Moduli spaces of flat connections and Poisson analogues of Kitaev models

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Contents

TQFTs and moduli spaces

TQFTs of Reshetikhin-Turaev and Turaev-Viro type have Hamiltonian counterparts:

TQFT	**Hamiltonian version**
Reshetikhin-Turaev	Combinatorial quantization /Hopf algebra gauge theory
Turaev-Viro	Kitaev models

Transition to Hamiltonian version:

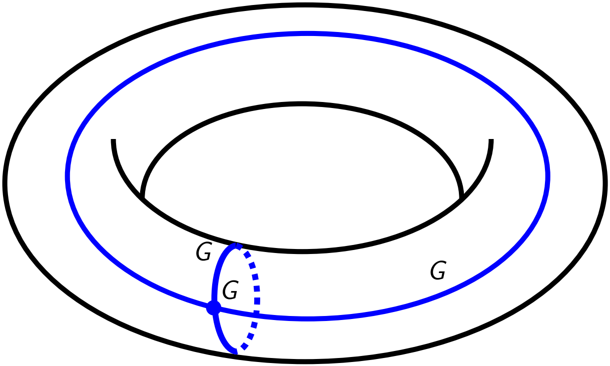
2d manifold Σ (oriented)	embedded graph decorated with Hopf algebra elements
3d cobordism	graph transformation

Hamiltonian theory	**Poisson analogue**
Combinatorial quantization /Hopf algebra gauge theory	moduli spaces of flat connections
Kitaev models	?

Aim:

- construct Poisson analogue of Kitaev models
- investigate relation of this analogue with moduli spaces of flat connections

- First: Consider transition moduli spaces → combinatorial quantization
- [Fock and Rosly, 1999]: Symplectic structure of moduli space of flat connections $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ realized by graph embedded into surface Σ decorated with elements of Poisson-Lie group G.



$\rightarrow$ Combinatorial quantization obtained via dictionary.

Let:

- Γ: graph with edge set E and vertex set V
- K: semisimple finite dimensional Hopf algebra
- G: Poisson-Lie group

Simplified dictionary:

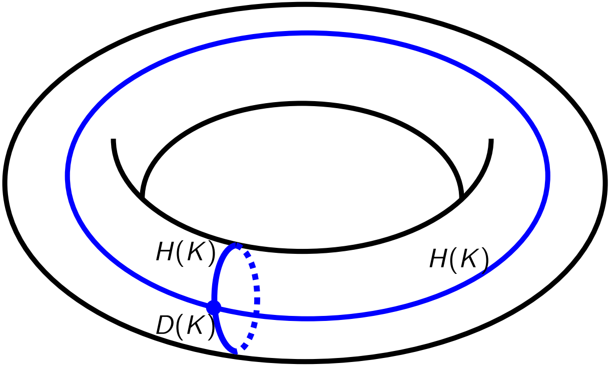
	moduli space	**quantum analogue**
decoration on edges	G	K
graph connections	$G^{\times E}$	$K^{\otimes E}$
gauge trafos.	$\mathcal{G} = G^{\times V}$	$\mathcal{K} = K^{\otimes V}$
observables	$\mathcal{G}$-invariant functions on $G^{\times E}$	$\mathcal{K}$-invariant elements of $K^{*\otimes E}$

Kitaev models and Poisson analogues

Kitaev models ([Kitaev, 2003], [Buerschaper et al., 2013])

Ingredients:

- Finite dimensional semisimple Hopf algebra K
- At each edge $e \in E$:
  copy of **Heisenberg double** $H(K) = K\#K^*$
  $\Rightarrow$ Algebra $H(K)^{\otimes E}$
- At each vertex $v \in V$:
  $D(K)$-right module algebra structure on $H(K)^{op \otimes E}$ for **Drinfeld double** $D(K)$ ($\cong K \otimes K^*$ as vector space)


H(K)
H(K)
D(K)

Poisson analogue of Kitaev models

New dictionary (simplified):

Kitaev models	**Poisson analogue**
Hopf algebra K	Poisson-Lie group G
Drinfeld double $D(K)$	classical double $D(G)$
Heisenberg double $H(K)$	classical Heisenberg double $H(G)$
algebra $H(K)^{\otimes E}$	Poisson product space $H(G)^{\times E}$
$D(H)^{\otimes V}$-module algebra structure on $H(K)^{\otimes E}$	Poisson $D(G)^{\times V}$-space structure on $H(G)^{\times E}$

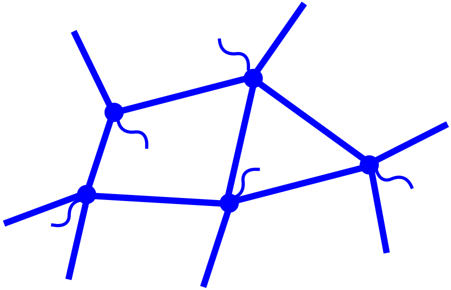
From Poisson analogues to moduli spaces

[Meusburger, 2017]: Hopf algebra gauge theory for $D(K) \widehat{=}$ Kitaev model for K.
Similar result expected for Poisson analogues.

$\rightarrow$ **Conjecture:** $(H(G)^{\times E}, \{\,,\}_{\pi}) \cong (H(G)^{\times E}, \{\,,\}_{FR})$
(with Poisson structure $\{\,,\}_{FR}$ introduced by Fock and Rosly)

We need a **regular ciliated** graph Γ:

- **ciliated:** linear ordering of edges at each vertex $v \in V$, expressed with a *cilium* at v
- **regular:** each **face** of Γ contains *exactly* one cilium (and some technical conditions)



Consider:

- Poisson-Lie group G with Lie bialgebra $\mathfrak{g}$
- dual Poisson-Lie group G^* with Lie bialgebra $\mathfrak{g}^*$

The **classical double** $D(G)$

- contains G and G^* (as Poisson-Lie subgroups). Assume $D(G) \cong G \times G^*$ as manifold.
- is **quasi-triangular**: For $f, g \in C^\infty(D(G))$

$$\{f, g\}_{D(G)}(d) = \langle df \otimes dg, \omega(d) \rangle$$

with $\omega(d) \in T_d D(G)^{\otimes 2}$ given by

$$\omega(d) = (TL_d^{\otimes 2} - TR_d^{\otimes 2})\, r$$

where $r = \mathrm{id}_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}^*$.

Definition: The **classical Heisenberg double** is defined by

- $H(G) = D(G)$ (as manifold)
- with Poisson structure

$$\{f, g\}_{H(G)}(d) = \langle df \otimes dg, \omega_H(d)\rangle$$

where

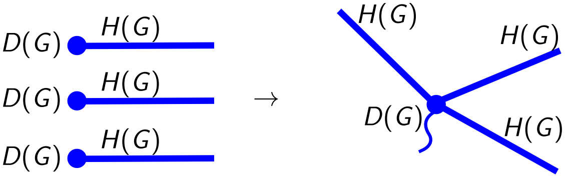
$$\omega_H(d) = -(TL_d^{\otimes 2} + TR_d^{\otimes 2})r\,.$$

$H(G)$ is a Poisson $D(G)$-space with *two* Poisson actions:

$$\triangleright : D(G) \times H(G) \to H(G) \quad (d, h) \mapsto dh$$
$$\triangleright' : D(G) \times H(G) \to H(G) \quad (d, h) \mapsto hd^{-1}$$

Fock-Rosly Poisson structure on $H(G)^{\times E}$:

- $D(G)$ quasi-triangular $\Rightarrow$ category of Poisson $D(G)$-spaces is monoidal.
- At each vertex $v \in V$: Take product of Poisson $D(G)$-spaces $H(G)$ at incident edges.
- Obtain Poisson $D(G)^{\times V}$-space $(H(G)^{\times E}, \{\,,\}_{FR})$.



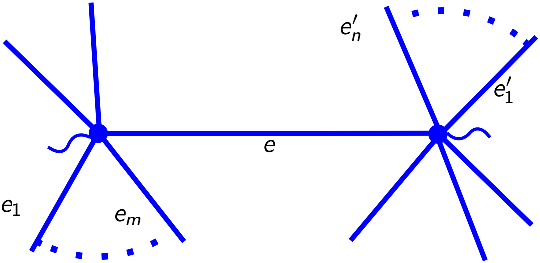
Theorem: Let Γ be ciliated and regular and G a Poisson-Lie group such that $D(G) \cong G \times G^*$ as manifold. Then there is an isomorphism of Poisson manifolds

$$\Phi : (H(G)^{\times E}, \{\,,\}_\pi) \to (H(G)^{\times E}, \{\,,\}_{FR}).$$

For $\overline{d} = (d_e)_{e \in E} \in H(G)^{\times E}$ and an edge $e \in E$ it is given by

$$\Phi(\overline{d})_e = \pi_G(d_{e'_1}) \cdots \pi_G(d_{e'_n})\, d_e\, \pi_G(d_{e_m})^{-1} \cdots \pi_G(d_{e_1})^{-1}$$

where $\pi_G : H(G) \to G$ is the projection on the G-component of $H(G)$.



Idea: Consider specific holonomies in Γ.

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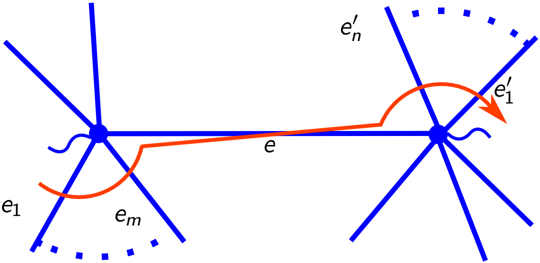
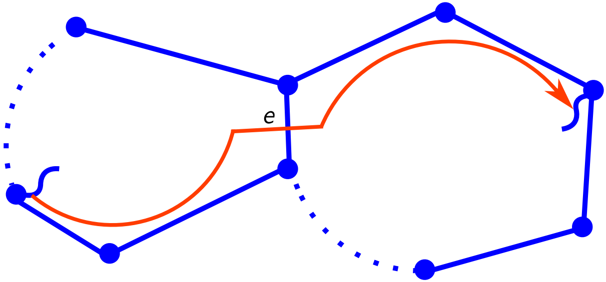


Idea: Consider specific holonomies in Γ.

For the inverse map $\Psi : (H(G)^{\times E}, \{\,,\}_{FR}) \to (H(G)^{\times E}, \{\,,\}_{\pi})$ similar holonomies, but in the dual graph:



Conclusion:

- Identified a classical analogue of Kitaev models. All structures from Kitaev models have Poisson analogues.
- Decoupled moduli space of flat connections for gauge group $D(G)$ (generalization of decoupling transformation by [Alekseev and Malkin, 1995])

Outlook:

- "Defects" in this Poisson/symplectic framework
- Graph transformations
- Poisson-Analogue to TQFTs

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