Non-formal Deformation Quantization and abstract O^*-Algebras

Based on joint work with Stefan Waldmann

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On Noncommutativity and Physics:
Hopf algebras in Noncommutative Geometry

Outline

- Formal star product on Poisson manifold M:
  Associative $\mathbb{C}[\![\hbar]\!]$-linear product $\star$ on $\mathcal{C}^\infty(M)[\![\hbar]\!]$, such that

$$f \star g = fg + \mathcal{O}(\hbar)$$
$$\frac{1}{\mathrm{i}\hbar}[\,f\,,\,g\,]_\star = \{\,f\,,\,g\,\} + \mathcal{O}(\hbar)$$

for all $f, g \in \mathcal{C}^\infty(M)$.

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- Convergent star product:
  Subalgebra $\mathscr{A} \subseteq \mathcal{C}^\infty(M)$ with locally convex topology and formal star product $\star$ on M such that $f \star g$ converges for all $f, g \in \mathscr{A}$ and all (or sufficiently many) $\lambda \in \mathbb{C}$.

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- Ideally: Pointwise complex conjugation as *-involution.

Exponential star products on $\mathbb{R}^n$

$$f \star_{\Lambda,\hbar} g = \sum_{r=0}^{\infty} \frac{\hbar^r}{r!} \sum_{\substack{i_1,\dots,i_r=1 \\ j_1,\dots,j_r=1}}^{2n} \Lambda^{i_1,j_1} \dots \Lambda^{i_n,j_n} \frac{\partial^r f}{\partial x^{i_1} \dots \partial x^{i_n}} \frac{\partial^r g}{\partial x^{j_1} \dots \partial x^{j_n}}$$

with $\Lambda \in \mathbb{C}^{n \times n}$ such that $-2\mathrm{i}\Lambda_{\mathrm{asym}} = \pi$.

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- For general $f, g \in \mathcal{C}^\infty(M)$: only defined as formal star product.
- Convergent on polynomials $f, g \in \mathbb{C}[x^1, \dots, x^n]$.
- Complex conjugation is *-involution if Λ is Hermitian (e.g. Weyl- or Wick star product).

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Only trivial submultiplicative seminorm on ccr

If $\|\cdot\|$ is a submuliplicative seminorm on a unital associative algebra $\mathscr{A}$, and $P, Q \in \mathscr{A}$ fulfil $[P, Q] := PQ - QP = \mathbb{1}$, then $\|\cdot\| = 0$:
PROOF:

$$n!\|\mathbb{1}\| = \|[\dots[P^n, Q], \dots, Q]\| \leq 2^n \|P\|^n \|Q\|^n$$

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$\Rightarrow \quad \mathscr{A}$ cannot be lmc-*-algebra.

Given such a *-algebra $\mathscr{A}$...

- Understand functionals calculus.
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Heuristically, pure states of commutative *-algebras are characters:

Definition

A *state* on $\mathscr{A}$ is an $\omega \in \mathscr{A}^*$ such that $\langle \omega , a^* a \rangle \geq 0$ for all $a \in \mathscr{A}$ and $\langle \omega , \mathbb{1} \rangle = 1$.
A *pure state* on $\mathscr{A}$ is an extreme point of the convex set of states.
A *character* of $\mathscr{A}$ is a unital *-homomorphism $\omega \colon \mathscr{A} \to \mathbb{C}$.

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- Are characters and pure states identical for commutative $\mathscr{A}$?

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Theorem: Topologisation for constant Poisson structures

There exists a coarsest locally convex topology on $\mathbb{C}[x^1, \ldots, x^n]$ such that $\star$ for all Hermitian $\Lambda \in \mathbb{C}^{n\times n}$ and all $\hbar \geq 0$, the *-involution (complex conjugation) and the evaluation functional at 0, $\delta_0 \colon \mathbb{C}[x^1, \ldots, x^n] \to \mathbb{C}$, $f \mapsto \delta_0(f) := f(0)$, are continuous. This topology is the one defined by the norms coming from the inner products $\langle f \mid g \rangle_{\Lambda,\hbar} := \delta_0(f^* \star g)$ for all positive definite $\Lambda \in \mathbb{C}^{n\times n}$ and $\hbar > 0$. (S., Waldmann; 2018)

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- Equivalence transformations are continuous.
- There are many continuous positive linear functionals.
- $\star$-exponentials of linear functions exist in completion.
- Everything remains valid when $\mathbb{R}^n$ is replaced by nuclear space V, i.e. when $\mathbb{C}[x^1, \dots, x^n]$ is replaced by $\mathcal{S}^\bullet(V)$.

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  - Ω^+_{H} is a cone.
  - $a \triangleright \omega \in \Omega^+_{\mathrm{H}}$ for all $a \in \mathscr{A}$ and $\omega \in \Omega^+_{\mathrm{H}}$, where $\langle\, a \triangleright \omega\,,\, b\,\rangle := \langle\, \omega\,,\, a^* b\, a\,\rangle$ for all $b \in \mathscr{A}$.
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- A morphism of abstract O^*-algebras $(\mathscr{A}, \Omega^+_{\mathrm{H}})$ and $(\mathscr{B}, \mathcal{R}^+_{\mathrm{H}})$ is a unital *-homomorphism $M \colon \mathscr{A} \to \mathscr{B}$ such that $M^*(\psi) := \psi \circ M \in \Omega^+_{\mathrm{H}}$ for all $\psi \in \mathcal{R}^+_{\mathrm{H}}$.

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- A representation as operators of an abstract O^*-algebra $(\mathscr{A}, \Omega^+_{\mathrm{H}})$ is a morphism from $(\mathscr{A}, \Omega^+_{\mathrm{H}})$ to $\big(\mathcal{L}^*(\mathfrak{H}), \mathcal{X}(\mathfrak{H})\big)$ for some pre-Hilbert space $\mathfrak{H}$, where $\mathcal{L}^*(\mathfrak{H})$ is the *-algebra of adjointable (in the algebraic sense) endomorphisms of $\mathfrak{H}$ and $\mathcal{X}(\mathfrak{H})$ generated by the positive linear functionals on $\mathcal{L}^*(\mathfrak{H})$ of the form $\mathcal{L}^*(\mathfrak{H}) \ni a \mapsto \langle\, \chi_\xi \,,\, a \,\rangle := \langle \xi \,|\, a(\xi) \rangle$ with $\xi \in \mathfrak{H}$.

Properties of representations

Let $\mathscr{A}$, $\mathscr{B}$ be locally convex *-algebras with at least separately continuous multiplication. $\Phi\colon \mathscr{A} \to \mathscr{B}$ continuous unital *-homomorphism.

- Construct $\Omega^+_{\mathrm{H}} \subseteq \mathscr{A}^{*,+}_{\mathrm{H}}$ as the cone of continuous positive linear functionals on $\mathscr{A}$. Analogous $\mathcal{R}^+_{\mathrm{H}} \subseteq \mathscr{B}^{*,+}_{\mathrm{H}}$.
- Then $\Phi\colon (\mathscr{A}, \Omega^+_{\mathrm{H}}) \to (\mathscr{B}, \mathcal{R}^+_{\mathrm{H}})$ is a morphism of abstract O^+-algebras
  $\Rightarrow$ functoriality of the construction.
- Then a unital *-homomorphism $\Psi\colon \mathscr{A} \to \mathcal{L}^*(\mathfrak{H})$ is weakly continuous if and only if Ψ is a representation as operators of $(\mathscr{A}, \Omega^+_{\mathrm{H}})$.
  $\Rightarrow$ representations of locally convex *-algebras can be studied via abstract O^*-algebras.

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Similar constructions for strongly continuous representations of locally convex *-algebras, representations of ordered *-algebras,...

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Theorems: (S., Waldmann; 2018 und S; 2018)

- Nelsons criterium for essential self-adjointness can be transfered to abstract O^*-algebras. This yields:
- All Hermitian elements of up to degree 2 and all positive Hermitian elements up to degree 4 are essentially self-adjoint in all continuous representations.

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- All Hermitian elements of up to degree 2 and all positive Hermitian elements up to degree 4 are essentially self-adjoint in all continuous representations.
- Pure states and characters of commutative abstract O^*-algebras coincide under very general assumptions. This yields:
- The continuous pure states of the classical limit are just the continuous characters, which are the evaluation functionals on points of $\mathbb{R}^n$ (or on points of the topological dual of V).

Thank you for your attention!