Weak quasi Hopf algebras, tensor C^*-categories and conformal field theory

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Introduction

Hopf algebras and their generalisations are often understood as useful 'linearizations' of semisimple tensor categories. Two most studied classes are:

- quasi-Hopf algebras (Drinfeld, 90s)
- weak Hopf algebras (Bohm, Nill, Szlachanyi, 00s)

Aim of talk

- Review of general theory of weak quasi Hopf algebras extending QHA (Drinfeld, Majid, Mack-Schomerus, Haring-Oldenburg).

- Introduce a special subclass (w-Hopf algebras) with 'trivial' associator. Examples include $A(\mathfrak{sl}_N, \ell)$ constructed in 2015 'orthogonal' to $\mathrm{SU}_q(N)$ for $q > 0$. New examples.
- Construction of semisimple f.d. weak quasi Hopf C^*-algebras $A(\mathfrak{g}, \ell)$, etc, associated to quantum group/ VOA/conformal net fusion categories under certain conditions.

- Application to construction of tensor C^*-categories associated to the level k affine vertex operator algebras $V_{\mathfrak{g}_k}$

Tensor category

A tensor category $\mathcal{C}$ is a $\mathbb{C}$-linear category with a bifunctor (tensor product) $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, unit object ι, associativity morphisms

$$\alpha_{\rho,\sigma,\tau} : (\rho \otimes \sigma) \otimes \tau \to \rho \otimes (\sigma \otimes \tau)$$

satisfying a list of axioms including the pentagon relation

$$\begin{array}{ccccc}
((\rho \otimes \sigma) \otimes \tau) \otimes \upsilon & \xrightarrow{\alpha \otimes 1} & (\rho \otimes (\sigma \otimes \tau)) \otimes \upsilon & \xrightarrow{\alpha} & \rho \otimes ((\sigma \otimes \tau) \otimes \upsilon) \\
\downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle 1 \otimes \alpha} \\
(\rho \otimes \sigma) \otimes (\tau \otimes \upsilon) & & \xrightarrow[\alpha]{\hspace{6em}} & & \rho \otimes (\sigma \otimes (\tau \otimes \upsilon))
\end{array}$$

- A tensor *-category has a *-involution $(T : \rho \to \sigma) \to (T^* : \sigma \to \rho)$ s.t. $(S \otimes T)^* = S^* \otimes T^*$ and α is unitary.
- In a tensor C^*-category morphism spaces are Banach spaces s.t. $\|T^* T\| = \|T\|^2$.

Examples: Vec, Herm, Hilb, resp.

Examples of fusion categories

- $U_q(\mathfrak{g})$ Drinfeld-Jimbo-Lusztig quantum groups at $q = e^{i\pi/\ell d}$,

  $$\mathcal{C}(\mathfrak{g}, \ell) := \mathrm{Tilt}/\mathrm{Neg} \quad \text{Andersen tensor category}$$

  It is a modular fusion category (Kirillov, Turaev-Wenzl), and admits a unitary structure (Wenzl-Xu).

- $\mathrm{Rep}(V_{\mathfrak{g}_k})$ category of modules of level k affine VOA. It is a rigid braided tensor category, in fact modular (Huang-Lepowski).

- More generally, may consider $\mathrm{Rep}(V)$, for a VOA V, which is a modular category if V is subject to specific conditions (Huang).

- $I \subset S^1 \to \mathcal{A}(I)$ completely rational conformal net (DHR, Gabbiani-Frölich), then Rep($\mathcal{A}$) is a modular C^*-tensor category (Kawahigashi-Longo-Müger).

Relations between the three settings

- There is a general construction

$$\{\text{strongly local VOA}\} \to \{\text{DHR conformal nets}\}$$

  under which an affine VOA is taken to a loop group conformal net (Carpi-Kawahigashi-Longo-Weiner).

- Known connections between quantum grp fusion category and affine VOA will be recalled.

Quasi-Hopf algebras (Drinfeld)

$A = (A, \Delta, \varepsilon, \Phi, S)$ with $\Delta : A \to A \otimes A$ unital coproduct, $\Phi \in A^{\otimes 3}$ associator

$$\Phi \Delta \otimes 1 \circ \Delta(a) = 1 \otimes \Delta \circ \Delta(a) \Phi$$

Φ satisfies a 3-cocycle condition

A quasi-Hopf algebra is a Hopf algebra if and only if $\Phi = I \otimes I \otimes I$.

Twist operation: $F \in A^{\otimes 2}$ invertible with $\varepsilon \otimes 1(F) = I = 1 \otimes \varepsilon(F)$ gives a new $A_F = (A, \Delta_F, \varepsilon, \Phi_F)$,

$$\Delta_F(a) = F\Delta(a)F^{-1}, \quad \Phi_F = I \otimes F 1 \otimes \Delta(F) \Phi \Delta \otimes 1(F^{-1}) F^{-1} \otimes I,$$

Φ_F 3-coboundary deformation of Φ

Main fact: $\mathrm{Rep}(A)$ is a rigid tensor category; and $\mathrm{Rep}(A) \simeq \mathrm{Rep}(A_F)$ tensor equivalence

Example

$A = \mathrm{Fun}(G)$, $\quad G$ finite group,

$$\Delta(f)(g,h) = f(gh),$$

$$\Phi_\omega(g,h,k) = \omega(g,h,k), \quad \omega : G^3 \to \mathbb{C}^* \text{ 3-cocycle}$$

$\omega = \delta(\mu)$ 3-coboundary $\to A_{\mu^{-1}} = \mathrm{Fun}(G)$ ordinary Hopf algebra structure.

$\mathrm{Vec}_G^\omega := \mathrm{Rep}(A)$ are classified, up to tensor equivalence, by $H^3(G, \mathbb{C}^\times)$. These exhaust pointed tensor categories (all irreducibles are invertible).

Drinfeld-Kohno

Let $\mathfrak{g}$ be a simple complex Lie algebra. $U(\mathfrak{g})[[h]]$ deformation of $U(\mathfrak{g})$, $\Delta(x) = x \otimes I + I \otimes x$; $t \in \mathfrak{g} \otimes \mathfrak{g}$ tensor coming from the Killing form,

$$R_t = e^{ht/2}.$$

Drinfeld constructed

$$\Phi_{KZ} \quad \text{(Drinfeld associator)}$$

from previous work in CFT (Knizhnik-Zamolodgikov).

He obtained a quasi-bialgebra $A_{\mathfrak{g},t}$ with associator Φ_{KZ} and R-matrix R_t. Then he showed existence of a twist F s.t.

$$(A_{g,t})_F = U_h(\mathfrak{g}), \quad (R_t)_F = R_h.$$

In particular, Φ_{KZ} is a 3-coboundary

$$\Phi_{KZ} = 1 \otimes \Delta(F^{-1}) I \otimes F^{-1} F \otimes I \Delta \otimes 1(F)$$

This also implies rigidity of $\mathrm{Rep}(A_{\mathfrak{g},t})$.

Fusion categories and quasi Hopf algebras

- A fusion category $\mathcal{C}$ admits a unique ring homomorphism on the Grothendieck ring $\mathrm{Gr}(\mathcal{C}) \to \mathbb{R}$ which is positive on the irreducibles. This is the Perron-Frobenius dimension function,

$$X \to \mathrm{FPdim}(X)$$

(Etingof-Nikshych-Ostrik)

- It follows that $\mathcal{C} \simeq \mathrm{Rep}(A)$ for a quasi Hopf algebra A if and only if

$$\mathrm{FPdim}(X) \in \mathbb{N},$$

for all irred. X, and in this case A is unique up to twist.

- However, this is not the case for many models (e.g. Ising model).
- To describe general fusion categories, two main classes are known, the weak Hopf algebras, and the weak quasi Hopf algebras. We review both and explain relation.

Weak Hopf algebras (Bohm, Nill, Szlachanyi)

Let $R =$ be a f.d. semisimple $\mathbb{C}$-algebra and $\mathcal{F} : \mathcal{C} \to R\text{-bimod}$ fiber functor. Consider the composite functor

$$\mathcal{F}' : \mathcal{C} \to R - \text{bimod} \to \text{Vec}$$

Theorem (Bohm, Szlachanyi): Then $A = \text{Nat}\mathcal{F}'$ has a natural structure of weak Hopf algebra over R and $\mathcal{C} \simeq \text{Rep}(A)$

- For any fusion category $\mathcal{C}$, $\mathcal{F}$ exists with $R = \oplus_{\lambda \in \text{Irr}\mathcal{C}} M_{n_\lambda}(\mathbb{C})$, n_λ arbitrary.
- $n_\lambda = 1$ gives a face algebra (Hayashi)
- Weak Hopf algebras play a crucial role in establishing important properties of fusion categories, e.g. Ocneanu rigidity (Etingof-Nikshych-Ostrik)

Weak quasi-Hopf algebras (Mack-Schomerus)

Mack and Schomerus extended the definition of quasi-Hopf algebras to weak quasi-Hopf algebras $A = (A, \Delta, \varepsilon, \Phi, S)$ with modifications:

- $\Delta(I) \neq I$
- $\Phi \in A^{\otimes 3}$ associator has domain $\Delta \otimes 1\Delta(I)$ and range $1 \otimes \Delta\Delta(I)$
- Twist operation is now defined by $F \in A^{\otimes 2}$ partially invertible with domain $\Delta(I)$ and some range $q = FF^{-1}$

It follows

- Twisted algebra still makes sense $A_F = (A, \Delta_F, \varepsilon, \Phi_F)$
- Twisting still leaves $\mathrm{Rep}(A)$ unchanged.
- They described an example associated to Verlinde fusion rules, using $U_q(\mathfrak{sl}_2)$ at $q = e^{i\pi/\ell}$.
- Note: notion of a 3-coboundary associator not addressed.

Weak quasi tensor functor (Majid, Haring-Oldenburg)

A weak quasi-tensor functor $F : \mathcal{C} \to \mathcal{C}'$ between tensor categories is a $\mathbb{C}$-linear functor together with two natural transformations:

$$F_2 : F_\rho \otimes F_\sigma \to F_{\rho\otimes\sigma}, \quad \text{‘projection’}$$

$$F_2^{-1} : F_{\rho\otimes\sigma} \to F_\rho \otimes F_\sigma \quad \text{‘inclusion’}$$

such that $F_2 F_2^{-1} = 1$.

- no compatibility property with associativity morphisms,
- If F_2 isomorphism $\to$ quasi-tensor functors (Majid),
- In general, $P_2 := F_2^{-1} F_2 : F(\rho) \otimes F(\sigma) \to F(\rho) \otimes F(\sigma)$ is a non-trivial idempotent

Tannakian results

Let $\mathcal{C}$ be a semisimple tensor category with f.d. morphism spaces and $\mathcal{F} : \mathcal{C} \to \mathrm{Vec}$ a faithful weak quasi-tensor functor. Then

$A = \mathrm{Nat}(\mathcal{F})$ is a discrete weak quasi-bialgebra $A \simeq \bigoplus M_{n_i}(\mathbb{C})$ and

$$\mathcal{C} \simeq \mathrm{Rep}(A).$$

Extra structure of $\mathcal{C}$ can be carried over to extra structure on A:

braiding$\to$ R-matrix
rigidity$\to$ antipode
involution$\to$ involution
$C^* \to C^*$ (understood in a twisted sense on A:
$\Delta(a^*) = \Omega^{-1}\Delta(a)^*\Omega$ etc, if F_2 is not a partial unitary)

Generality of weak quasi-Hopf algebras

Definition: A weak (integral) dimension function is a function

$$D : \mathrm{Gr}(\mathcal{C}) \to \mathbb{N}, \quad D(\rho\sigma) \leq D(\rho)D(\sigma), \quad D(\rho) = D(\rho^\vee)$$

- For any fusion category $\mathcal{C}$, a weak dimension function D exists and is non-unique (there are infinitely many)
- A weak quasi tensor functor $\mathcal{C} \to$ Vec is determined, up to twist and isomorphism, by the associated weak dimension function D.
- In case of a tensor C^*-category, $\mathcal{C} \to$ Hilb is determined up to an extra 1-twist of the C^*-structure

Summary: For fusion categories $\mathcal{C}$, there is a correspondence

$$\{(\mathcal{C}, D)\} \to \{\text{f.d. semis. weak quasi bialg. } A \text{ up to twist}\}$$

Relation WHA and WQHA

In the theory of WHA one has:

Proposition (Bohm-Szlachanyi): A weak Hopf algebra has multiplicative ε (antimultiplicative S) if and only if it is a Hopf algebra

In analogy for WQH:

Proposition: A weak quasi-Hopf algebra is coassociative with $\Phi = \Delta^2(I)$ if and only if it is a Hopf algebra

Hence, strict coassociativity of Δ together with $\Phi = \Delta^2(I)$ its not a good 'trivial associator ' in the weak case (no weak example).

It also follows: $\{\text{weak quasi Hopf}\} \cap \{\text{weak Hopf}\} = \{\text{Hopf}\}$

Quantum groups at roots of unity

Let $U_q(\mathfrak{g})$ be the specialisation of $U_x(\mathfrak{g})$ at $q \in \mathbb{T}$ in the sense Lusztig.
It is known (Wenzl) to be a complex Hopf algebra with *-involution, twisted by an Ω derived from R-matrix.
Interested in

$$x \to q = e^{i\pi/\ell d}, \quad \ell \in \mathbb{N}, \quad \ell > h^\vee, \quad d = \frac{\langle \alpha_l, \alpha_l \rangle}{\langle \alpha_s, \alpha_s \rangle}$$

Then $U_q(\mathfrak{g})$ is not semisimple.

- Let Tilt_ℓ be the category of modules admitting a Weyl filtration, together with their duals. Negligible modules and morphisms appear in Tilt_ℓ:

$$q-\dim(V) = 0.$$

Fusion category $\mathcal{F}(\mathfrak{g}, \ell)$

The quotient category

$$\mathcal{F}(\mathfrak{g}, \ell) = \mathsf{Tilt}_\ell/\mathsf{Neg}$$

has morphisms $(W, W')/\mathsf{Neg}(W, W')$. Known to be:

- Semisimple tensor category with simple objects labelled by

  $$\Lambda_\ell := \{\lambda \in \Lambda^+ : \langle \lambda + \rho, \theta \rangle < d\ell\}, \quad \langle \alpha_s, \alpha_s \rangle = 2$$

  (Andersen, Gelfand-Kazhdan)
- modular (Kirillov, Turaev)
- tensor C^*-category (Wenzl-Xu)

Examples $A(\mathfrak{g}, \ell)$ from quantum groups

Now, $D(V_\lambda) = \dim(V_\lambda)$ is a weak dimension function. Hence we have

$$A(\mathfrak{g}, \ell), \quad \text{weak quasi Hopf } C^*\text{-algebra}$$

with irreducible reps of classical dimension:

$$A(\mathfrak{g}, \ell) \simeq \oplus_{\lambda \in \Lambda_\ell{}^+} M_{\dim_{V_\lambda}}(\mathbb{C})$$

$$\text{Rep}(A(\mathfrak{g}, \ell)) \simeq \text{Tilt}_\ell / \text{Neg}.$$

A special subclass: w-Hopf algebras

Defn: Given a weak quasi bialgebra A, the associator Φ is called a 3-coboundary if there is a twist $F \in A \otimes A$, in the weak sense, such that

$$\Phi = 1 \otimes \Delta(F^{-1}) I \otimes F^{-1} F \otimes I \Delta \otimes 1(F).$$

Let $\mathcal{C}$ be a fusion category and D a weak dimension function on $\mathrm{Gr}(\mathcal{C})$. The question of whether the associator Φ of the weak quasi Hopf algebra A associated to $(\mathcal{C}, D)$ be a 3-coboundary depends only on $(\mathcal{C}, D)$.

If this is the case, the weak quasi Hopf algebra A_F has associator of the form

$$\Phi' = 1 \otimes \Delta' \Delta'(I)\ \Delta' \otimes 1 \Delta'(I).$$

These are called w-Hopf algebras.

$$\{\text{Hopf}\} \subset \{w - \text{Hopf}\} \subset \{\text{weak quasi Hopf}\}$$

Categorical definition of *w*-Hopf algebras

- A discrete w-Hopf algebra corresponds, via duality, to a weak quasi tensor functor $(F, F_2) : \mathcal{C} \to \mathrm{Vec}$ making the diagram,

$$\begin{array}{ccccc}
F((\rho\sigma)\tau) & \xrightarrow{F_2^{-1}} & F(\rho\sigma) \otimes F(\tau) & \xrightarrow{F_2^{-1} \otimes 1} & (F(\rho)F(\sigma))F(\tau) \\
\downarrow{\scriptstyle F(\alpha)} & & & & \downarrow \\
F(\rho(\sigma\tau)) & \xleftarrow{F_2} & F(\rho) \otimes F(\sigma\tau) & \xleftarrow{1 \otimes F_2} & F(\rho)(F(\sigma)F(\tau))
\end{array}$$

  together with an analogous diagram for α^{-1}, commutative.

This can be used to construct new examples from old ones:

Given a w-Hopf A, a rigid full tensor subcategory of $\mathrm{Rep}(A)$ gives rise to a quotient w-Hopf algebra $A \twoheadrightarrow B$, and a rigid tensor subcategory of $\mathrm{Rep}(A)$ with the same objects defines larger w-Hopf algebras $A \hookrightarrow B$.

- Trivial associators are rigid: two w-Hopf algebras with isomorphic algebras and coproducts are isomorphic as weak quasi Hopf algebras.
- 2-cocycles may be defined. A twist of a w-Hopf algebra by a 2-cocycle is another such.
- The antipode of a w-Hopf algebra can be defined in the same way as for Hopf algebras and is unique.
- weak tensor functors preserve right and left dual objects.
- Explicit general commutation relations between S and * can be derived: $S(a) = \omega S^{-1}(a^*)^* \omega^{-1}$, $\omega = m(S \otimes 1(\Omega^{-1}))$.
- analogue of Podles-Woronowicz nondegeneracy conditions hold: $\Delta(I) A \otimes A = \Delta(A) I \otimes A, \quad A \otimes A \Delta(I) = A \otimes I \Delta(A)$.
- Ribbon structures understood in the framework of weak quasi Hopf algebras turn out computationally manageable.

Tensor * category $\mathrm{Rep}_h(A)$ or C^*-category $\mathrm{Rep}^+(A)$

Let $\mathrm{Rep}_h(A)$ be the category of f.d. *-representations on nondegenerate Hermitian spaces.

If $\Omega \geq 0$, let $\mathrm{Rep}^+(A)$ be the full subcategory of *-rep on Hilbert spaces.

tensor product form: $(\xi \otimes \xi', \eta \otimes \eta') := (\xi \otimes \xi', \Omega\eta \otimes \eta')_{\mathrm{product}}$.

Prop: $\mathrm{Rep}_h(A)$ is a tensor *-category, $\mathrm{Rep}^+(A)$ is a tensor C^*-category. Associativity morphisms are unitary.

Twist invariance: $\mathrm{Rep}^+(A) \simeq \mathrm{Rep}^+(A_F)$ unitary tensor *-equivalence. Similarly in the Hermitian case.

Theorem: If A is a discrete w-Hopf pre-C^*-algebra, $\Omega \geq 0$ then $\mathrm{Rep}^+(A)$ is rigid.

Main results on w-Hopf algebras

Theorem Let A be a semisimple f.d. weak quasi Hopf algebra with a 3-coboundary associator. Then A has a compatible C^*-structure. Hence Rep(A) is tensor equivalent to a tensor C^*-category.

Examples of non-unitarizable fusion categories are known (Yang-Lee category, examples of Frolich-Kerler and Rowell from quantum groups). Hence no weak dim function can correspond to a w-Hopf algebra.

Theorem For $\mathfrak{g} = \mathfrak{sl}_N$, the associator of $A(\mathfrak{sl}_N, \ell)$ is a 3-coboundary. That is, $A(\mathfrak{sl}_N, \ell)$ can be twisted to a w-Hopf C^*-algebra.

First proven in 2015. New proof independent of Wenzl-Xu theory, and reproving Wenzl-Xu in this specific case.

More examples of w-Hopf algebras

• $\mathrm{Vec}_{\mathbb{Z}_2}$ has infinitely many weak integral dimension functions corresponding to w-Hopf algebras.

The Grothendieck ring of $\mathcal{C}(\mathfrak{s}l_2, 2+k)$ is the Verlinde fusion ring with base $V_0 = I, V_1, \dots, V_k$. $X = V_k$ generates a subring $\mathbb{Z}\mathbb{Z}_2$. Then X generates a pointed full tensor subcategory of $\mathcal{C}(\mathfrak{s}l_2, 2+k)$. We thus have a new w-Hopf algebra B_k with $D(X) = k+1$ arising as a quotient of $A(\mathfrak{s}l_2, 2+k)$. Hence

$$B_k \simeq \mathbb{C} \oplus M_{k+1}(\mathbb{C}).$$

These tensor subcategories are known to be tensor equivalent to $\mathrm{Vec}^{\omega}_{\mathbb{Z}_2}$ with ω trivial for k even and non trivial for k odd.

This is in contrast with uniqueness of dimension function in fusion categories.

Constructing tensor C^*-categories in CFT

In CFT one would like to explicitly construct unitary tensor structures on the categories at hand. If a tensor category is known to be equivalent to a tensor C^*-category, (that is, unitarizable), it is not clear how to use this information to transfer the unitary structure. The main instance is the case of fusion categories of affine VOAs and quantum groups. The difficulty resides in the notion of tensor equivalence.

Let V be a vertex operator algebra. Under certain rationality assumptions due to Huang, the category $\mathrm{Rep}(V)$ of V-modules is known to be a modular tensor category. There is a notion of unitary VOA (Carpi-Kawahigashi-Longo-Weiner; Dong-Lin) which allows to speak about

$$\mathrm{Rep}^+(V) = \text{category of unitary } V\text{-modules}$$

It is a C^*-category. Assume that every V-module is unitarizable. Then the forgetful

$$\mathrm{Rep}^+(V) \to \mathrm{Rep}(V)$$

is a $\mathbb{C}$-linear equivalence from a C^*-category to a tensor category.

An important problem is that of making $\mathrm{Rep}^+(V)$ into a C^*-tensor category tensor equivalent to $\mathrm{Rep}(V)$.

Affine VOAs

Let $\mathfrak{g}$ complex simple f.d. Lie algebra, k positive integer, $V_{\mathfrak{g}_k}$ level k affine unitary vertex operator algebra, so $\mathrm{Rep}(V_{\mathfrak{g}_k})$ is a modular tensor category (Huang, Lepowski) .

$V_{\mathfrak{g}_k}$ is a unitary VOA with irreducible modules labelled by $\Lambda^{(k)}$

$$\Lambda^{(k)} = \{\lambda \in \Lambda^+ : (\lambda, \theta) \leq k\}$$

FKL theory: Quantum groups and affine VOA

A connection between the fusion (quotient) category of $U_q(\mathfrak{g})$ at $q = e^{i\pi/\ell d}$ and affine VOA has been developed in a series of papers by Finkelberg-Kazhdan-Lusztig:

$$\mathrm{Rep}(V_{\mathfrak{g}_k}) \simeq \mathcal{F}(\mathfrak{g}, \ell), \qquad \ell = k + \check{h}$$

excluding perhaps some exceptional cases.

Hence $\mathrm{Rep}(V_{\mathfrak{g}_k})$ is equivalent to a tensor C^*-category, by Wenzl-Xu theory.

We use the theory of weak quasi Hopf algebras to explicitly upgrade the C^*-category $\mathrm{Rep}^+(V_{\mathfrak{g}_k})$ to a tensor C^*-category equivalent to $\mathrm{Rep}(V_{\mathfrak{g}_k})$

Theorem: Let $V_{\mathfrak{g}_k}$ be the affine level k vertex operator algebra. Then the C^*-category $\mathrm{Rep}^+(V_{\mathfrak{g}_k})$ of unitary representations of $V_{\mathfrak{g}_k}$ admits the structure of a C*-tensor category with unitary braided symmetry s.t. the forgetful $\mathrm{Rep}^+(V_{\mathfrak{g}_k}) \to \mathrm{Rep}(V_{\mathfrak{g}_k})$ is a braided tensor equivalence.

Note: B. Gui proved an analogous result for $\mathfrak{s}l_N$, $N \geq 2$ and $\mathfrak{s}o_{2N}$, $N \geq 3$ by a completely different method.

Abstract setting

$\mathcal{C}$ tensor category; $\mathcal{C}^+$ C^*-category s.t. every object is a finite direct sum of irreducibles and $\mathcal{F} : \mathcal{C}^+ \to \mathcal{C}$ linear equivalence.

Interpretation: 'every representation of $\mathcal{C}$ is equivalent to a unitary representation'. $\mathcal{F}$ forgets the unitary structure. We wish to construct a tensor product on the C^*-category $\mathcal{C}^+$ of unitary reps.

Defn: The tensor structure of $\mathcal{C}$ is C^*-transportable if $\mathcal{C}^+$ can be upgraded to a tensor C^*-category and $\mathcal{F}$ to a tensor equivalence.

If $\mathcal{C}$ has a weak dimension function we may construct a weak quasi-Hopf algebra A with a C^*-structure and a commutative diagram of equivalences unique up to twists

$$\begin{array}{ccc} \mathcal{C}^+ & \xrightarrow{\mathcal{E}^+} & \mathrm{Rep}^+(A) \\ \downarrow \mathcal{F} & & \downarrow \\ \mathcal{C} & \xrightarrow{\mathcal{E}} & \mathrm{Rep}(A) \end{array}$$

$\mathcal{E}^+$ *-equivalence, $\mathcal{E}$ tensor equivalence.

Prop: The tensor structure of $\mathcal{C}$ is C^*-transportable to $\mathcal{C}^+$ via $\mathcal{F}$ if and only if A can be upgraded to an Ω-involutive algebra with $\Omega \geq 0$. All equivalences become tensorial.

Given Ω, $\mathrm{Rep}^+(A)$ becomes a tensor category, and we may transfer its tensor structure to $\mathcal{C}^+$ making $\mathcal{E}^+$ into a unitary tensor *-equivalence.

Theorem: C^*-transportability holds if $\mathcal{C}$ has a weak dimension function and is unitarizable.

Examples:

- $\mathcal{C}$ admits a weak tensor functor to Vec or
- $\mathcal{C}$ is a unitarizable fusion category

Back to affine VOAs

If $\lambda \in \mathrm{Rep}(V_{\mathfrak{g}_k})$ is irreducible,

$$H_\lambda = \bigoplus_{n \in \mathbb{Z} \geq 0} H_n^\lambda,$$

$H_n^\lambda = \mathrm{Ker}(L_0^\lambda - (h+n)1_{H_\lambda})$ for a unique scalar h, H_n^λ f.d., $H_0^\lambda \neq 0$.

We define a linear functor

$$\mathcal{F} : \mathrm{Rep}(V_{\mathfrak{g}_k}) \to \mathrm{Vec},$$

$$\lambda \to H_0^\lambda, \quad T \to T \restriction_{H_0^\lambda}$$

which is faithful (minimum energy functor). Let

$$A(V_{\mathfrak{g}_k}) = \mathrm{Nat}(\mathcal{F})$$

it is a semisimple associative algebra and may be identified with Zhu algebra of CFT.
Verlinde's fusion rules imply that $D(\lambda) = \mathrm{dim}(H_0^\lambda)$ is a weak dimension function. Thus Zhu algebra $A(V_{\mathfrak{g}_k})$ admits the structure of a weak quasi Hopf algebra and $\mathrm{Rep}(V_{\mathfrak{g}_k}) \simeq \mathrm{Rep}(A(V_{\mathfrak{g}_k}))$.
This functor has already appeared in the literature with no mention to tensor structure (Dong-Mason; Huang-Yang; Zhu)

Unitarity of V gives $A(V_{\mathfrak{g}_k})$ a canonical *-algebra structure.

Every unitary $V_{\mathfrak{g}_k}$-module M induces a C^*-representation on $A(V_{\mathfrak{g}_k})$. This makes $A(V_{\mathfrak{g}_k})$ into a f.d. C^*-algebra.

FKL equivalence theorem implies that Zhu algebra and the algebras from quantum groups are related by a twist F:

$$A(V_{\mathfrak{g}_k})_F \simeq A(\mathfrak{g}, \ell), \quad \ell = k + h^\vee$$

Hence $A(V_{\mathfrak{g}_k})$ inherits a positive Ω-involution. We may apply abstract unitarizability results to this case.

Conclusions and conjectures

- WQHA are general in fusion categories, useful for questions of unitarizability, e.g. for affine vertex operator algebras
- There is a special subclass, the w-Hopf algebras, which is cohomologically motivated, enjoys a complete theory; many ideas of the compact quantum groups can be extended; it includes q-group/level k affine VOA over $\mathrm{SU}(N)$, and new examples arise from old ones. It is unclear how general they are among fusion categories.
- If existence of w-Hopf algebras can be shown for quantum groups of other Lie types, these may be used to reprove Wenzl-Xu theory in those cases and, conjecturally, a connection between fusion categories of quantum groups/affine VOAs with those of loop group conformal nets in the SU(N)-case may be established.