Quantization of Poisson-Lie groups and of (easy) Poisson groupoids

Pavol Ševera

Deformation quantization problem for Hopf algebras

Ingredients

- a *commutative* Hopf algebra $(\mathcal{H}, m_0, \Delta_0, S_0, 1, \epsilon)$
- a compatible Poisson bracket $\{,\} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ ($\Delta_0 : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is a Poisson algebra morphism)

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The problem

Find "universal" (functorial) deformations

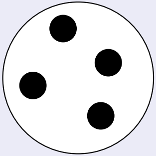
$$m_\hbar = \sum_{n=0}^{\infty} \hbar^n m_n \qquad \Delta_\hbar = \sum_{n=0}^{\infty} \hbar^n \Delta_n \qquad S_\hbar = \sum_{n=0}^{\infty} \hbar^n S_n$$

s.t. $(\mathcal{H}, m_\hbar, \Delta_\hbar, S_\hbar, 1, \epsilon)$ is a Hopf algebra and $m_1 - m_1^{op} = \{,\}$
[For $\mathcal{H} = (U\mathfrak{g})^*$: Etingof-Kazhdan 1995]

The method in a nutshell: holonomies on a surface

(not supposed to be understandable at this point)

Hopf holonomies on a disk ...



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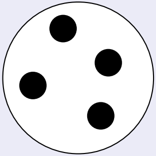

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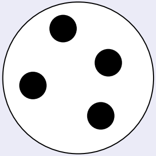

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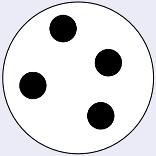

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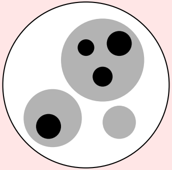

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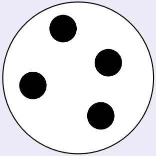

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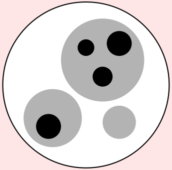

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Quantization: obtain the B_n action via the KZ connection (or from a Drinfeld associator)

The nerve of a group G

holonomies in the "commutative world"

X a finite set

$$F(X) = \{g : X \times X \to G \mid g_{ij}g_{jk} = g_{ik} \;\&\; g_{ii} = 1\,(\forall i, j, k \in X)\}$$

$$F(X) \cong G^{|X|-1}, \text{ e.g. } \quad \bullet \xrightarrow{g_{12}} \bullet \xrightarrow{g_{23}} \bullet \xrightarrow{g_{34}} \bullet \quad (|X| = 4)$$

functoriality: $f : X \to Y \quad \rightsquigarrow \quad f^* : F(Y) \to F(X)$

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From a nerve to its group

If F is the nerve of G then $G = F(\bullet\bullet)$

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From a nerve to its group

If F is the nerve of G then $G = F(\bullet\bullet)$
F is a nerve iff $F(\bullet^n) \to F(\bullet\bullet)^{n-1}$ is an iso.
The product:
$F(\bullet\bullet) \times F(\bullet\bullet) \cong F(\bullet\bullet\bullet) \to F(\bullet\bullet)$

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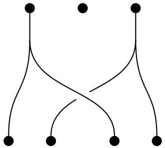
Colliding braids and Hopf algebras

Hopf algebras in terms of braids

BrSet - “braided maps”:



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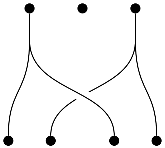

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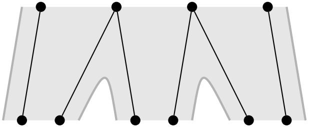

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Hopf algebras (with invertible S) in a BMC $\mathcal{C}$ are equivalent to braided lax-monoidal functors $F : \mathrm{BrSet} \to \mathcal{C}$ such that $F(\bullet\bullet)^{n-1} \to F(\bullet^n)$ is an iso and $1_{\mathcal{C}} \to F() \to F(\bullet)$ are isos

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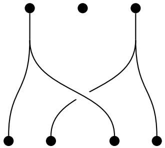

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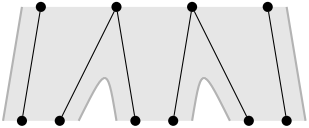

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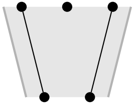
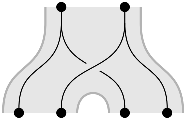
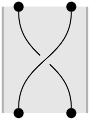
$\mathcal{H} = F(\bullet\bullet)$, $\Delta =$  , $m =$  , $S =$ 

Hopf holonomies at last

Constructing the nerve of a Hopf algebra

a Hopf algebra $\mathcal{H} \in \mathcal{C} \quad \rightsquigarrow \quad$ a functor $F : \mathsf{BrSet} \to \mathcal{C}$

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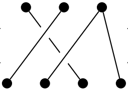
$F(\bullet^n) = \mathcal{H}^{n-1}$ $\qquad$ $F\Big($  $\Big) : \mathcal{H}^3 \to \mathcal{H}^2$ is

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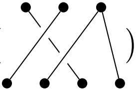
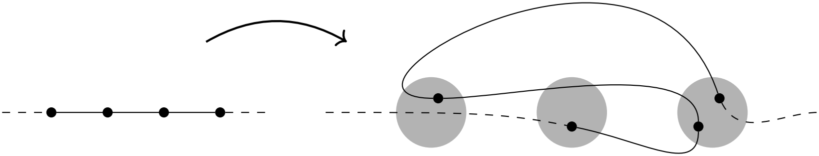
$F(\bullet^n) = \mathcal{H}^{n-1}$ $\qquad$ $F\big($  $\big) : \mathcal{H}^3 \to \mathcal{H}^2$ is



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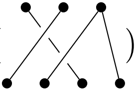
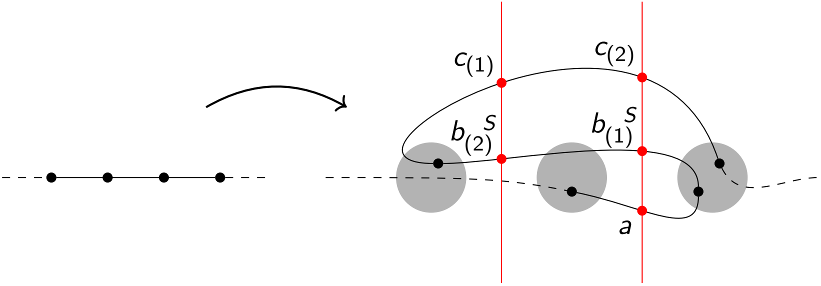
$F(\bullet^n) = \mathcal{H}^{n-1}$ $\qquad$ $F\Big($  $\Big) : \mathcal{H}^3 \to \mathcal{H}^2$ is

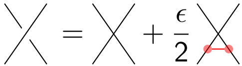


$$a \otimes b \otimes c \quad \mapsto \quad b_{(2)}{}^S c_{(1)} \otimes a b_{(1)}{}^S c_{(2)}$$

The semiclassical picture: chord diagrams

Poisson Hopf algebras in terms of infinitesimal braids (chord diagrams)

Chord diagrams, or "semiclassical braids":



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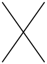
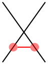
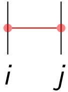
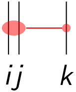
 $=$  $+ \frac{\epsilon}{2}$  $(\epsilon^2 = 0)$



$= t^{ij} = t^{ji}$,



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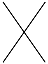
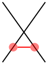
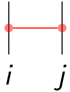
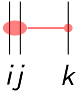
 $=$  $+\ \frac{\epsilon}{2}$  $(\epsilon^2 = 0)$



$= t^{ij} = t^{ji}$, 

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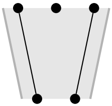
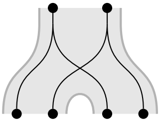
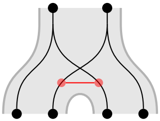
$\mathcal{H} = F(\bullet\bullet)$, $\Delta =$ , $m =$ , $\{,\} =$ 

Quantization: KZ connection and associators

KZ connection becomes Gauss-Manin connection

Knizhnik-Zamolodchikov connection

$$A_n^{KZ} = \hbar \sum_{1\leq i<j\leq n} t^{ij} \frac{d(z_i - z_j)}{z_i - z_j} \qquad dA_n^{KZ} + [A_n^{KZ}, A_n^{KZ}]/2 = 0$$

Quantization of Poisson Hopf algebras

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Better: $\Phi \in K\langle\langle X, Y \rangle\rangle$ a Drinfeld associator, BrSet $\to$ ChordSet via

 $\mapsto$  $\circ \exp(\hbar t^{12}/2)$  $\mapsto$   $\mapsto \Phi(\hbar t^{12}, \hbar t^{23})$

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(I guess it's still not understandable)

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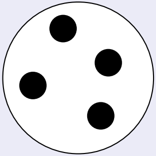

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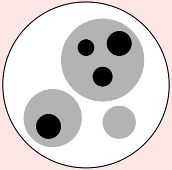

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or simply quantization of twists

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Easy Poisson groupoids

a Lie groupoids $\Gamma \rightrightarrows M$ with a Poisson structure on Γ
such that $\Gamma_{x,y} \subset \Gamma$ is a Poisson submanifold $\forall x, y \in M$
and s.t. the composition $\Gamma_{xy} \times \Gamma_{y,z} \to \Gamma_{x,z}$ is a Poisson map

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$F : \mathrm{FinSet}, \mathrm{ChordSet}, \mathrm{BrSet} \to \mathcal{C}$ a braided lax-monoidal functor
s.t. $F(\bullet\bullet) \otimes_{F(\bullet)} F(\bullet\bullet) \otimes_{F(\bullet)} \cdots \otimes_{F(\bullet)} F(\bullet\bullet) \to F(\bullet^n)$ is an iso

Algebraic/quantum/quantization

Commutative algebra $B = F(\bullet)$, Poisson/NC algebra $A = F(\bullet\bullet)$,
$\epsilon : A \to B$ (units*), central maps $\eta_{L,R} : B \rightrightarrows A$ (source*,target*),
coassociative $\Delta : A \to A \otimes_B A$ (composition*), antipode $S : A \to A$

(Easy) Poisson groupoids

or simply quantization of twists

Groupoids have nerves, too - which Poisson structures on Lie groupoids can we quantize in this way?

Easy Poisson groupoids

a Lie groupoids $\Gamma \rightrightarrows M$ with a Poisson structure on Γ
such that $\Gamma_{x,y} \subset \Gamma$ is a Poisson submanifold $\forall x, y \in M$
and s.t. the composition $\Gamma_{xy} \times \Gamma_{y,z} \to \Gamma_{x,z}$ is a Poisson map

F : $\mathrm{FinSet}, \mathrm{ChordSet}, \mathrm{BrSet} \to \mathcal{C}$ a braided lax-monoidal functor
s.t. $F(\bullet\bullet) \otimes_{F(\bullet)} F(\bullet\bullet) \otimes_{F(\bullet)} \cdots \otimes_{F(\bullet)} F(\bullet\bullet) \to F(\bullet^n)$ is an iso

Algebraic/quantum/quantization

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THANKS!