Noncommutativity in Unified Theories and Gravity

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Bayrischzell 2018

1. *Higher-Dimensional Unified Gauge Theories and Coset Space Dimensional Reduction*
2. *Fuzzy Extra Dimensions and Renormalizable Chiral Unified Theories*
3. *Gravity as a gauge theory in 3 and 4 dimensions*
4. *Gravity as a noncommutative gauge theory on fuzzy spaces*

- Kaluza - Klein observation: Dimensional Reduction of a pure gravity theory on $M^4 \times S^1$ leads to a $U(1)$ gauge theory coupled to gravity in four dimensions. The extra dimensional gravity provided a geometrical unified picture of gravitation and electromagnetism.

- Generalization to $M^D = M^4 \times B$, with B a compact Riemannian space with a non-Abelian isometry group S leads after dim. reduction to gravity coupled to Y-M in four dims.

Kerner '68

Cho - Freund '75

Problems

- No classical ground state corresponding to the assumed M^D.
- Adding fermions in the original action, it is impossible to obtain chiral fermions in four dims.

  Witten '85

- However by adding suitable matter fields in the original action, in particular Y-M one can have a classical stable ground state of the required form and massless chiral fermions in four dims.

  Horvath - Palla - Cremmer - Scherk '77

Original motivation

Use higher dimensions

- to unify the gauge and Higgs sectors
- to unify the fermion interactions with gauge and Higgs fields

⋆ Supersymmetry provides further unification (fermions in adj. reps)

Forgacs - Manton '79
Manton '81
Chapline - Slansky '82
Kubyshin - Mourao - Rudolph - Volobujev '89
Kapetanakis - Z '92
Manousselis - Z '01 –' 08

Further successes

(a) chiral fermions in 4 dims from vector-like reps in the higher dim theory

(b) the metric can be deformed (in certain non-symmetric coset spaces) and more than one scales can be introduced

(c) Wilson flux breaking can be used

(d) Softly broken susy chiral theories in 4 dims can result from a higher-dimensional susy theory

Theory in D dims $\to$ Theory in 4 dims

1. Compactification

$$\begin{array}{ccccc} M^D & \to & M^4 & \times & B \\ | & & | & & | \\ x^M & & x^\mu & & y^a \end{array}$$

B - a compact space
$\dim B = D - 4 = d$

2. Dimensional Reduction

Demand that $\mathcal{L}$ is independent of the extra y^a coordinates

- One way: Discard the field dependence on y^a coordinates
- An elegant way: Allow field dependence on y^a and employ a symmetry of the Lagrangian to compensate

  Obvious choice: Gauge Symmetry

Allow a non-trivial dependence on y^a, but impose the condition that a symmetry transformation by an element of the isometry group S of B is compensated by a gauge transformation.

$\Rightarrow$ $\mathcal{L}$ independent of y^a just because is gauge invariant.

Integrate out extra coordinates

CSDR $B = S/R$

$$S: \quad Q_A = \{\underbrace{Q_i}_{R}, \underbrace{Q_a}_{S/R}\}$$

$$[Q_i, Q_j] = f_{ij}{}^k Q_k \quad , \quad [Q_i, Q_a] = f_{ia}{}^b Q_b$$

$$[Q_a, Q_b] = f_{ab}{}^i Q_i + f_{ab}{}^c Q_c$$

where f_{ab}^c vanishes in symmetric S/R.

Consider a Yang-Mills-Dirac theory in D dims based on group G defined on $M^D \to M^4 \times S/R$, $D = 4 + d$

$$g^{MN} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & -g^{ab} \end{pmatrix} \qquad \eta^{\mu\nu} = diag(1, -1, -1, -1)$$

$$d = dimS - dimR \qquad g^{ab} - \text{coset space metric}$$

$$A = \int d^4x d^d y \sqrt{-g} \Big[-\frac{1}{4} Tr(F_{MN} F_{K\Lambda}) g^{MK} g^{N\Lambda} + \frac{i}{2} \overline{\psi} \Gamma^M D_M \psi \Big]$$

$$D_M = \partial_M - \theta_M - A_M \ , \ \ \theta_M = \frac{1}{2} \theta_{MN\Lambda} \Sigma^{N\Lambda}$$

where θ_M is the spin connection of M^D and ψ is in rep F of G

We require that any transformation by an element of S acting on S/R is compensated by gauge transformations.

$$
\begin{aligned}
A_\mu(x,y) =& g(s) A_\mu(x, s^{-1} y) g^{-1}(s) \\
A_a(x,y) =& g(s) J_a{}^b A_b(x, s^{-1} y) g^{-1}(s) \\
&+ g(s) \partial_a g^{-1}(s) \\
\psi(x,y) =& f(s) \Omega \psi(x, s^{-1} y) f^{-1}(s)
\end{aligned}
$$

g, f: gauge transformations in the adj, F of G corresponding to the s transformation of S acting on S/R

$J_a{}^b$: Jacobian for s

Ω: Jacobian + local Lorentz rotation in tangent space

Above conditions imply constraints that D-dims fields should obey.

Solution of constraints:

- 4-dim fields
- Potential
- Remaining gauge invariance

1) The 4-dim gauge group:

$$H = C_G(R_G)$$
$$\text{i.e.} \quad G \supset R_G \times H$$

where G is the higher-dim group and H is the 4-dim group.

2) Scalar fields:

$$\begin{aligned} S &\supset R \\ adjS &= adjR + v \\ G &\supset R_G \times H \\ adjG &\supset (adjR, 1) + (1, adjH) + \Sigma(r_i, h_i) \end{aligned}$$

If $v = \sum s_i$

when $s_i = r_i \Rightarrow$ h_i survives in 4-dims.

3) Fermions:

$$G \supset R_G \times H$$
$$F = \Sigma(t_i, h_i)$$

spinor of $SO(d)$ under R

$$\sigma_d = \Sigma\sigma_j$$

for every $t_i = \sigma_i \Rightarrow$ h_i survives in 4-dims.

Possible to obtain a chiral theory in 4 dims even starting with Weyl (+ Majorana) fermions in vector-like reps of G in $D = 4n + 2$ dims.

Soft Supersymmetry Breaking by CSDR over non-symmetric CS.

We have examined the dim reduction of a supersymmetric E_8 over the 3 existing $6-$dim CS:

$$G_2/SU(3)\,, \quad Sp(4)/(SU(2)\times U(1))_{\text{non-max}}\,, \quad SU(3)/U(1)\times U(1)$$

$\Rightarrow$ Softly Broken Supersymmetric Theories in 4 dims without any further assumption

Non-symmetric CS admit torsion and the two latter more than one radii.

Reduction of 10-dim, $N = 1$, E_8 over $S/R = SU(3)/U(1) \times U(1) \times Z_3$

Irges - Z '11

Dimensional reduction $E_8 \to E_6 \times U(1) \times U(1) \xrightarrow[\text{spont. breaking}]{\text{geometrical}} E_6$

Wilson flux breaking leads in 4-dims to

$$\mathcal{N} = 1 \ , \quad SU(3)_C \times SU(3)_L \times SU(3)_R$$

with matter superfields in

$$(\overline{3}, 1, 3)_{(3,1/2)}, \qquad (3, \overline{3}, 1)_{(0,-1)}, \qquad (1, 3, \overline{3})_{(-3,1/2)}$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$\begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix} = q^c, \quad \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix} = q, \quad \begin{pmatrix} N & E^c & \nu \\ E & N^c & e \\ \nu^c & e^c & S \end{pmatrix} = \lambda$$

and soft supersymmetry breaking terms.

Further supersymmetry and gauge symmetry breaking

Consider the vevs in the scalars of $\lambda^{(1)}$, $\lambda^{(2)}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & V \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ V & 0 & 0 \end{pmatrix}$$

$$\lambda^{(1)} : SU(3)_L \times SU(3)_R \times U(1)_A \times U(1)_B \to SU(2)_L \times SU(2)_R \times U(1)$$
$$\lambda^{(2)} : SU(3)_L \times SU(3)_R \times U(1)_A \times U(1)_B \to SU(2)_L \times SU(2)_R \times U(1)'$$

their combination gives

$$SU(3)_L \times SU(3)_R \times U(1)_A \times U(1)_B \to SU(2)_L \times U(1)_Y$$

Electroweak breaking proceeds by $\begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & V \end{pmatrix}$

Note that before EW breaking, supersymmetry is broken by D and F-terms, in addition to its breaking by soft terms.

Given that the trilinear soft supersymmetry breaking terms $\sim \frac{1}{R}$, where R is the radius of the extra dimensions, we proceed along two directions:

- R - very small $\rightarrow$ susy breaking at high scales $\rightarrow$ a version of split supersymmerty
- R - large $\rightarrow$ calculation of K-K contributions in the various parameters of the 4-dim theory.

$$M^D = M^4 \times (S/R)_F \,,$$

where $(S/R)_F$ is a finite matrix manifold, e.g. fuzzy sphere S_F^2.

Aschieri - Madore - Manousselis - Z '04, '05
Manolakos - Z '16

Instead of considering the algebra of functions:

$$Fun(M^D) \sim Fun(M^4) \times Fun(S/R) \,,$$

we consider the algebra:

$$A = Fun(M^4) \times M_N \,,$$

where M_N is a finite dim NC (non-com) algebra of matrices that approximates the functions on $(S/R)_F$.
On A we still have the action of symmetry group S $\rightarrow$ we can apply CSDR.

Madore '91

Nice example of $(S/R)_F$ is the fuzzy sphere S^2_F, a matrix approximation of S^2. The algebra of functions on S^2 (spanned by spherical harmonics) is truncated at a given angular momentum and becomes finite dimensional. The algebra becomes that of $N \times N$ matrices.

The associativity of the algebra is nicely achieved by relaxing commutativity.

The algebra of functions on S^2 can be generated by the coordinates of $\mathbb{R}^3$ modulo the relation

$$\sum_{a=1}^{3} x_a^2 = r^2 \,.$$

Scalar functions on S^2 can be expanded:

$$f(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_{lm}(\theta, \phi) \, ,$$

where the spherical harmonics, $Y_{lm}(\theta, \phi)$, can be expressed in terms of the cartesian coordinates $x_a, a = 1, 2, 3$ in $\mathbb{R}^3$:

$$Y_{lm}(\theta, \phi) = \sum_{a} f^{(lm)}_{a_1 \ldots a_l} x^{a_1} \ldots x^{a_l} \, ,$$

where $f^{(lm)}_{a_1 \ldots a_l}$ is the $SO(3)$ traceless symmetric tensor of rank l .

Similarly, we can expand $N \times N$ matrices of a matrix theory on a fuzzy sphere:

$$\hat{f} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} f_{lm} \hat{Y}_{lm} \, , \quad \hat{Y}_{lm} = r^{-l} \sum_{a} f^{(lm)}_{a_1 \ldots a_l} \hat{x}^{a_1} \ldots \hat{x}^{a_l} \, ,$$

where $f^{(lm)}_{a_1 \ldots a_l}$ are the same as in S^2 and

$$\hat{x}_a = r\frac{i}{\sqrt{N^2-1}}X_a\,, \quad \hat{x}^\dagger = \hat{x}_a\,,$$

are $N \times N$ hermitian matrices proportional to the $N-$dim rep of the $SU(2)$ generators, which satisfy the relations:

$$\sum_{a=1}^{3} \hat{x}_a \hat{x}_a = r^2\,, \quad [X_a, X_b] = \epsilon_{abc} X_c\,.$$

$\hat{Y}_{lm}$ - fuzzy spherical harmonics obeying:

$$Tr_N(\hat{Y}^\dagger_{lm}\hat{Y}_{l'm'}) = \delta_{ll'}\delta_{mm'}\,.$$

Obviously, it holds:

$$\hat{f} = \sum_{l=0}^{N-1}\sum_{m=-l}^{l} f_{lm}\hat{Y}_{lm} \to \sum_{l=0}^{N-1}\sum_{m=-l}^{l} f_{lm}Y_{lm}\,.$$

Similarly,

$$\frac{1}{N}Tr_N \to \frac{1}{4\pi}\int d\Omega\,, \quad d\Omega = sin\theta d\theta d\phi\,.$$

Differential calculus on S_F^2

In addition, on S_f^2 there is a natural $SU(2)$ covariant differential calculus. The derivations of a function f along X_a are given by:

$$e_a(f) = [X_a, f] , \quad a = 1, 2, 3 ,$$

i.e. this calculus is 3−dimensional.

These are essentially the angular momentum operators:

$$J_a f = i e_a f = i[X_a, f] ,$$

which satisfy the $SU(2)$ Lie algebra:

$$[J_a, J_b] = i\epsilon_{abc} J_c .$$

In the limit $N \to \infty$ the e_a becomes:

$$e_a = \epsilon_{abc} x_b \partial_c ,$$

i.e. 2−dimensional.

The exterior derivative is given by:

$$df = [X_a, f]\theta^a \,,$$

where θ^a are $1-$forms dual to e_a , $\langle e_a, \theta^b \rangle = \delta_a^b$.

$1-$forms are generated by θ^a:

$$\omega = \sum_{a=1}^{3} \omega_a \theta^a \,,$$

where ω any $1-$form.

$1-$form on $M^4 \times S_F^2$:

$$A = A_\mu dx^\mu + A_a \theta^a \,,$$

with $A_\mu = A_\mu(x^\mu, x_a)$, $A_a = A_a(x^\mu, x_a)$.

Non Commutative gauge fields and transformations

Madore, Wess et al '00

Consider a field $\phi(X_a)$ on a fuzzy space described by non-com coordinates, X_a. An infinitesimal gauge transformation:

$$\delta\phi(X_a) = \lambda(X_a)\phi(X_a)\,,$$

where $\lambda(X_a)$ is a gauge transformation parameter:

- $U(1)$ if $\lambda(X_a)$ is antihermitian function of X_a
- $U(P)$ if $\lambda(X_a)$ is valued in Lie algebra of $P \times P$ matrices

Coordinates X_a are invariant under gauge transformation, i.e. $\delta X_a = 0$. Therefore:

- $\delta(X_a\phi) = X_a\lambda(X_a)\phi \neq \lambda(X_a)X_a\phi$
- $\delta(\phi_a\phi) = \lambda(X_a)\phi_a\phi\,,$ $\quad -\phi_a$: covariant coords

  which holds if: $\delta(\phi_a) = [\lambda(X_a), \phi_a]$
- $\underbrace{\phi_a}_{\text{NC analogue of cov. der.}} = X_a + \underbrace{A_a}_{\text{interpreted as gauge fields}}$

Note that the transformation of A_a is:

$$\delta A_a = -[X_a, \lambda] + [\lambda, A_a]\,,$$

supporting the interpretation of A_a as gauge field.

Correspondingly, define:

$$\begin{aligned} F_{ab} &= [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - C^c{}_{ab} A_c \\ &= [\phi_a, \phi_b] - C^c{}_{ab} \phi_c\,, \end{aligned}$$

an analogue of the field strength tensor.

Its transformation is given by:

$$\delta F_{ab} = [\lambda, F_{ab}]$$

Also, for a spinor ψ in the adjoint rep, the transformation is:

$$\delta \psi = [\lambda, \psi]$$

Actions in higher dimensions seen as 4−dim actions (expansion in KK modes)

$G = U(P)$ on $M^4 \times (S/R)_F$

$A_{YM} = \frac{1}{4}\int d^4x Tr\, tr_G F_{MN}F^{MN}$

$\uparrow$

integration

over $(S/R)_F$

$F_{MN} \rightarrow (F_{\mu\nu}, F_{\mu b}, F_{ab})$

- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$
- $F_{\mu a} = \partial_\mu A_a - [X_a, A_\mu] + [A_\mu, A_a] = \partial_\mu \phi_a + [A_\mu, \phi_a] = D_\mu \phi$
- $F_{ab} = [\phi_a, \phi_b] - C^c{}_{ab}\phi_c$

$$\rightarrow A_{YM} = \int d^4x Tr\, tr_G \left(\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}(D_\mu\phi_a)^2 - V(\phi)\right),$$

$$V(\phi) = -\frac{1}{4} Tr\, tr_G \sum_{ab} F_{ab} F_{ab}$$
$$= -\frac{1}{4} Tr\, tr_G \sum_{ab} ([\phi_a, \phi_b] - C^c{}_{ab}\phi_c)([\phi_a, \phi_b] - C^c{}_{ab}\phi_c) .$$

$\hookrightarrow$ The infinitesimal G gauge transformations with parameter $\lambda(x^\mu, x^a)$ can be interpreted as M^4 gauge transformation

$$\lambda(x^\mu, X^a) = \lambda(x^\mu, X^a)^\alpha \mathcal{T}^\alpha = \lambda(x^\mu)^{h,\alpha} T^h \mathcal{T}^\alpha ,$$

- $\mathcal{T}^\alpha$ – generators of $U(P)$
- $\lambda(x^\mu, X^a)^\alpha$ – $N \times N$ matrices $\rightarrow$ expressible as:

KK modes of $\lambda(x^\mu, X^a)^\alpha$	–	$\lambda(x^\mu)^{\alpha,h} T^h$	–	T^h - generators of $U(N)$

Considering on equal footing the indices h, α, we interpret $\lambda^{h,\alpha}(x^\mu)$ as a field valued in the tensor product:

$$\mathrm{Lie}(U(N)) \otimes \mathrm{Lie}(U(P)) = \mathrm{Lie}(U(NP))\,.$$

Similarly, we write the gauge field A_ν as:

$$A_\nu(x^\mu, X^a) = A_\nu^\alpha(x^\mu, X^a)\mathcal{T}^\alpha = A_\nu^{h,\alpha}(x^\mu)T^h\mathcal{T}^\alpha$$

and interpret it as $\mathrm{Lie}(U(NP))$ valued gauge field on M^4.

Similarly for ϕ_a. Then we reduce the number of gauge fields and scalars by applying the CSDR principle.

Major difference among ordinary and fuzzy CSDR

- 4−dim gauge theory appears already spontaneously broken

$\hookrightarrow$ in 4 dims appears only the physical Higgs that survives SSB

$\hookrightarrow$ Yukawa sector

(i) massive fermions

(ii) interactions among fermions and physical Higgs fields

$\Rightarrow$ if we obtain in fuzzy - CSDR the SM $\rightarrow$ large extra dims

Fundamental differences among ordinary and fuzzy CSDR

- A non-abelian gauge group is not necessary in high dims.

The presence of a $U(1)$ in the higher - dim theory is enough to obtain non-abelian gauge theories in 4 dims.

•• The theory is renormalizable in the sense that divergencies can be removed by a finite number of counterterms.

4−dim $SU(N)$ gauge theory

We have constructed a renormalizable 4−dim $SU(N)$ gauge theory with suitable multiplet of scalar fields.

Aschieri - Gram/los - Steinacker - Z '06, '07

The symmetry breaking pattern and low energy gauge group are determined dynamically in terms of a few free parameters of the potential.

Depending on these parameters, the final gauge group can be $SU(n)$ or $SU(n_1) \times SU(n_2) \times U(1)$.

We explicitly found the tower of massive K-K modes, consistent with an interpretation as dimensionally reduced higher - dim gauge theory over an S_F^2.

The minima of the potential where vevs of scalars, $\langle\phi_a\rangle$, form the coordinates (generators) of a NC manifold (e.g S_F^2, CP_F^N).

$\hookrightarrow$ interpreted as spontaneously generated fuzzy extra dims.

Fluctuations around the vacuum:
internal components of a higher - dim gauge field

$$\underbrace{\phi_a}_{\text{covariant coordinates}} = \underbrace{\langle\phi_a\rangle}_{\text{coordinates}} + \underbrace{A_a}_{\text{gauge fields}}$$

with a finite K-K tower of massive states.

- Intermediate scales $\rightsquigarrow$ Gauge theory on $M^4 \times M_{\text{fuzzy}}$
- Low energy physics governed by zero modes.

- At high scales the theory behaves again as a 4−dim gauge theory maintaining renormalizability.

$\Rightarrow$ Main features of dim reduction are realized within the framework of renormalizable 4−dim gauge theory.

Potential problem with chirality:
Best case: only models with mirror fermions (not excluded exp)

Steinacker, Z '07

Chatzistavrakidis, Steinacker, Z '09

Chiral models demand additional requirements, e.g. orbifolding

Nice example: $SU(N)^3$ chiral models leading after further spontaneous breakings to $SU(3)^3$ and MSSM.

Chatzi/dis, Steinacker, Z '10, '11, '12

$\mathcal{N} = 4$ SYM

Particle content in $\mathcal{N} = 1$ language

1. a $SU(3N)$ vector supermultiplet
2. three adjoint chiral supermultiplets Φ^i

And in components:

1. $SU(3N)$ gauge bosons A_μ
2. six adjoint real scalars ϕ_a (or three complex)
3. four adjoint Majorana fermions

The theory has a global R - symmetry, $SU(4)_R$ under which the fields transform:

1. gauge fields as singlets
2. real scalars as 6
3. fermions as 4

Orbifolding by $\mathbb{Z}_3$ embedded in $SU(3)$ as:

$$SU(4)_R \supset SU(3) \times U(1)_R$$

$$\begin{aligned} 6 &= 3_2 + \bar{3}_{-2} \\ 4 &= 1_3 + 3_{-1} \end{aligned}$$

Kachru, Silverstein '98

leads to $\mathcal{N} = 1$ theory.

$\mathbb{Z}_3$ acts non-trivially on the various fields depending on their reps under the R-symmetry and the gauge group.

Orbifold projection keeps the fields which are invariant under the combined $\mathbb{Z}_3$ action. *(see e.g. Bailin + Love Phys. Rept '99)*

The projected theory is $\mathcal{N}=1, SU(N)^3$ gauge theory with chiral superfields in:

$$3\Big((N,\bar{N},1)+(\bar{N},1,N)+(1,N,\bar{N})\Big)$$

i.e. chiral theory ! with 3 families !!!

However, the $\mathcal{N}=4$ superpotential

$$W_{\mathcal{N}=4}=Tr(\epsilon_{ijk}\Phi^i\Phi^j\Phi^k)$$

is projected and gives the scalar potential:

$$V_{\mathcal{N}=1}(\phi)=\frac{1}{4}Tr\left([\phi^i,\phi^j]^\dagger[\phi_i,\phi_j]\right),$$

with minimum for vanishing vevs

$\hookrightarrow$ No vacuum of NC - type !

Natural mechanism, aim for:

1. fuzzy vacua
2. (potentially) realistic theory

require introduction of $\mathcal{N}=1$ Soft Supersymmetry Breaking (SSB) terms, i.e. those that explicitly break $\mathcal{N}=1$, but do not introduce quadratic divergences (*Girandello - Grisaru '81*):

scalar mass terms, trilinear scalar interaction, gaugino masses.

$\hookrightarrow$ Full potential is:

$$V = V_{\mathcal{N}=1} + V_{SSB} + V_D$$

$\nwarrow$

$$D - terms \geq 0$$

and can be brought in the form:

$$V = \frac{1}{4}(F^{ij})^\dagger F^{ij} + V_D\,, \quad \text{with} \quad F^{ij} = [\phi^i, \phi^j] - i\epsilon^{ijk}\phi^{k\dagger}\,.$$

The minimum is obtained when

$$[\phi^i, \phi^j] = i\epsilon^{ijk}\phi^{k\dagger}$$
$$\phi^i\phi^{i\dagger} = R^2$$

compatible with $\mathbb{Z}_3$ *projection*

Defining $\phi^i = \Omega\tilde{\phi}$, with $\Omega \neq 1\,, \quad \Omega^3 = 1\,, \quad \Omega^\dagger = \Omega^{-1};$

$\tilde{\phi}^{i\dagger} = \tilde{\phi}^i$, i.e. $\phi^{i\dagger} = \Omega\phi^i$

$\hookrightarrow$ $[\tilde{\phi}^i, \tilde{\phi}^j] = i\epsilon^{ijk}\tilde{\phi}^k\,; \quad \tilde{\phi}^i\tilde{\phi}^i = R^2\,,$

i.e. ordinary fuzzy sphere.

The ϕ^is with fluctuations around the vacuum:

$$\phi^i = \begin{pmatrix} \lambda^i_{(N-n)} + A^i & 0 & 0 \\ 0 & \omega(\lambda^i_{(N-n)} + A'^i) & 0 \\ 0 & 0 & \omega^2(\lambda^i_{(N-n)} + A''^i) \end{pmatrix}$$

with $\omega = 2\pi/3$.

The gauge symmetry $SU(N)^3$ is broken down to $SU(n)^3$.
Moreover, there exists a finite K-K tower of massive states.

Particle Physics Models

Considering the embedding

$$SU(N) \supset SU(N-3) \times SU(3) \times U(1)$$

$$\hookrightarrow \; SU(N) \; \rightarrow \; SU(3)^3$$

$$SU(3)_c \times SU(3)_L \times SU(3)_R$$

$$3 \cdot ((3, \bar{3}, 1) + (\bar{3}, 1, 3) + (1, 3, \bar{3}))$$

Embedding in Matrix Models

$\hookrightarrow \mathbb{Z}_3-$ Orbifold Matrix M.

$$\mathbb{Z}_3 \subset SU(3) \times U(1) \times SO(6) \subset SO(9,1)$$

Aoki - Iso Suyama '02

Three-Dimensional Gravity on noncommutative spaces based on SU(2) and SU(1,1)

- Description of 4-d gravity as a gauge theory
- Description of 3-d gravity (with and without cosmological constant) as a gauge theory
- Translation to the noncommutative regime
- Determination of the fuzzy spaces we work on
- Gauge theory of their isometry groups
- Results: Transformation of fields, curvatures, action, commutative limit

Chatz/dis, Jonke, Jurman, Manousselis, Manolakos, Z. '18

Vielbein formulation of GR

$\downarrow$

gauge theory of the Poincare group,

$ISO(1,3)$

$\downarrow$

consists of 10 generators:

see for details:
Utiyama '56, Kibble '61,
McDowell-Mansuri '77,
Kibble - Stelle '85

- 4 of local translations, P_a
- 6 Lorentz transformations, M_{ab}

The generators satisfy the commutation relations:

$$
\begin{aligned}
[M_{ab}, M_{cd}] &= \eta_{ac} M_{db} - \eta_{bc} M_{da} - \eta_{ad} M_{cb} + \eta_{bd} M_{ca} \\
[P_a, M_{bc}] &= \eta_{ab} P_c - \eta_{ac} P_b \\
[P_a, P_b] &= 0
\end{aligned}
$$

where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$.

Gauging: For each generator $\rightsquigarrow$ introduction of a gauge field:

- The vielbein ${e_\mu}^a$ for translations
- The spin connection ${\omega_\mu}^{ab}$ for Lorentz transformations

Therefore, the gauge connection would be:

$$A_\mu = {e_\mu}^a(x) P_a + \frac{1}{2}{\omega_\mu}^{ab}(x) M_{ab}$$

A_μ transforms in the adjoint rep:

$$\delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon]$$

where ϵ is a parameter valued in $\mathfrak{iso}(1,3)$:

$$\epsilon = \xi^a(x) P_a + \frac{1}{2}\lambda^{ab}(x) M_{ab}$$

The transformations of the gauge fields, e, ω derive from:

$$\delta(e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab} M_{ab}) =$$
$$\partial_\mu(\xi^a P_a + \frac{1}{2}\lambda^{ab} M_{ab}) + [e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab} M_{ab}, \xi^c P_c + \frac{1}{2}\lambda^{cd} M_{cd}]$$

and their expressions are:

$$\delta e_\mu{}^a = \partial_\mu \xi^a - e_\mu{}^b \lambda^a{}_b + \omega_\mu{}^{ab} \xi_b$$
$$\delta \omega_\mu{}^{ab} = \partial_\mu \lambda^{ab} - \lambda^a{}_c \omega_\mu{}^{cb} + \lambda^b{}_c \omega_\mu{}^{ca}$$

Curvature tensors are obtained using the standard formula:

$$R_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Writing $R_{\mu\nu} = T_{\mu\nu}{}^a P_a + \frac{1}{2} R_{\mu\nu}{}^{ab} M_{ab}$, we calculate:

$$T_{\mu\nu}{}^a P_a + \frac{1}{2} R_{\mu\nu}{}^{ab} M_{ab} = \partial_\mu \left(e_\nu{}^a P_a + \frac{1}{2}\omega_\nu{}^{ab} M_{ab} \right) - (\mu \leftrightarrow \nu)$$
$$+ \left[e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab} M_{ab}, e_\nu{}^c P_c + \frac{1}{2}\omega_\nu{}^{cd} M_{cd} \right]$$

The expressions of the tensors are:

$$T_{\mu\nu}{}^{a} = \partial_\mu e_\nu{}^{a} - \partial_\nu e_\mu{}^{a} + e_\mu{}^{b}\omega_{\nu b}{}^{a} - e_\nu{}^{b}\omega_{\mu b}{}^{a}$$
$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} - \omega_\mu{}^{cb}\omega_\nu{}^{a}{}_{c} + \omega_\mu{}^{ac}\omega_{\nu c}{}^{b}$$

Action is needed to complete the picture:

- Built out of Poincare covariants we just constructed
- Analogy with Yang-Mills theory suggests an action of the form:

$$\mathcal{S} = \int d^4\xi R_{ab}{}^{cd} R^{ab}{}_{cd}$$

- Not the right choice $\rightsquigarrow$ -4 dimension $\rightsquigarrow$ no space for a dim/ful parameter
- Action should have the "wrong" dims
- The right choice is:

$$\mathcal{S}_E = \frac{1}{16\pi G}\int d^4\xi R_{ab}{}^{ab} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R_{ab}{}^{ab}$$

Therefore, the Einstein action can be written as:

$$\mathcal{S}_E = \frac{1}{16\pi G}\int d^4x\sqrt{-g}e^{\mu}{}_{a}e^{\nu}{}_{b}(\partial_\mu\omega_\nu{}^{ab}-\partial_\nu\omega_\mu^{ab}+\omega_\mu{}^{ac}\omega_{\nu c}{}^{b}-\omega_\nu{}^{ac}\omega_{\mu c}{}^{b})$$

- $\rightsquigarrow$ Functional of both the vielbeins and the spin connections
- $\rightsquigarrow$ First order formulation of GR equations

Varying with respect to the fields $\rightsquigarrow$ e.o.m.:

- with respect to ω $\rightsquigarrow$ torsion-free condition
  - ✓ Torsion-free condition holds when scalars coupled to gravity
  - ✗ Torsion non-zero when spinors coupled to gravity
- with respect to e $\rightsquigarrow$ Einstein field equations (no matter)

Therefore, we conclude:

- Form of Einstein action: $A^2(dA + A^2)$
- Such action does not exist in gauge theories
- In that sense, gravity *cannot be* considered as gauge theory.

The algebra Witten '88

- 3-d Gravity: gauge theory of $\mathfrak{iso}(1,2)$ (Poincare - isometry of M^3)
- Presence of Λ: dS or AdS algebras, i.e. $\mathfrak{so}(1,3), \mathfrak{so}(2,2)$
- Corresponding generators: $P_a, J_{ab}, a = 1,2,3$ (translations, LT)
- Satisfy the following CRs:

$$[J_{ab}, J_{cd}] = 4\eta_{[a[c}J_{d]b]}\,,\ [P_a, J_{bc}] = 2\eta_{a[b}P_{c]}\,,\ [P_a, P_b] = \Lambda J_{ab}$$

- CRs valid in *arbitrary* dim; particularly in 3-d:

$$[J_a, J_b] = \epsilon_{abc}J^c\,, \quad [P_a, J_b] = \epsilon_{abc}P^c\,, \quad [P_a, P_b] = \Lambda\epsilon_{abc}J^c$$

- After the redefinition: $J^a = \frac{1}{2}\epsilon^{abc}J_{bc}$

The gauging procedure

- Intro of a gauge field for each generator: $e_\mu{}^a, \omega_\mu{}^a$(transl, LT)
- The Lie-valued 1-form gauge connection is:

$$A_\mu = e_\mu{}^a(x) P_a + \omega_\mu{}^a(x) J_a$$

- Transforms in the adjoint rep, according to the rule:

$$\delta A_\mu = \partial_\mu \epsilon + [A_\mu, \epsilon]$$

- The gauge transformation parameter is expanded as:

$$\epsilon = \xi^a(x) P_a + \lambda^a(x) J_a$$

- *Combining* the above $\rightarrow$ transformations of the fields:

$$\begin{aligned} \delta e_\mu{}^a &= \partial_\mu \xi^a - \epsilon^{abc}(\xi_b \omega_{\mu c} + \lambda_b e_{\mu c}) \\ \delta \omega_\mu{}^a &= \partial_\mu \lambda^a - \epsilon^{abc}(\lambda_b \omega_{\mu c} + \Lambda \xi_b e_{\mu c}) \end{aligned}$$

Curvatures and action

- Curvatures of the fields are given by:

$$R_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

- Tensor $R_{\mu\nu}$ is also Lie-valued:

$$R_{\mu\nu}(A) = T_{\mu\nu}{}^a P_a + R_{\mu\nu}{}^a J_a$$

- *Combining* the above $\rightarrow$ curvatures of the fields:

$$\begin{aligned} T_{\mu\nu}{}^a &= 2\partial_{[\mu} e_{\nu]}{}^a + 2\epsilon^{abc}\omega_{[\mu b} e_{\nu]c} \\ R_{\mu\nu}{}^a &= 2\partial_{[\mu}\omega_{\nu]}{}^a + \epsilon^{abc}(\omega_{\mu b}\omega_{\nu c} + \Lambda e_{\mu b} e_{\nu c}) \end{aligned}$$

- The Chern-Simons action functional of the Poincare, dS or AdS algebra is found to be *identical* to the 3-d E-H action:

$$\mathcal{S}_{CS} = \frac{1}{16\pi G}\int \epsilon^{\mu\nu\rho}(e_\mu{}^a(\partial_\nu\omega_{\rho a} - \partial_\rho\omega_{\nu a}) + \epsilon_{abc} e_\mu{}^a \omega_\nu{}^b \omega_\rho{}^c + \frac{\Lambda}{3}\epsilon_{abc} e_\mu{}^c e_\nu{}^b e_\rho{}^c) \equiv S_{EH}$$

Nc gauge theories (revisited)

- Operators $X_\mu \in \mathcal{A}$ satisfy the CR: $[X_\mu, X_\nu] = i\theta_{\mu\nu}$, $\theta_{\mu\nu}$ arbitrary
- Lie-type nc: $[X_\mu, X_\nu] = iC_{\mu\nu}{}^\rho X_\rho$
- Natural intro of nc gauge theories through *covariant nc coordinates*: $\mathcal{X}_\mu = X_\mu + A_\mu$ *Madore-Schraml-Schupp-Wess '00*
- Obeys a covariant gauge transformation rule: $\delta\mathcal{X}_\mu = i[\epsilon, \mathcal{X}_\mu]$
- A_μ transforms in analogy with the gauge connection: $\delta A_\mu = -i[X_\mu, \epsilon] + i[\epsilon, A_\mu]$, ($\epsilon$ - the gauge parameter)
- Definition of a (Lie-type) nc *covariant field strength tensor*: $F_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - iC_{\mu\nu\rho}\mathcal{X}_\rho$
- Gauge theory could be abelian or nonabelian:
  - Abelian if ϵ is a function in $\mathcal{A}$
  - Nonabelian if ϵ is matrix valued $(\mathrm{Mat}(\mathcal{A}))$

Non-Abelian case

▷ *In nonabelian case, where are the gauge fields valued?*

- Let us consider the CR of two elements of an algebra:

$$[\epsilon, A] = [\epsilon^A T^A, A^B T^B] = \frac{1}{2}\{\epsilon^A, A^B\}[T^A, T^B] + \frac{1}{2}[\epsilon^A, A^B]\{T^A, T^B\}$$

- *Not* possible to restrict to a matrix algebra:
  last term neither *vanishes* in nc nor is an *algebra element*
- There are two options to overpass the difficulty:
  - Consider the universal enveloping algebra
  - Extend the generators and/or fix the rep so that the anticommutators close

▷ *We employ the second option*

$\mathbb{R}^3_\lambda$: A 3-d fuzzy space based on SU(2)

- Fuzzy sphere, S^2_F: Matrix approximation of ordinary sphere, S^2

Madore '92

- S^2 defined by coordinates of $\mathbb{R}^3$ modulo $\sum_{a=1}^3 x_a x^a = r^2$
- S^2_F defined by three rescaled angular momentum operators, $X_i = \lambda J_i$, J_i the Lie algebra generators in a unitary irreducible reps of $SU(2)$. The X_is satisfy:

$$[X_i, X_j] = i\lambda\epsilon_{ijk}X_k\,, \quad \sum_{i=1}^{3} X_i X_i = \lambda^2 j(j+1) := r^2\,, \lambda \in \mathbb{R}\,, 2j \in \mathbb{N}$$

- Allowing X_i to live in *reducible* rep: obtain the nc $\mathbb{R}^3_\lambda$, viewed as direct sum of S^2_F with all possible radii (each block of the rep is an irrep, i.e. a fuzzy sphere)

Hammou-Lagraa-Sheikh Jabbari '02

Vitale-Wallet '13, Vitale '14

- $\mathbb{R}^3_\lambda$: discrete foliation of $\mathbb{R}^3$ by multiple S^2_F of different radii
- In analogy: Lorentzian case: 3-d fuzzy space based on $SU(1,1)$

Jurman-Steinacker '14

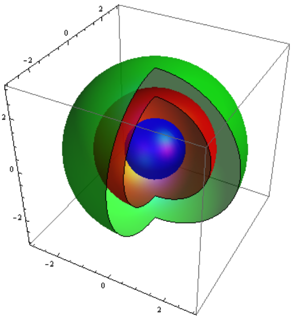

Figure: *Foliation of* $\mathbb{R}^3_\lambda$ *by fuzzy spheres*

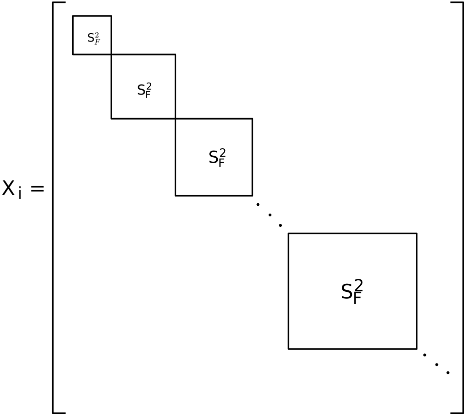

Figure: *Matrix (coordinate) of* $\mathbb{R}^3_\lambda$ *as a block diagonal form - each block is a fuzzy sphere*

- $\mathbb{R}^3_\lambda$ isometry group: $SO(4) \cong Spin(4) = SU(2) \times SU(2)$ ($SO(1,3)$ for the Lor. case)
- Anticommutators *do not close* $\rightarrow$ Fix the rep + extension of the algebra to $U(2) \times U(2)$ ($GL(2,\mathbb{C})$ for the Lor. case)
- Each $U(2)$: four 4x4 matrices as generators:

$$J_a^L = \begin{pmatrix} \sigma_a & 0 \\ 0 & 0 \end{pmatrix}, \; J_a^R = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_a \end{pmatrix}, \; J_0^L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix}, \; J_0^R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

- Identification of the correct nc dreibein and spin connection fields:

$$P_a = \frac{1}{2}(J_a^L - J_a^R), \; M_a = \frac{1}{2}(J_a^L + J_a^R), \; \mathbb{1} = J_0^L + J_0^R, \; \gamma_5 = J_0^L - J_0^R$$

- Calculations give the CRs and aCRs

$$[P_a, P_b] = i\epsilon_{abc}M_c, [P_a, M_b] = i\epsilon_{abc}P_c, [M_a, M_b] - i\epsilon_{abc}M_c, [\gamma_5, P_a] = [\gamma_5, M_a] = 0$$

$$\{P_a, P_b\} = \tfrac{1}{2}\delta_{ab}\mathbb{1}, \{P_a, M_b\} = \tfrac{1}{2}\delta_{ab}\gamma_5, \{M_a, M_b\} = \tfrac{1}{2}\delta_{ab}\mathbb{1}, \{\gamma_5, M_a\} = 2P_a, \{\gamma_5, P_a\} = 2M_a$$

Nc gauge theory of U(2)xU(2)

- Proceed with the gauging of $U(2) \times U(2)$
- Determine the covariant coordinate: $\mathcal{X}_\mu = X_\mu + \mathcal{A}_\mu$
  $\mathcal{A}_\mu = \mathcal{A}^i_\mu(X_a) \otimes T^i$ the $\mathfrak{u}(2) \times \mathfrak{u}(2)$-valued gauge connection
- Gauge connection expands on the generators as:
  $$\mathcal{A}_\mu = e_\mu{}^a(X) \otimes P_a + \omega_\mu{}^a(X) \otimes M_a + A_\mu(X) \otimes i\mathbb{1} + \tilde{A}_\mu(X) \otimes \gamma_5$$
- Gauge parameter, ϵ, expands similarly:
  $\epsilon = \xi^a(X) \otimes P_a + \lambda^a(X) \otimes M_a + \epsilon_0(X) \otimes i\mathbb{1} + \tilde{\epsilon}_0(X) \otimes \gamma_5$
- Covariant transf rule: $\delta\mathcal{X}_\mu = [\epsilon, \mathcal{X}_\mu] \rightarrow$ transf of the gauge fields:

$$\delta e_\mu{}^a = -i[X_\mu + A_\mu, \xi^a] - 2\{\xi_b, \omega_{\mu c}\}\epsilon^{abc} - 2\{\lambda_b, e_{\mu c}\}\epsilon^{abc} + i[\epsilon_0, e_\mu{}^a] - 2i[\lambda^a, \tilde{A}_\mu] - 2i[\tilde{\epsilon}_0, \omega_\mu{}^a]$$

$$\delta\omega_\mu{}^a = -i[X_\mu + A_\mu, \lambda^a] + \tfrac{1}{2}\{\xi_b, e_{\mu c}\}\epsilon^{abc} - 2\{\lambda_b, \omega_{\mu c}\}\epsilon^{abc} + i[\epsilon_0, \omega_\mu{}^a] + \tfrac{i}{2}[\xi^a, \tilde{A}_\mu] + \tfrac{i}{2}[\tilde{\epsilon}_0, e_\mu{}^a]$$

$$\delta A_\mu = -i[X_\mu + A_\mu, \epsilon_0] - i[\xi_a, e_\mu{}^a] + 4i[\lambda_a, \omega_\mu{}^a] - i[\tilde{\epsilon}_0, \tilde{A}_\mu]$$

$$\delta\widetilde{A}_\mu = -i[X_\mu + A_\mu, \tilde{\epsilon}_0] + 2i[\xi_a, \omega_\mu{}^a] + 2i[\lambda_a, e_\mu{}^a] + i[\epsilon_0, \tilde{A}_\mu]$$

- Definition of curvature:

$$\mathcal{R}_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - i\lambda\epsilon_{\mu\nu}{}^{\rho}\mathcal{X}_\rho$$

- Curvature tensor can be expanded in the $\mathfrak{u}(2) \times \mathfrak{u}(2)$ generators:

$$\mathcal{R}_{\mu\nu} = T^a_{\mu\nu} \otimes P_a + R^a_{\mu\nu} \otimes M_a + F_{\mu\nu} \otimes i\mathbb{1} + \tilde{F}_{\mu\nu} \otimes \gamma_5$$

- The expressions of the various tensors are:

$$T^a_{\mu\nu} = i[X_\mu + A_\mu, e_\nu{}^a] - i[X_\nu + A_\nu, e_\mu{}^a] + \tfrac{i}{2}\{e_{\mu b}, \omega_{\nu c}\}\epsilon^{abc} + \tfrac{i}{2}\{\omega_{\mu b}, e_{\nu c}\}\epsilon^{abc} + [\omega_\mu{}^a, \tilde{A}_\nu] - [\omega_\nu{}^a, \tilde{A}_\mu] - i\lambda\epsilon_{\mu\nu}{}^{\rho} e_\rho{}^a$$

$$R^a_{\mu\nu} = i[X_\mu + A_\mu, \omega_\nu{}^a] - i[X_\nu + A_\nu, \omega_\mu{}^a] + \tfrac{i}{2}\{\omega_{\mu b}, \omega_{\nu c}\}\epsilon^{abc} + \tfrac{i}{2}\{e_{\mu b}, e_{\nu c}\}\epsilon^{abc} + [e_\mu{}^a, \tilde{A}_\nu] - [e_\nu{}^a, \tilde{A}_\mu] - i\lambda\epsilon_{\mu\nu}{}^{\rho}\omega_\rho{}^a$$

$$F_{\mu\nu} = i[X_\mu + A_\mu, X_\nu + A_\nu] - \tfrac{i}{4}[e_\mu{}^a, e_{\nu a}] - \tfrac{i}{4}[\omega_\mu{}^a, \omega_{\nu a}] - i[\tilde{A}_\mu, \tilde{A}_\nu] - i\lambda\epsilon_{\mu\nu}{}^{\rho}(X_\rho + A_\rho)$$

$$\tilde{F}_{\mu\nu} = i[X_\mu + A_\mu, \tilde{A}_\nu] - i[X_\nu + A_\nu, \tilde{A}_\mu] + \tfrac{1}{4}[e_\mu{}^a, \omega_{\nu a}] + \tfrac{1}{4}[\omega_\mu{}^a, e_{\nu a}] - i\lambda\epsilon_{\mu\nu}{}^{\rho}\tilde{A}_\rho$$

Gere-Vitale-Wallet '14

- The action we propose is Chern-Simons type:

$$\mathcal{S} = \frac{1}{g^2}\mathrm{Trtr}\left(\frac{i}{3}\epsilon^{\mu\nu\rho}\mathcal{X}_\mu\mathcal{X}_\nu\mathcal{X}_\rho + \frac{\lambda}{2}\mathcal{X}_\mu\mathcal{X}^\mu\right)$$

- Tr: Trace over matrices X; tr: Trace over the algebra
- The action can be written as:

$$\mathcal{S} = \frac{1}{6g^2}\mathrm{Trtr}(i\epsilon^{\mu\nu\rho}\mathcal{X}_\mu\mathcal{R}_{\nu\rho}) + \mathcal{S}_\lambda$$

- where $\mathcal{S}_\lambda = +\frac{\lambda}{6g^2}\mathrm{Trtr}(\mathcal{X}_\mu\mathcal{X}^\mu)$
- Using the explicit form of the algebra trace, variation of the action leads to the equations of motion:

$$T_{\mu\nu}{}^a = 0\,, \quad R_{\mu\nu}{}^a = 0\,, \quad F_{\mu\nu} = 0\,, \quad \tilde{F}_{\mu\nu} = 0\,.$$

Summary & Future Plans

Summary

- 3-d gravity described as gauge theory
- Translation to nc regime (gauge theories through cov. coord.)
- 3-d nc spacetimes built from $SU(2)$ (and $SU(1,1)$)
- Gauge their symmetry groups
- Transformations of fields - Curvatures - Action
- 3-d gravity recovered at comm limit (Lorentzian case)

Future Plans

- Further analysis of the Lorentzian case space structure
- Move to the realistic 4-d case

Heckman-Verlinde '14, Buric-Madore '15

Thank you for your attention!

http://physics.ntua.gr/corfu2018/

The Lorentzian case

- In analogy: Lorentzian case: 3-d fuzzy space based on $SU(1,1)$

Jurman-Steinacker '14

- Fuzzy hyperboloids, dS_F^2, defined by three rescaled operators, $X_i = \lambda J_i$, (in appropriate irreps) satisfying:

$$[X_i, X_j] = i\lambda C_{ij}^k X_k\,, \quad \sum_{i,j} \eta_{ij} X_i X_j = \lambda^2 j(j-1)\,,$$

- where C_{ij}^k are the structure constants of $\mathfrak{su}(1,1)$
- Difference to previous case: Non-compact group, i.e. no finite-dim UIRs but infinite-dim
- Again, letting X_i live in (infinite-dim) *reducible* reps: Block diagonal form - each block being a dS_F^2
- 3-d Minkowski spacetime foliated with leaves being dS_F^2 of different radii

The Lorentzian case *Aschieri-Castellani '09*

- Consideration of the foliated M^3 with $\lambda > 0$
- Relevant isometry group: $SO(3,1)$
- Consider the corresponding spin group: $SO(3,1) \cong Spin(3,1) = SL(2,\mathbb{C})$
- Anticommutators *do not close* $\rightarrow$ Fix at spinor rep generated by: $\sum_{AB} = \frac{1}{2}\gamma_{AB} = \frac{1}{4}[\gamma_A, \gamma_B]\,, A = 1,\ldots 4$
- Satisfying the CRs and aCRs:

  $[\gamma_{AB}, \gamma_{CD}] = 8\eta_{[A[C}\gamma_{D]B]}\,, \{\gamma_{AB}, \gamma_{CD}\} = 4\eta_{C[B}\eta_{A]D}\mathbb{1} + 2i\epsilon_{ABCD}\gamma_5$
- Inclusion of γ_5 and identity in the algebra $\rightarrow$ extension of $SL(2,\mathbb{C})$ to $GL(2,\mathbb{C})$ with set of generators: $\{\gamma_{AB}, \gamma_5, i\mathbb{1}\}$

SO(3) notation

- In $SO(3)$ notation: $\gamma_{a4} \equiv \gamma_a$ and $\tilde{\gamma}^a \equiv \epsilon^{abc}\gamma_{bc}$,$a = 1, 2, 3$
- The CRs and aCRs are now written:

$$[\tilde{\gamma}^a, \tilde{\gamma}^b] = -4\epsilon^{abc}\tilde{\gamma}_c\,,\ [\gamma_a, \tilde{\gamma}_b] = -4\epsilon_{abc}\gamma^c,\ [\gamma_a, \gamma_b] = \epsilon_{abc}\tilde{\gamma}^c,\ [\gamma^5, \gamma^{AB}] = 0$$
$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = -8\eta^{ab}\mathbb{1},\ \{\gamma_a, \tilde{\gamma}^b\} = 4i\delta_a^b\gamma_5,\ \{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbb{1}\,,$$
$$\{\gamma^5, \gamma^a\} = i\tilde{\gamma}_a\,,\ \{\tilde{\gamma}^5, \gamma^a\} = -4i\gamma_a$$

- Proceed with the gauging of $GL(2, \mathbb{C})$
- Determine the covariant coordinate: $\mathcal{X}_\mu = X_\mu + \mathcal{A}_\mu$
  $\mathcal{A}_\mu = \mathcal{A}_\mu^i(X_a) \otimes T^i$ the $\mathfrak{gl}(2, \mathbb{C})$-valued gauge connection
- Gauge connection expands on the generators as:

  $$\mathcal{A}_\mu = e_\mu{}^a(X) \otimes \gamma_a + \omega_\mu{}^a(X) \otimes \tilde{\gamma}_a + A_\mu(X) \otimes i\mathbb{1} + \tilde{A}_\mu(X) \otimes \gamma_5$$

- Gauge parameter, ϵ, expands similarly:
  $\epsilon = \xi^a(X) \otimes \gamma_a + \lambda^a(X) \otimes \tilde{\gamma}_a + \epsilon_0(X) \otimes i\mathbb{1} + \tilde{\epsilon}_0(X) \otimes \gamma_5$

Kinematics

- Covariant transf rule: $\delta\mathcal{X}_\mu = [\epsilon, \mathcal{X}_\mu] \rightarrow$ transf of the fields:

$$\delta e_\mu{}^a = -i[X_\mu + A_\mu, \xi^a] - 2\{\xi_b, \omega_{\mu c}\}\epsilon^{abc} - 2\{\lambda_b, e_{\mu c}\}\epsilon^{abc} + i[\epsilon_0, e_\mu{}^a] - 2i[\lambda^a, \tilde{A}_\mu] - 2i[\tilde{\epsilon}_0, \omega_\mu{}^a]$$

$$\delta\omega_\mu{}^a = -i[X_\mu + A_\mu, \lambda^a] + \tfrac{1}{2}\{\xi_b, e_{\mu c}\}\epsilon^{abc} - 2\{\lambda_b, \omega_{\mu c}\}\epsilon^{abc} + i[\epsilon_0, \omega_\mu{}^a] + \tfrac{i}{2}[\xi^a, \tilde{A}_\mu] + \tfrac{i}{2}[\tilde{\epsilon}_0, e_\mu{}^a]$$

$$\delta A_\mu = -i[X_\mu + A_\mu, \epsilon_0] - i[\xi_a, e_\mu{}^a] + 4i[\lambda_a, \omega_\mu{}^a] - i[\tilde{\epsilon}_0, \tilde{A}_\mu]$$

$$\delta\widetilde{A}_\mu = -i[X_\mu + A_\mu, \tilde{\epsilon}_0] + 2i[\xi_a, \omega_\mu{}^a] + 2i[\lambda_a, e_\mu{}^a] + i[\epsilon_0, \tilde{A}_\mu]$$

- Abelian limit: $e_\mu{}^a = \omega_\mu{}^a = \tilde{A}_\mu = 0$:
  $\delta A_\mu = -i[X_\mu, \epsilon_0] + i[\epsilon_0, A_\mu] \rightarrow$ trans rule of a nc Maxwell gauge field
- Commutative limit: Y-M and gravity fields disentangle and inner derivation becomes $[X_\mu, f] \rightarrow -i\partial_\mu f$:

$$\delta e_\mu{}^a = -\partial_\mu\xi^a - 4\xi_b\omega_{\mu c}\epsilon^{abc} - 4\lambda_b e_{\mu c}\epsilon^{abc}$$

$$\delta\omega_\mu{}^a = -\partial_\mu\lambda^a + \xi_b e_{\mu c}\epsilon^{abc} - 4\lambda_b\omega_{\mu c}\epsilon^{abc}$$

- After the redefinitions: $\gamma_a \rightarrow \frac{2i}{\sqrt{\Lambda}}P_a$, $\tilde{\gamma}_a \rightarrow -4J_a$, $4\lambda^a \rightarrow \lambda^a$, $\xi^a\frac{2i}{\sqrt{\Lambda}} \rightarrow -\xi^a, e^a_\mu \rightarrow \frac{\sqrt{\Lambda}}{2i}e^a_\mu, \omega^a_\mu \rightarrow -\frac{1}{4}\omega^a_\mu$ $\rightarrow$ 3-d gravity

- Definition of curvature:

$$\mathcal{R}_{\mu\nu} = [\mathcal{X}_\mu, \mathcal{X}_\nu] - i\lambda C_{\mu\nu}{}^{\rho}\mathcal{X}_\rho$$

- Curvature tensor can be expanded in the $GL(2,\mathbb{C})$ generators:

$$\mathcal{R}_{\mu\nu} = T^a_{\mu\nu} \otimes \gamma_a + R^a_{\mu\nu} \otimes \tilde{\gamma}_a + F_{\mu\nu} \otimes i\mathbb{1} + \tilde{F}_{\mu\nu} \otimes \gamma_5$$

- The expressions of the various tensors are:

$$T^a_{\mu\nu} = i[X_\mu + A_\mu, e_\nu{}^a] - i[X_\nu + A_\nu, e_\mu{}^a] - 2\{e_{\mu b}, \omega_{\nu c}\}\epsilon^{abc} - 2\{\omega_{\mu b}, e_{\nu c}\}\epsilon^{abc} - 2i[\omega_\mu{}^a, \tilde{A}_\nu] + 2i[\omega_\nu{}^a, \tilde{A}_\mu] - i\lambda C_{\mu\nu}{}^{\rho} e_\rho{}^a$$

$$R^a_{\mu\nu} = i[X_\mu + A_\mu, \omega_\nu{}^a] - i[X_\nu + A_\nu, \omega_\mu{}^a] - 2\{\omega_{\mu b}, \omega_{\nu c}\}\epsilon^{abc} + \tfrac{1}{2}\{e_{\mu b}, e_{\nu c}\}\epsilon^{abc} + \tfrac{i}{2}[e_\mu{}^a, \tilde{A}_\nu] - \tfrac{i}{2}[e_\nu{}^a, \tilde{A}_\mu] - i\lambda C_{\mu\nu}{}^{\rho}\omega_\rho{}^a$$

$$F_{\mu\nu} = i[X_\mu + A_\mu, X_\nu + A_\nu] - i[e_\mu{}^a, e_{\nu a}] + 4i[\omega_\mu{}^a, \omega_{\nu a}] - i[\tilde{A}_\mu, \tilde{A}_\nu] - i\lambda C_{\mu\nu}{}^{\rho}(X_\rho + A_\rho)$$

$$\tilde{F}_{\mu\nu} = i[X_\mu + A_\mu, \tilde{A}_\nu] - i[X_\nu + A_\nu, \tilde{A}_\mu] + 2i[e_\mu{}^a, \omega_{\nu a}] + 2i[\omega_\mu{}^a, e_{\nu a}] - i\lambda C_{\mu\nu}{}^{\rho}\tilde{A}_\rho$$

- Commutative limit: *Coincidence* with the expressions of 3-d gravity after applying the redefinitions

The action

Gere-Vitale-Wallet '14

- The action we propose is Chern-Simons type:

$$\mathcal{S} = \frac{1}{g^2}\mathrm{Trtr}\left(\frac{i}{3}C^{\mu\nu\rho}\mathcal{X}_\mu\mathcal{X}_\nu\mathcal{X}_\rho - \frac{\lambda}{2}\mathcal{X}_\mu\mathcal{X}^\mu\right)$$

- Tr: Trace over matrices X; tr: Trace over the algebra
- The action can be written as:

$$\mathcal{S} = \frac{1}{6g^2}\mathrm{Trtr}(iC^{\mu\nu\rho}\mathcal{X}_\mu\mathcal{R}_{\nu\rho}) + \mathcal{S}_\lambda$$

- where $\mathcal{S}_\lambda = -\frac{\lambda}{6g^2}\mathrm{Trtr}(\mathcal{X}_\mu\mathcal{X}^\mu)$
- Using the explicit form of the algebra trace:

$$\mathrm{Tr}C^{\mu\nu\rho}\left(e_{\mu a}T^a_{\nu\rho} - 4\omega_{\mu a}R^a_{\nu\rho} - (X_\mu + A_\mu)F_{\nu\rho} + \tilde{A}_\mu\tilde{F}_{\nu\rho}\right)$$

- Commutative limit: First two term *identical* to 3-d gravity (after redefinition)