Quantisation ambiguities and non-commutative waves in 3d gravity

Bernd Schroers
Maxwell Institute and Department of Mathematics
Heriot-Watt University, Edinburgh, UK
b.j.schroers@hw.ac.uk

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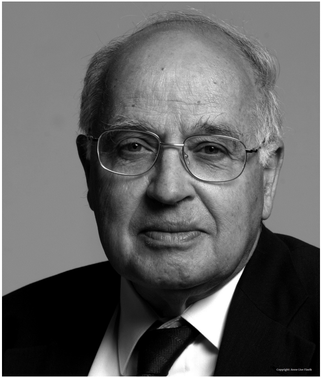
Motivation

- Review systematic way of quantising 3d gravity in Chern-Simons formulation
- Exhibit Hopf algebras and non-commutative spaces which arise
- Discuss some recent results regarding quantisation ambiguities
- Discuss some recent results for non-commutative waves describing gravitational anyons

References

1. Prince Osei and BJS, Classical r-matrices for the generalised Chern-Simons formulation of 3d gravity, Class. Quant. Grav. 35 (2018) 075006
2. Sergio Inglima and BJS, Non-commutative waves for gravitational anyons, arXiv:1804.05782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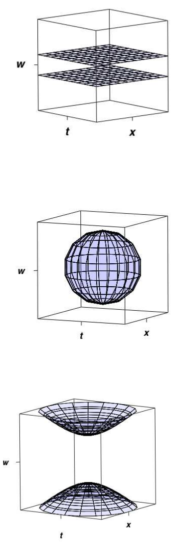
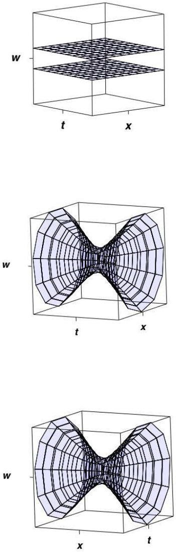
3d Gravity in Chern-Simon formulation



(a) Euclidean



(b) Lorentzian

Local isometry groups of 3d gravity

Cos. constant	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\Lambda = 0$	$SU(2) \ltimes \mathbb{R}^3$	$SL(2,\mathbb{R}) \ltimes \mathbb{R}^3$
$\Lambda > 0$	$SU(2) \times SU(2)$	$SL(2,\mathbb{C})$
$\Lambda < 0$	$SL(2,\mathbb{C})$	$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

Isometry Lie algebras

With metric $\eta_{ab} = \text{diag}(1, -1, -1)$, and

$$\lambda = -c^2\Lambda = \theta^2$$

isometry Lie algebra $\mathfrak{g}_\lambda$ is

$$[J_a, J_b] = \epsilon_{abc}J^c, \quad [J_a, P_b] = \epsilon_{abc}P^c, \quad [P_a, P_b] = \lambda\epsilon_{abc}J^c.$$

Have bilinear pairings

$$t(J_a, J_b) = 0, \quad t(P_a, P_b) = 0, \quad t(J_a, P_b) = \eta_{ab},$$

and

$$s(J_a, J_b) = \eta_{ab}, \quad s(P_a, P_b) = \lambda\eta_{ab}, \quad s(J_a, P_b) = 0.$$

Most general such form parametrised in terms of $\tau = \alpha + \theta\beta$

$$(\cdot, \cdot)_\tau = \alpha t(\cdot, \cdot) + \beta s(\cdot, \cdot),$$

which is non-degenerate if $\tau\bar{\tau} = \alpha^2 - \lambda\beta^2 \neq 0$.

Chern-Simons formulation

Cartan geometry combines spin connection ω with frame field e:

$$A = \omega_a J^a + e_a P^a,$$

The Chern-Simons action for the gauge field A is

$$I_\tau(A) = \int_M (A \wedge dA)_\tau + \frac{1}{3}(A \wedge [A, A])_\tau.$$

Integrating by parts and ignoring boundary terms, this can be expanded as

$$\begin{aligned} I_\tau(A) = \alpha \int_M \left(2e^a \wedge R_a + \frac{\lambda}{3}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right) \\ + \beta \int_M \left(\omega^a \wedge d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a \wedge \omega^b \wedge \omega^c + \lambda e^a \wedge T_a \right). \end{aligned}$$

With $\alpha = \dfrac{1}{16\pi G}$ this is ‘Einstein + cosmological’ and ‘Immirzi term’

Fock-Rosly compatibility I

Fock-Rosly define Poisson structure on extended phase space of holonomies in terms of classical r-matrix

$$r \in \mathfrak{g}_\lambda \otimes \mathfrak{g}_\lambda$$

satisfying a compatibility condition involving Casimir associated to τ:

$$K_\tau = \frac{\alpha}{\tau\bar{\tau}}(J_a \otimes P^a + P_a \otimes J^a) - \frac{\beta}{\tau\bar{\tau}}(\lambda J_a \otimes J^a + P_a \otimes P^a).$$

Seek $r' \in \mathfrak{g}_\lambda \wedge \mathfrak{g}_\lambda$ so that $r = r' + K$ satisfies CYBE, or

$$[[r', r']] = -[[K_\tau, K_\tau]].$$

Expanding in terms of antiy-symmetric matrices A and C, and matrix B

$$r' = A_{ba} J^a \otimes J^b + B_{ba} P^a \otimes J^b - B_{ba} J^b \otimes P^a + C_{ba} P^a \otimes P^b$$

In terms of

$$\mu + \theta\nu = \frac{1}{\tau^2} \Leftrightarrow \mu = \frac{\alpha^2 + \lambda\beta^2}{(\alpha^2 - \lambda\beta^2)^2}, \quad \nu = -\frac{2\alpha\beta}{(\alpha^2 - \lambda\beta^2)^2},$$

condition is

$$\begin{aligned}
\frac{1}{2}\mathrm{tr}(A^2) - \frac{\lambda}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right) &= \mu\lambda, \\
\mathrm{tr}(CB) &= \nu, \\
(B - \mathrm{tr}(B)\mathrm{id})(B + B^t) + \frac{1}{2}\left(\mathrm{tr}(B)^2 - \mathrm{tr}(B^2)\right)\mathrm{id}& \\
-CA + \lambda(C^2 - \frac{1}{2}\mathrm{tr}(C^2)\mathrm{id}) &= -\mu\,\mathrm{id}, \\
-A(B + B^t) + (B^t - \mathrm{tr}(B)\mathrm{id})\,(\lambda C - A) - \mathrm{tr}(AB)\mathrm{id} &= -\lambda\nu\,\mathrm{id}.
\end{aligned}$$

Some compatible gravitational r-matrices[1]

Set $\beta = 0$, and define 3d Planck mass and length

$$m_P = \frac{1}{4G}, \quad \ell_P = \frac{\hbar}{m_P} = 4\hbar G$$

Classical doubles

$$r = \frac{2\pi}{m_P}(P_a \otimes J^a) + m^c \epsilon_{abc} J^a \otimes J^b, \quad \vec{m}^2 = -\lambda$$

Momentum co-multiplication when $\lambda = 0, \vec{m} = 0$

$$\delta P_a = \frac{2\pi}{m_P} \epsilon_{abc} P^b \otimes P^c$$

Position commutators

$$[X_a, X_b] = 2\pi \ell_P \epsilon_{abc} X_c$$

[1] C Meusburger and BJS 2008; Ballesteros, Herranz and Meusburger 2013

Bicrossproduct Poisson-Lie algebra for $\lambda = 0$:

$$r' = q^a \epsilon_{abc}(P^b \otimes J^c - J^c \otimes P^b), \quad \vec{q}^2 = -\left(\frac{4\pi}{m_P}\right)^2$$

Momentum co-multiplication

$$\delta P_a = \frac{4\pi}{m_P}(\vec{n} \cdot \vec{P}) \wedge P_a, \quad \vec{n}^2 = -1.$$

Position commutators

$$[X_a, X_b] = 4\pi \ell_P (n_a X_b - n_b X_a).$$

Symmetry is κ-Poincaré algebra[2] with *spacelike* deformation parameter.

[2] Lukierski, Nowicki, Ruegg and Tolstoi 1991

Quantum isometry groups in 3d quantum gravity, $q = e^{-\frac{\hbar G\sqrt{\Lambda}}{c}}$

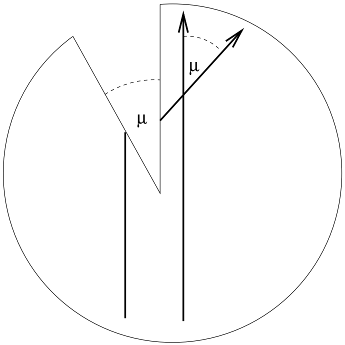
Cos. const.	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\Lambda = 0$	$D(U(su(2)))$	$D(U(su(1,1)))$
$\Lambda > 0$	$D(U_q(su(2)))$, q root of unity	$D(U_q(su(1,1)))$ $q \in \mathbb{R}$
$\Lambda < 0$	$D(U_q(su(2)))$, $q \in \mathbb{R}$	$D(U_q(sl(2,\mathbb{R})))$, $q \in U(1)$

Non-commutative waves for gravitational anyons ($\Lambda = 0$)

Fractional spin in 3d gravity[3]

Spacetime surrounding a particle of mass *m* is cone with deficit angle

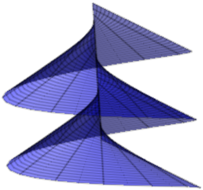
$$\mu = 8\pi G m$$



Scattering is ‘classical Aharonov-Bohm scattering’

[3] ’t Hooft 1988

Fractional spin in 3d gravity[4]



Spacetime surrounding a particle of mass m and spin s is 'twisted cone'

Simple quantisation argument for wavefunction on angular range $[0, 2\pi - \mu)$ gives spin values

$$s = \frac{k}{1 - \frac{\mu}{2\pi}}, \qquad k \in \mathbb{Z}.$$

[4] Bais, Mulller and BJS 2002

The quantum double $D(G)$ of a Lie group G is a ribbon-Hopf algebra. As a vector space $D(G) = C(G \times G)$, with multiplication, co-product, unit, co-unit, antipode and ribbon element given by

$$\begin{aligned}
(F_1 \bullet F_2)(g,u) &= \int_G F_1(v, vuv^{-1}) F_2(v^{-1}g, u) dv, \\
1(g,u) &= \delta_e(g), \\
(\Delta F)(g_1, u_1; g_2, u_2) &= F(g_1, u_1 u_2) \delta_{g_1}(g_2), \\
\epsilon(F) &= \int_G F(g,e) dg, \\
(SF)(g,u) &= F(g^{-1}, g^{-1}u^{-1}g), \\
F^*(g,u) &= \overline{F(g^{-1}, g^{-1}ug)}. \\
c(g,u) &= \delta_g(u),
\end{aligned} \tag{1}$$

The Lorentz group and its covers

The proper, orthochronous part of the Lorentz group in 2+1 dimensions is double covered by

$$\mathrm{SU}(1,1) \simeq \mathrm{SL}(2,\mathbb{R})$$

Can parametrise $u \in SU(1,1)$ in terms of $\omega \in [0,4\pi), \gamma \in D \subset \mathbb{C}$ as

$$u(\omega,\gamma) = \frac{1}{\sqrt{1-|\gamma|^2}} \begin{pmatrix} e^{i\frac{\omega}{2}} & \bar{\gamma}e^{-i\frac{\omega}{2}} \\ \gamma e^{i\frac{\omega}{2}} & e^{-i\frac{\omega}{2}} \end{pmatrix} = \frac{1}{\sqrt{1-|\gamma|^2}} \begin{pmatrix} 1 & \bar{\gamma} \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} e^{i\frac{\omega}{2}} & 0 \\ 0 & e^{-i\frac{\omega}{2}} \end{pmatrix}.$$

The universal cover is not a matrix group, but can be identified with the interior of an infinite cylinder $\mathbb{R} \times D$, with group multiplication

$$\omega = \omega_1 + \omega_2 + \frac{1}{i} \ln \left(\frac{1 + \bar{\gamma}_1 \gamma_2 e^{-i\omega_1}}{1 + \gamma_1 \bar{\gamma}_2 e^{i\omega_1}} \right),$$
$$\gamma = (\gamma_1 + \gamma_2 e^{-i\omega_1})(1 + \bar{\gamma}_1 \gamma_2 e^{-i\omega_1})^{-1}.$$

Central elements $\Omega = (2\pi, 0)$ projects to $-\mathrm{id}$ under $\pi : \widetilde{SU}(1,1) \to SU(1,1)$ and $\ker \pi = \{\Omega^{2n} | n \in \mathbb{Z}\}$.

The universal cover of the Poincaré group

$$P_3^\infty = \widetilde{\mathrm{SU}}(1,1) \ltimes \mathfrak{su}(1,1)^*,$$

UIRs are determined by orbits in $(\mathfrak{su}(1,1)^*)^* = \mathfrak{su}(1,1)$ and centraliser representations.

Parametrise time-like elements as

$$p = v(-8\pi G m J^0)v^{-1} = -8\pi G\,\vec{p}\cdot\vec{J}.$$

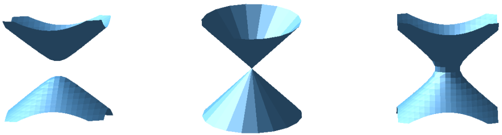


Figure: Adjoint Orbits of $\mathrm{SU}(1,1)$

The carrier space for UIRs describing massive particles is

$$V^A_{ms} = \Big\{\psi \colon \widetilde{\mathrm{SU}}(1,1) \to \mathbb{C} | \psi(\omega+\alpha,\gamma) = e^{-is\alpha}\psi(\omega,\gamma)\ \forall\alpha\in\mathbb{R},$$
$$\forall(\omega,\gamma)\in\widetilde{\mathrm{SU}}(1,1), \int_{\widetilde{\mathrm{SU}}(1,1)/\tilde{N}^T} |\psi^2| d\nu < \infty\Big\}.$$

The action of $((\omega,\gamma),a)\in P_3^\infty$ is

$$\left(\pi^{\mathrm{eq}}_{ms}((\omega,\gamma),a)\psi\right)(v) = \exp\left(i\langle a, \mathrm{Ad}_{((\omega,\gamma)^{-1}v)}(-\mu J^0)\rangle\right)\psi\left((\omega,\gamma)^{-1}v\right).$$

Recall discrete series UIRs of $\widetilde{\mathrm{SU}}(1,1)$. Let $\mathcal{H}_{l\pm}$, $l \in \mathbb{R}^+$ be the completion of holomorphic (anti-holomorphic) functions on the unit disc, and define

$$(D_{l+}(\omega,\gamma)f)(z) = e^{il\omega}(1-|\gamma|^2)^l(1+\bar{\gamma}z)^{-2l}f(z\cdot\pi(\omega,\gamma)).$$

and

$$(D_{l-}(\omega,\gamma)f)(\bar{z}) = e^{-il\omega}(1-|\gamma|^2)^l(1+\gamma\bar{z})^{-2l}f(\overline{z\cdot\pi(\omega,\gamma)}).$$

Write corresponding infinesimal generators as $d_{l\pm}(J^a)$ and let $|0\rangle : z \mapsto 1$ be the vacuum.

Define covariant field by picking (ω,γ) so that $p = -\mu\,\mathrm{Ad}_{(\omega,\gamma)}(s^0)$

$$\tilde{\phi}_\pm \colon O^T_m \to \mathcal{H}_{l\pm}, \qquad \tilde{\phi}_\pm(p) = \psi(\omega,\gamma)D_{l\pm}(\omega,\gamma)\,|0\rangle_l\,.$$

This is well-defined if $s = \pm l$.

The anyonic Majorana equation[5]

The covariant representation on $L^2(\mathfrak{su}(1,1))$ defined via

$$(\pi^{co}_{ms}((\omega,\gamma),a)\tilde{\phi}_\pm)(p) = \exp(i\langle a, \mathrm{Ad}_{(\omega,\gamma))^{-1}}p\rangle)D_{l\pm}((\omega,\gamma))\tilde{\phi}_\pm(\mathrm{Ad}_{(\omega,\gamma)^{-1}}p).$$

is not irreducible. We need spin constraint

$$(d_{l\pm}(p) - i\mu s)\tilde{\phi}_\pm(p) = 0,$$

and the mass constraint:

$$(\vec{p}^2 - m^2)\tilde{\phi}_\pm(p) = 0.$$

Choose $l+$ for $s > 0$ and $l-$ for $s < 0$. Then m and p^0 automatically have the same sign!

Fourier transforms produces field $\phi_\pm : \mathbb{R}^{2,1} \to \mathcal{H}_{l\pm}$ satisfying

[5] Majorana 1932, Plyushchay 1991

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Fourier transforms produces field $\phi_\pm : \mathbb{R}^{2,1} \to \mathcal{H}_{l\pm}$ satisfying

$$\boxed{(\Box + m^2)\phi_\pm = 0, \qquad (d_{l\pm}(J^a)\partial_a - ms)\,\phi_\pm = 0}$$

[5] Majorana 1932, Plyushchay 1991

The Lorentz double[6]

Replace momentum space $\mathfrak{su}(1,1)^*$ of the Poincaré group by $C(\widetilde{\mathrm{SU}}(1,1))$ to deform

$$P_3^\infty \to D(\widetilde{\mathrm{SU}}(1,1))$$

This means

- Lorentz symmetry is still $\widetilde{\mathrm{SU}}(1,1)$. Anyonic spin!
- Momentum space is now $\widetilde{\mathrm{SU}}(1,1)$. What does this mean?
- Double is still ribbon-Hopf algebra.

[6] Bais, Koornwinder, Muller 1997; Bais, Muller, BJS 2002

Classification now in terms of conjugacy classes and centraliser representations. For massive particles, require elliptic conjugacy classes

$$E(\mu) = \left\{ v(\mu, 0)v^{-1} \;\middle|\; v \in \widetilde{\mathrm{SU}}(1,1),\ \mu \in (\mathbb{R} \setminus 2\pi\mathbb{Z}) \right\}.$$

and get UIRs on the same carrier space as for the P_3^∞ UIRs:

$$(\Pi^{\mathrm{eq}}_{ms}(F)\psi)(v) = \int_{\widetilde{\mathrm{SU}}(1,1)} F\left(g, g^{-1}v(\mu, 0)v^{-1}g\right)\psi(g^{-1}v)dg,$$

Define covariant, infinite-component fields

$$\tilde{\phi}_{\pm}\colon E(\mu) \to \mathcal{H}_{l\pm}, \qquad \tilde{\phi}_{\pm}(u) = \psi(v) D_{l\pm}(v) \left|0\right\rangle_l .$$

Here v is chosen so that $u = v(\mu, 0)v^{-1}$ and $s = l$ for $\tilde{\phi}_+$ and $s = -l$ for $\tilde{\phi}_-$.

The covariant fields satisfy the following spin constraint

$$\left(D_{l\pm}(u) - e^{i\mu s}\right) \tilde{\phi}_{\pm}(u) = 0,$$

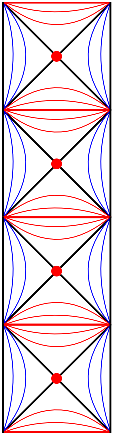
which can be expressed in terms of the ribbon element as

$$\Pi^{\text{co}}_{ms}(c)\tilde{\phi}_{\pm} = e^{i\mu s}\tilde{\phi}_{\pm}.$$

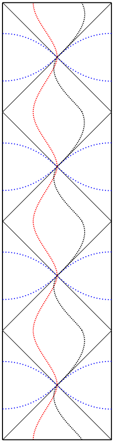
For UIR on $L^2(\widetilde{\mathrm{SU}}(1,1))$ also need a mass constraint

$$\frac{1}{2}\mathrm{tr}(\pi(\omega,\gamma)) = \cos\left(\frac{\mu}{2}\right), \qquad \text{int}\left(\frac{\omega}{2\pi}\right) = \text{int}\left(\frac{\mu}{2\pi}\right).$$

Parametrising curved momentum space



(a) Selected conjugacy classes



(b) Selected exponential curves

Proposition (Parametrisation of $\widetilde{\mathrm{SU}}(1,1)$)

Every element $(\omega, \gamma) \in \widetilde{\mathrm{SU}}(1,1)$ can be uniquely expressed in terms of the 2π-rotation Ω and the exponential map via

$$(\omega, \gamma) = \Omega^n \widetilde{\exp}(p), \quad p = -(8\pi G)\, \vec{p} \cdot \vec{J} \in \mathfrak{su}(1,1),\; n \in \mathbb{Z},$$

with

$$\vec{p}^2 < \frac{1}{(4G)^2}, \text{ and } p^0 > 0 \text{ if } \vec{p}^2 > 0.$$

Group Fourier transform[7]

We want to generalise

$$\mathrm{L}^2(\mathfrak{g}) \to \mathrm{L}^2(\mathfrak{g}^*).$$

to

$$\mathrm{L}^2(G) \to \mathrm{L}^2_\star(\mathfrak{g}^*).$$

Need 'non-commutative plane waves'

$$E : G \times \mathfrak{g}^* \to \mathbb{C},$$

satisfying the following normalisation and completeness relations

$$E(e;x) = 1,\ E(u^{-1};x) = \bar{E}(u;x) = E(u;-x),\ \delta_e(u) = \frac{1}{(2\pi)^d}\int_{\mathfrak{g}^*} E(u;x)\,dx,$$

Then can define

$$E(u_1;x) \star E(u_2;x) = E(u_1 u_2;x).$$

and

$$\mathcal{F} : L^2(G) \to L^2_\star(\mathfrak{g}^*), \quad \phi(x) = \mathcal{F}(\tilde{\phi})(x) = \int_G E(u;x)\tilde{\phi}(u)\,du,$$

[7] Rieffel 1990; Freidel and Livine 2006; Guedes, Oriti and Raasakka 2013

Definition

We define non-commutative plane waves for $\widetilde{\mathrm{SU}}(1,1)$ as the maps

$$E : \widetilde{\mathrm{SU}}(1,1) \times (\mathfrak{su}(1,1)^* \times S^1) \to \mathbb{C},$$
$$E(u; x, \varphi) = \frac{1}{\rho(p)} e^{i(\langle x,p\rangle + n\varphi)}, \tag{2}$$

where $p \in \mathfrak{su}(1,1)$ and $n \in \mathbb{Z}$ are the parameters determining u, the function ρ is defined via $du = \rho(p) d^3\vec{p}$ and $\varphi \in [0, 2\pi)$ is an angular coordinate on the circle S^1.

The integer n labels particle types, and the parameter φ is a dual angular variable.

The mass constraint is split according to

$$\mu = \mu_0 + 2\pi n, \qquad \mu_0 \in (0, 2\pi)$$

into a Klein-Gordon equation for the fractional part

$$\left(\Box + \left(\frac{\mu_0}{2\pi}\right)^2 m_P^2\right)\phi(x,\varphi) = 0.$$

and a differential condition on the angular dependence of ϕ for the integer part:

$$-i\frac{\partial}{\partial\varphi}\phi(x,\varphi) = n\phi(x,\varphi).$$

The spin constraint involves Atiyah's exponentiated Dirac operator [8].

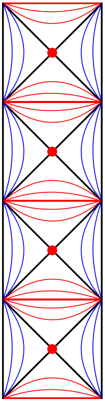
$$\left(e^{2\pi i \ell_P \, d_{l\pm}(J^a)\partial_a} - e^{i\mu_0 s}\right)\phi_\pm(x,\varphi) = 0.$$

[8] Atiyah and Moore 2009

Fusion and braiding

Fusion rules generalising the 'Gott-pair' production are encoded in

$$e^{i(\langle x,p_1\rangle+n_1\varphi)}\star e^{i(\langle x,p_2\rangle+n_2\varphi)}=e^{i(\langle x,p(u_1u_2)\rangle+n(u_1u_2)\varphi)}.$$



The cone constraint revisited

The quantisation condition

$$s = \frac{k}{1 - \frac{\mu}{2\pi}}, \quad k \in \mathbb{Z},$$

follows from

$$\Pi_{\mu s}(c) = \Pi_{\mu s}(\Omega) \Rightarrow e^{i\mu s} = e^{2\pi i s}.$$

After group Fourier transform, and using $\mu = \mu_0 + 2\pi n$, this becomes

$$e^{2\pi i \ell_P \, d_{l\pm}(J^a)\partial_a}\phi = D_{l\pm}(\Omega^{1-n})\phi.$$

Noting that, for the discrete series,

$$(id_{l+}(J^0))^2 - (id_{l+}(J^1))^2 - (id_{l+}(J^1))^2 = s(s-1),$$

this condition means

$$\boxed{\text{Translation by } (2\pi s) \times \text{Planck length} \;=\; \text{Rotation by } \; 2\pi(1-n)}$$

Conclusion

- There is a systematic way of deriving non-commutative geometry from 3d quantum gravity
- The outcome is not unique but related by twist - e.g., quantum double and spacelike κ-Poincaré both possible
- Fourier transform of UIRs of quantum double amounts to Rieffel deformation quantisation of spin spacetime
- Non-commutative waves provide new ‘spacetime’ picture of 3d quantum gravity as braided QFT