# Lie algebroid gauge theories and applications to T-duality

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# Reference

Talk based on:

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Hidden Isometry of "T-duality without Isometry"
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Mark Bugden, "A Tour of T-duality – Geometric and Topological Aspects of T-dualities", PhD Thesis 2018

Kyle Wright, "Generalised Geometries and Lie Algebroid Gauging in String Theory", PhD Thesis 2018

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# Reference

Inspired by:

A. Kotov and T. Strobl,
Gauging without initial symmetry,
J. Geom. Phys 99 (2016) 184-189, arXiv:1403.8119

A. Chatzistavrakidis, A. Deser and L. Jonke,
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JHEP 01 (2016) 154, arXiv:1509.01829

# The 2D (non-linear) sigma model

A 2D non-linear sigma model describes maps $X$ from a 2-dimensional surface ('worldsheet') $\Sigma$ to an $N$-dimensional manifold $M$ ('target'), equipped with additional structure

For example

$$S[X] = \frac{1}{2}\int_{\Sigma} G_{ij}(X)\, dX^i \wedge \star dX^j + B_{ij}(X)\, dX^i \wedge dX^j$$

# Symmetries of sigma model

Given a set of vector fields $v_a(X) = v_a^i(X)\partial_i$ forming a Lie algebra $\mathfrak{g}$

$$[v_a, v_b] = C^c{}_{ab} v_c$$

Consider the infinitesimal transformations

$$\delta_\epsilon X^i = v_a^i(X)\,\epsilon^a$$

we have

$$\delta_\epsilon S = \int_\Sigma \epsilon^a \left( (\mathcal{L}_{v_a} G)_{ij}\, dX^i \wedge \star dX^j + (\mathcal{L}_{v_a} B)_{ij}\, dX^i \wedge dX^j \right)$$

The sigma model action is invariant under these transformations if

$$\mathcal{L}_{v_a} G = 0\,, \qquad \mathcal{L}_{v_a} B = 0$$

If this is the case, we can *gauge* the model by promoting the global symmetry to a local one (i.e. take $\epsilon \in C^\infty(\Sigma, \mathfrak{g})$)

# The gauged action

Introducing gauge fields $A \in \Omega^1(\Sigma, \mathfrak{g})$ the gauged action is given by

$$S[X, A] = \frac{1}{2} \int_\Sigma G_{ij}(X)\, DX^i \wedge \star DX^j + B_{ij}(X)\, DX^i \wedge DX^j$$

where

$$DX^i = dX^i - v_a^i A^a$$

are the covariant derivatives.

# Gauge invariance

The gauged action $S[X, A]$ is invariant with respect to the following (local) gauge transformations:

$$\begin{aligned} \delta_\epsilon X^i &= v_a^i\, \epsilon^a \\ \delta_\epsilon A &= d\epsilon + [A, \epsilon] = (d\epsilon^a + C^a{}_{bc}\, A^b\, \epsilon^c)\, T_a \end{aligned}$$

where $T_a$ is a basis of $\mathfrak{g}$.

Now suppose we want the gauged sigma model to be equivalent to the the ungauged model. Then we need to 'fix the gauge'

# Gauge fixing

Introduce the curvature $F \in \Omega^2(\Sigma, \mathfrak{g})$

$$F = dA + A \wedge A = (dA^a + \tfrac{1}{2} C^a{}_{bc} A^b \wedge A^c) T_a = F^a T_a$$

and an 'auxiliary field' $\widehat{X} \in C^\infty(\Sigma, \mathfrak{g}^*)$, with infinitesimal transformation rules

$$\delta_\epsilon F^a = C^a{}_{bc} F^b \epsilon^c$$
$$\delta_\epsilon \widehat{X}_a = -C^c{}_{ab} \widehat{X}_c \epsilon^b$$

# Gauge fixing

Consider the action

$$S[X, A, \widehat{X}] = \frac{1}{2} \int_\Sigma \Big( G_{ij}(X)\, DX^i \wedge \star DX^j + B_{ij}(X)\, DX^i \wedge DX^j \Big) + \int_\Sigma \widehat{X}_a\, F^a$$

The equation of motion for $\widehat{X}_a$ gives $F^a = 0$.

To solve this equation we need to lift the action of $\mathfrak{g}$ to an action of the group G ($\mathfrak{g} = \mathrm{Lie}\, \mathrm{G}$)

# Example: Group manifold

Let $g : \Sigma \to \mathrm{G}$

$$S[g] = \frac{1}{2} \int_\Sigma (g^{-1} dg \overset{\wedge}{,} *g^{-1} dg)_G$$

Invariant under left action of $h \in \mathrm{G}$

$$S[hg] = S[g]$$

while

$$S[gh] = \frac{1}{2} \int_\Sigma (\mathrm{Ad}(h^{-1}) g^{-1} dg \overset{\wedge}{,} \mathrm{Ad}(h^{-1}) * g^{-1} dg)_G$$

So, invariant under right action of G if $G$ is Ad-invariant (Killing form)

# Example: Gauged

In that case we can gauge in the standard way, and obtain the gauged model (with $F$-term)

$$S[g, A, \widehat{X}] = \frac{1}{2} \int_\Sigma (g^{-1}Dg \overset{\wedge}{,} *g^{-1}Dg)_G + \int_\Sigma \langle \widehat{X}, F \rangle$$

where

$$\begin{aligned} g^{-1}Dg &= g^{-1}dg - A \\ F &= dA + A \wedge A \end{aligned}$$

and gauge symmetry, for $h \in \mathrm{G}$

$$\begin{aligned} g &\to gh \\ A &\to h^{-1}Ah + h^{-1}dh \\ \widehat{X} &\to \mathrm{Ad}^*(h^{-1})\widehat{X} \end{aligned}$$

# Example: Gauged model

Solving $F = 0$ gives $A = -dkk^{-1}$ for $k \in C^\infty(\Sigma, \mathrm{G})$, and substituting

$$g^{-1}Dg \to g^{-1}dg + dkk^{-1} = k\big((gk)^{-1}d(gk)\big)k^{-1}$$

I.e.

$$S[g, A = -dkk^{-1}] = S[gk]$$

so after 'fixing the gauge' we recover the ungauged model.

# (Non-abelian) T-duality

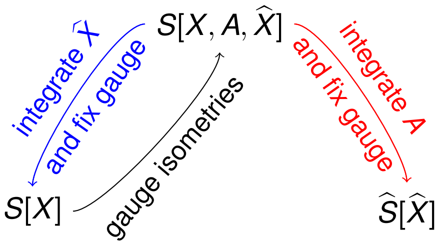
On the other hand, first solving the equation of motion for $A$, and then fixing the gauge, gives dual model

$$\widehat{S}[\widehat{X}] = \frac{1}{2}\int_{\Sigma} \widehat{G}^{ab}(\widehat{X})\, d\widehat{X}_a \wedge \star d\widehat{X}_b$$

with dual 'metric'

$$\widehat{G}^{-1}{}_{ab} = G_{ab} - C^c{}_{ab}\widehat{X}_c$$

# T-duality



$S[X, A, \widehat{X}]$
integrate $\widehat{X}$
and fix gauge
$S[X]$
gauge isometries
integrate $A$
and fix gauge
$\widehat{S}[\widehat{X}]$

# Gauging without isometries?

The existence of global symmetries is a very stringent requirement. A generic metric will not have any Killing vectors.

**Question:** Is it possible to follow the same procedure when the vector fields are not Killing vectors?

Kotov and Strobl[1] introduced a method of gauging a sigma model without requiring the model to possess isometries.

[1] [1403.8119]

# Gauging without isometry

Their method uses Lie algebroids, and generalises the standard gauging in two notable ways:

- The structure constants of the Lie algebra are promoted to structure functions:

$$[v_a, v_b] = C^c{}_{ab}(X)\, v_c$$

- The gauge invariance of the gauged action doesn't require the original vector fields to be isometries:

$$\mathcal{L}_{v_a} G \neq 0 \qquad \mathcal{L}_{v_a} B \neq 0$$

# Gauging without isometry

The set-up involves a Lie algebroid $Q$, a map $X : \Sigma \to M$,

$$\begin{array}{ccccc} X^*Q & \longrightarrow & Q & \xrightarrow{\rho} & TM \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma & \xrightarrow{X} & M & \xrightarrow{\cong} & M \end{array}$$

together with a gauge field

$$A \in \Omega^1(\Sigma, X^*Q)$$

a connection $\nabla$ on $Q$

$$\nabla : \Gamma(Q) \to \Gamma(T^*M \otimes Q) = \Omega^1(M) \otimes \Gamma(Q)$$

and infinitesimal gauge parameter $\epsilon \in C^\infty(\Sigma, X^*Q)$.

# Non-isometric T-duality

Upon choosing a basis $e_a$ of sections of $Q$, and defining matrix-valued one-forms $\omega^b{}_a$ by

$$\nabla e_a = \omega^b{}_a e_b$$

the conditions on $G$ and $B$ become

$$\mathcal{L}_{v_a} G = \omega^b{}_a \vee \iota_{v_b} G$$
$$\mathcal{L}_{v_a} B = \omega^b{}_a \wedge \iota_{v_b} B$$

where $v_a = \rho(e_a)$.

# The gauged action

The gauged action

$$S^{\omega}[X,A] = \frac{1}{2}\int_{\Sigma} G_{ij}\, DX^i \wedge \star DX^j + B_{ij}\, DX^i \wedge DX^j$$

is invariant under the modified (infinitesimal) gauge transformations

$$\delta_\epsilon X^i = v_a^i\,\epsilon^a$$
$$\delta_\epsilon A^a = d\epsilon^a + C^a{}_{bc} A^b \epsilon^c + \omega^a{}_{bi}\epsilon^b DX^i$$

# The gauged action, cont'd

**Problems:**

- Infinitesimal gauge transformations do not necessarily close. In fact

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{[\epsilon_1,\epsilon_2]} \quad \text{iff} \quad R^b{}_a = d\omega^b{}_a + \omega^b{}_c \wedge \omega^c{}_a = 0$$

- How to lift this to a global (groupoid) action?

# Non-isometric T-duality

Chatzistavrakidis, Deser, and Jonke[2] apply this non-isometric gauging procedure to T-duality

The curvature is now given by

$$F_\omega^a = dA^a + \frac{1}{2} C^a{}_{bc}(X) A^b \wedge A^c - \omega^a{}_{bi} A^b \wedge DX^i$$

and

$$\delta_\epsilon \widehat{X}_a = -C^c{}_{ab} \epsilon^b \widehat{X}_c + v_a^i \omega^c{}_{bi} \epsilon^b \widehat{X}_c$$

**Observation:** In all their examples their ‘non-isometric T-duality’ is equivalent to non-abelian T-duality.

[2] [1509.01829] and [1604.03739]

# A necessary condition for gauge invariance

We have

$$\delta_\epsilon(\widehat{X}_a F^a_\omega) = \widehat{X}_b(d\omega^b{}_a + \omega^b{}_c \wedge \omega^c{}_a)\epsilon^a + \mathcal{O}(A) + \mathcal{O}(A^2)$$

Hence, a necessary condition for gauge invariance of the non-isometrically gauged action with $F_\omega$-term, is that $\omega^b{}_a$ is flat

$$R^b{}_a = d\omega^b{}_a + \omega^b{}_c \wedge \omega^c{}_a = 0$$

This tells us that $\omega^b{}_a$ is of the form $K^{-1}dK$ for some $K^b{}_a(X)$.

# A field redefinition

Using this $K$, we can perform the following field redefinitions:

$$\widetilde{A}^a = K^a{}_b A^b$$
$$\widetilde{\widehat{X}}_a = \widehat{X}_b (K^{-1})^b{}_a$$
$$\widetilde{v}_a = v^i_b (K^{-1})^b{}_a$$

Note that

$$\widetilde{D}X^i = dX^i - \widetilde{v}^i_a \widetilde{A}^a = dX^i - v^i_a A^a = DX^i$$

# Equivalence!

The gauged action can now be rewritten in terms of the new fields $(X^i, \widetilde{A}^a, \widetilde{\widehat{X}}_a)$.

$$
\begin{aligned}
S^\omega[X, \widetilde{A}, \widetilde{\widehat{X}}] &= \frac{1}{2}\int_\Sigma G_{ij}\, DX^i \wedge \star DX^j + B_{ij}\, DX^i \wedge DX^j + \int_\Sigma \widetilde{\widehat{X}}_a \widetilde{F}^a \\
&= S[X, \widetilde{A}, \widetilde{\widehat{X}}]
\end{aligned}
$$

where

$$
\widetilde{F}^a = d\widetilde{A}^a + \frac{1}{2}\widetilde{C}^a{}_{bc}\widetilde{A}^b \wedge \widetilde{A}^c
$$

The gauge transformations become the usual non-abelian gauge transformations, and a short computation reveals

$$\mathcal{L}_{\widetilde{v}_a} G = 0 \qquad \mathcal{L}_{\widetilde{v}_a} B = 0$$

Finally, gauge invariance of the action also requires that the structure functions $\widetilde{C}^c{}_{ab}(X)$ be constants.

# Result

**Conclusion:** Infinitesimal gauge invariance of the non-isometrically gauged Lie algebroid sigma model implies that the connection $\nabla^\omega$ is flat, and that there exists a Lie algebra $\mathfrak{g}(Q,\omega)$, with constant structure functions $\widetilde{C}^a{}_{bc}$ which is equivalent to this model upon field redefinition ('change of basis of the Lie algebroid').

**Corollary:** Non-isometric T-duality (in this context) is equivalent to non-abelian T-duality.

# Example: revisited

If, in our example,

$$S[g] = \frac{1}{2} \int_\Sigma (g^{-1} dg \overset{\wedge}{,} *g^{-1} dg)_G$$

the metric $G$ is not Ad-invariant, then we can still non-isometrically gauge with respect to the right action.

It turns out that by performing the field redefinitions this model is equivalent to isometrically gauging the left action.

# What next?

Generalise even further.

Suppose

$$\mathcal{L}_{v_a} G = \omega^b{}_a \vee \iota_{v_b} G - \phi^b{}_a \vee \iota_{v_b} B$$
$$\mathcal{L}_{v_a} B = \omega^b{}_a \wedge \iota_{v_b} B - \phi^b{}_a \wedge \iota_{v_b} G$$

for some matrix valued 1-forms $\omega^b{}_a$ and $\phi^b{}_a$.

The gauged action is invariant under the following (local) gauge transformations:

$$\delta_\epsilon X^i = v_a^i \epsilon^a$$
$$\delta_\epsilon A^a = d\epsilon^a + C^a_{bc} A^b \epsilon^c + \omega^a{}_{bi} \epsilon^b DX^i + \phi^a{}_{bi} \epsilon^b \star DX^i .$$

# Closure of the gauge algebra

Closure of the infinitesimal gauge algebra imposes conditions on $\omega$ and $\phi$.

Introduce 1-form connection coefficients

$$(\Omega^{\pm})^a{}_b = \omega^a{}_b \pm \phi^a{}_b$$

which define vector bundle connections via $\nabla^{\pm} e_a = (\Omega^{\pm})^b{}_a \otimes e_b$ for some basis $\{e_a\}$ for $\Gamma(Q)$.

Now $[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] - \delta_{[\varepsilon_1,\varepsilon_2]} = 0$ iff

$$\nabla^{\pm}_i (T_{\nabla^{\pm}})^a{}_{bc} = 2 v^j_{[b|} (R_{\nabla^{\pm}})^a{}_{ji|c]} .$$

# Closure of the gauge algebra, cont'd

Or better yet define Lie algebroid connections

$$({}^{Q}\Omega^{\pm})^{a}{}_{bc} = v^{i}_{c}\omega^{a}{}_{ib} \pm v^{i}_{c}\phi^{a}{}_{ib} + C^{a}{}_{bc}.$$

Then the infinitesimal gauge algebra closes iff $R_{{}^{Q}\nabla^{\pm}} = 0$.

Can we find solutions to these equations?

# Poisson-Lie symmetry

Poisson Lie symmetry requires

$$(\mathcal{L}_{v_a}E)_{ij} = \widetilde{C}^{bc}{}_a v_b^m v_c^n E_{mj} E_{in}$$

where $E = G + B$.

The Poisson-Lie compatibility condition

$$2C^d{}_{[a|g}\widetilde{C}^{gl}{}_{b]} + 2\widetilde{C}^{dg}{}_{[b}C^f{}_{a]g} - C^g{}_{ab}\widetilde{C}^{dl}{}_g = 0$$

implies that we have a Drinfeld double. A Lie group D with Lie bialgebra Lie(D) $= \mathfrak{d} \cong \mathfrak{g} \oplus \tilde{\mathfrak{g}}$.

We find a solution to the generalised Killing equations with

$$(\Omega^+)^b{}_{ia} = \widetilde{C}^{bc}_a v_c^j E_{ji}\,, \quad (\Omega^-)^b{}_{\mu a} = 0$$

# Does this work?

We are not sure!

- The choices result in a closed infinitesimal gauge algebra
- We can add an $F$-term gauge invariantly (which seems to be related to some sort of ‘half-gauging’ on the double D)
- And here we’re stuck (for now)

THANKS