Drinfel'd Twist Deformation Quantization on Symplectic Manifolds

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Motivation (I)

Let $\mathbb{k} = \mathbb{R}$ or $\mathbb{C}$.

Definition (Star product)

A star product on a Poisson manifold $(M, \{\cdot,\cdot\})$ is a $\mathbb{k}[[\hbar]]$-bilinear associative binary operation $\star$ on $C^\infty(M)[[\hbar]]$ of the form:

$$f \star g = \sum_{k=0}^{\infty} \hbar^k B_k(f,g) \ , \qquad \forall\, f, g \in C^\infty(M),$$

where each $B_k : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a bidifferential operator,

$$B_0(f,g) = fg \ , \qquad B_1(f,g) - B_1(g,f) = \{f,g\} \, ,$$

and:

$$f \star 1 = 1 \star f = f.$$

The algebra $(C^\infty(M)[[\hbar]], \star)$ is called a deformation quantization of $(M, \{\cdot,\cdot\})$.

Motivation (II)

"Drinfel'd twists are tools to obtain deformation quantization via symmetries."

Lie algebra $\mathfrak{g}$ is *symmetry* of symplectic manifold M if

$$\exists\ \mathfrak{g} \to \mathfrak{X}(M) \text{ Lie algebra map.}$$

Consider the natural extension

$$\triangleright \colon \mathcal{U}(\mathfrak{g}) \to \mathrm{DiffOp}^{\bullet}(M).$$

Definition (Drinfel'd twist)

$\mathcal{F} \in \mathcal{U}(\mathfrak{g})^{\otimes 2}[[\hbar]]$ is a (Drinfel'd) twist, if the following three properties hold:

i.) $(\mathcal{F} \otimes 1) \cdot (\Delta \otimes \mathrm{id})(\mathcal{F}) = (1 \otimes \mathcal{F}) \cdot (\mathrm{id} \otimes \Delta)(\mathcal{F})$, (2-cocylce condition)

ii.) $(\epsilon \otimes \mathrm{id})(\mathcal{F}) = 1 = (\mathrm{id} \otimes \epsilon)(\mathcal{F})$, (normalization property)

iii.) $\mathcal{F} = 1 \otimes 1 + \mathcal{O}(\hbar)$.

Motivation (III)

Proposition

Let $\mathcal{F} \in \mathcal{U}(\mathfrak{g})^{\otimes 2}[[\hbar]]$ *be a twist. Then*

$$\mathcal{H}_{\mathcal{F}} := (\mathcal{U}(\mathfrak{g})[[\hbar]], \Delta_{\mathcal{F}}, S_{\mathcal{F}})$$

is a Hopf algebra, where $\Delta_{\mathcal{F}} := \mathcal{F} \cdot \Delta \cdot \mathcal{F}^{-1}$ *and* $S_{\mathcal{F}} := \mathrm{U} \cdot S \cdot \mathrm{U}^{-1}$*, with* $\mathrm{U} := \mathcal{F}_1 S(\mathcal{F}_2)$*.*

If $\mathfrak{g}$ *is a symmetry of a Poisson manifold* $(M, \{\cdot, \cdot\})$*, then*

$$(\mathcal{C}^{\infty}(M)[[\hbar]], \star_{\mathcal{F}})$$

is a $\mathcal{H}_{\mathcal{F}}$*-module algebra, where*

$$f \star_{\mathcal{F}} g := m(\mathcal{F}^{-1} \triangleright (f \otimes g)),$$

for all $f, g \in \mathcal{C}^{\infty}(M)$*.*

Motivation (IV)

Definition (Twist star product)

A star product $\star$ on a Poisson manifold $(M, \{\cdot,\cdot\})$ is called twist star product if there is a symmetry $\mathfrak{g}$ of M and a twist $\mathcal{F} \in \mathcal{U}(\mathfrak{g})^{\otimes 2}[[\hbar]]$ such that

$$\star = \star_{\mathcal{F}}.$$

Example

Consider $M = \mathbb{R}^2$ with coordinates (x, y) and the standard Poisson bracket. The Moyal-Weyl star product

$$f \star_m g := m(\exp(i\hbar\partial_x \wedge \partial_y)(f \otimes g))$$

on $\mathbb{R}^2$ is a twist star product $\star_m = \star_{\mathcal{F}}$ with inducing twist given by

$$\mathcal{F} := \exp(-i\hbar\partial_x \wedge \partial_y).$$

Motivation (IV)

Definition (Twist star product)

A star product $\star$ on a Poisson manifold $(M, \{\cdot, \cdot\})$ is called twist star product if there is a symmetry $\mathfrak{g}$ of M and a twist $\mathcal{F} \in \mathcal{U}(\mathfrak{g})^{\otimes 2}[[\hbar]]$ such that

$$\star = \star_{\mathcal{F}}.$$

Advantages:

- Star products in terms of the twist.
- Compatible deformation quantization of all module algebras.
- Connection to integrable systems: $\mathcal{F} = 1 \otimes 1 + \hbar r + \mathcal{O}(\hbar^2)$ with $\mathrm{CYB}(r) = 0$.

Problem: Twist star products are rare!

Goal of the talk: Find obstructions for twist star products!

Schedule

1. Motivation and definition of twist star products $\surd$

2. Obstructions for symplectic Riemann surfaces

   📎 Pierre Bieliavsky, Chiara Esposito, Stefan Waldmann, TW,
   Obstructions for twist star products,
   Lett. Math. Phys. (2017).

3. Obstructions via Morita equivalence

   📎 Francesco D'Andrea, TW,
   Twist star products and Morita equivalence,
   C. R. Acad. Sci. Paris, Ser. I 355 (2017) 1178–1184.

2. Obstructions for symplectic Riemann surfaces

Consider a star product $\star$ on a Poisson manifold $(M, \{\cdot, \cdot\})$.

Lemma

Let $\mathcal{F} = 1 \otimes 1 + \sum_{k=1}^{\infty} \hbar^k \mathcal{F}_k \in \mathcal{U}(\mathfrak{g})^{\otimes 2}[[\hbar]]$ *be a twist.*

i.) *Then* $r := \mathcal{F}_1^{21} - \mathcal{F}_1 \in \Lambda^2 \mathfrak{g}$ *is a classical* r*-matrix, i.e.* $\mathrm{CYB}(r) := [\![r, r]\!] = 0$.

ii.) *If* $\star = \star_{\mathcal{F}}$, *then* $\{f, g\} = m(r \triangleright (f \otimes g))$ *for all* $f, g \in \mathcal{C}^\infty(M)$.

iii.) *If* $(M, \{\cdot, \cdot\})$ *is symplectic, connected, compact and there exists a twist star product on* $(M, \{\cdot, \cdot\})$, *then* M *is a homogeneous space.*

Theorem (Bieliavsky-Esposito-Waldmann-TW, 2016)

There are no twist star products

i.) *on the symplectic Riemann surfaces of genus* > 1.

ii.) *on the symplectic* 2*-sphere.*

2. Obstructions for symplectic Riemann surfaces

Theorem

There are no twist star products

i.) *on the symplectic Riemann surfaces of genus* > 1.

ii.) *on the symplectic* 2*-sphere.*

Sketch of the proof.

i.) Riemann surfaces are connected and compact but not homogeneous for genus > 1.

ii.)
1. We can assume that $r \in \Lambda^2\mathfrak{g}$ is non-degenerate.
2. All transitive Lie group actions on $\mathbb{S}^2$ (up to equivalence) are by semisimple Lie groups.
3. There are no non-degenerate r-matrices on semisimple Lie algebras.

■

3. Obstructions via Morita equivalence (I)

Let $\star$ be star product on a sympl mfd $(M, \{\cdot,\cdot\})$ and $L \to M$ smooth complex line bundle.

<u>Notation</u>: $\mathcal{A} = (\mathcal{C}^\infty(M), \cdot)$, $\boldsymbol{\mathcal{A}} = (\mathcal{C}^\infty(M)[[\hbar]], \star)$.

$\Rightarrow \Gamma(L) \cong P\mathcal{A}^n$ is projective finitely generated, with idempotent $P = P^2 \in M_n(\mathcal{A})$.

$\Rightarrow \Gamma(L)[[\hbar]] \cong \mathbf{P}\boldsymbol{\mathcal{A}}^n$ is projective finitely generated, with idempotent $\mathbf{P} = \mathbf{P}^2 \in M_n(\boldsymbol{\mathcal{A}})$

given by $\mathbf{P} = \frac{1}{2} + (P - \frac{1}{2}) \star \frac{1}{\sqrt[\star]{1+4(P\star P - P)}} \in M_n(\boldsymbol{\mathcal{A}})$.

Theorem (Bursztyn-Waldmann, 2002)

There is a star product $\star'$ on $(M, \{\cdot,\cdot\})$ such that

$$(\mathcal{C}^\infty(M)[[\hbar]], \star') \cong \mathrm{End}_{\boldsymbol{\mathcal{A}}}(\Gamma(L)[[\hbar]])$$

is an isomorphism of $\mathbb{C}[[\hbar]]$-algebras.

Moreover, $\star \sim \star'$ if and only if $c_1(L) = 0$.

3. Obstructions via Morita equivalence (I)

Theorem (Bursztyn-Waldmann, 2002)

There is a star product $\star'$ *on* $(M, \{\cdot, \cdot\})$ *such that*

$$(\mathcal{C}^\infty(M)[[\hbar]], \star') \simeq \mathrm{End}_{\mathcal{A}}(\Gamma(L)[[\hbar]]) \tag{1}$$

is an isomorphism of $\mathbb{C}[[\hbar]]$*-algebras.*
Moreover, $\star \sim \star'$ *if and only if* $c_1(L) = 0$*.*

Definition (Morita equivalence for star products)

Two star products $\star, \star'$ on $(M, \{\cdot, \cdot\})$ are Morita equivalent if there is L such that (1) is an isomorphism of $\mathbb{C}[[\hbar]]$-algebras.

Remark

The definition coincides with the ring-theoretic definition of Morita equivalence on star product algebras.

3. Obstructions via Morita equivalence (II)

Theorem (D'Andrea-TW, 2017)

Let $(M, \{\cdot, \cdot\})$ *be a symplectic manifold, which is a homogeneous* G*-space. Mutually exclusive are:*

- i.) *There is a* G*-equivariant* $L \to M$ *with* $c_1(L) \neq 0$*.*
- ii.) *There is a twist star product on* $(M, \{\cdot, \cdot\})$ *based on* $\mathcal{U}(\mathfrak{g})[[\hbar]]$*, where* $\mathrm{Lie}(G) = \mathfrak{g}$*.*

Corollary

There are no twist star products on symplectic $\mathbb{CP}^{n-1}$ *based on* $\mathcal{U}(\mathfrak{gl}_n(\mathbb{C}))[[\hbar]]$ *or any sub-bialgebra.*

Proof.

The tautological line bundle on $\mathbb{CP}^{n-1}$ has non-trivial Chern class and is $\mathrm{GL}_n(\mathbb{C})$-equivariant. ∎

3. Obstructions via Morita equivalence (IV)

Theorem (D'Andrea-TW, 2017)

$(M, \{\cdot,\cdot\})$ *symplectic manifold, which is a homogeneous* G*-space. Mutually exclusive are:*

i.) *There is a* G*-equivariant* $L \to M$ *with* $c_1(L) \neq 0$.

ii.) *There is a twist star product on* $(M, \{\cdot,\cdot\})$ *based on* $\mathcal{U}(\mathfrak{g})[[\hbar]]$*, where* $\mathrm{Lie}(G) = \mathfrak{g}$.

Sketch of the proof.

1. If $L \to M$ is G-equivariant $\Rightarrow \Gamma(L)$ is $\mathcal{U}(\mathfrak{g})$-equivariant.

2. If also $\exists\, \mathcal{F}$ on $\mathcal{U}(\mathfrak{g})[[\hbar]] \Rightarrow \Gamma(L)[[\hbar]]$ is $H_{\mathcal{F}}$-equivariant $(\mathcal{C}^\infty(M), \star_{\mathcal{F}})$-bimodule with

$$\lambda_{\mathcal{F}}(f \otimes s) := \lambda(\mathcal{F}^{-1} \rhd (f \otimes s)),$$

where $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma(L)$.

3. $\lambda_{\mathcal{F}} \colon (\mathcal{C}^\infty(M)[[\hbar]], \star_{\mathcal{F}}) \to \mathrm{End}_{(\mathcal{C}^\infty(M)[[\hbar]], \star_{\mathcal{F}})}(\Gamma(L)[[\hbar]])$ is an isomorphism of $\mathbb{C}[[\hbar]]$-algebras.

4. $\Rightarrow\ c_1(L) = 0$.

Open questions

- Are there more (general) obstructions for twist star products on symplectic manifolds?
- Are there obstructions for (non-symplectic) Poisson manifolds?
- Are there more examples of twist star products?
- Is there a classification?

Thank you for your attention!