of symmetries

$$\begin{array}{ccc} A & & {}_\sigma A \\ {\scriptstyle H}\big\uparrow & \rightsquigarrow\overset{\textit{twisting}}{\underset{\sigma\ \text{ on }\ K}{}}\rightsquigarrow & {\scriptstyle H}\big\uparrow \\ B = A^{coH} & & {}_\sigma B \simeq ({}_\sigma A)^{co(H)} \end{array}$$

  or double deformation, H_γ-Galois ${}_\sigma B \subseteq {}_\sigma A_\gamma$.

- Gauge group of a generic Hopf-Galois extension