Reduction and Morita Theory for Coisotropic Triples

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13.04.2019

Based on joint work with Chiara Esposito and Stefan Waldmann.
arXiv:1812.09703

Motivation

Coisotropic Reduction

Geometry

Poisson manifold:

$$(M, \pi_M)$$

Coisotropic submanifold:

$$C \overset{\iota}{\hookrightarrow} M$$

Characteristic distribution:

$$\{X_f(p) \in T_pM \mid p \in M, f\big|_C = 0\}$$

Reduced manifold:

$$M_{\mathrm{red}} = C/\sim$$

Algebra

Poisson algebra:

$$\mathscr{A} = (\mathscr{C}^\infty(M), \{\,\cdot\,,\,\cdot\,\})$$

Coisotropic ideal:

$$\mathscr{J} = \{f \in \mathscr{C}^\infty(M) \mid f\big|_C = 0\}$$

Normalizer:

$$\mathrm{N}(\mathscr{J}) = \{f \in \mathscr{C}^\infty(M) \mid \{f, \mathscr{J}\} \subseteq \mathscr{J}\}$$

Reduced algebra:

$$\mathscr{A}_{\mathrm{red}} = \mathrm{N}(\mathscr{J})/\mathscr{J} = \mathscr{C}^\infty(M_{\mathrm{red}})$$

Quantized Reduction

$\iota \colon C \hookrightarrow M$ coisotropic submanifold

$$\begin{aligned} \mathscr{A} &= (\mathscr{C}^\infty(M), \cdot) & \rightsquigarrow \quad & \boldsymbol{\mathscr{A}} = (\mathscr{C}^\infty(M)[\![\lambda]\!], \star) \\ \iota^* &\colon \mathscr{C}^\infty(M) \longrightarrow \mathscr{C}^\infty(C) & \rightsquigarrow \quad & \boldsymbol{\iota}^* \colon \mathscr{C}^\infty(M)[\![\lambda]\!] \longrightarrow \mathscr{C}^\infty(C)[\![\lambda]\!] \end{aligned}$$

Define:

$$\begin{aligned} \boldsymbol{\mathfrak{J}} &:= \ker(\boldsymbol{\iota}^*) \subseteq \mathscr{C}^\infty(M)[\![\lambda]\!] \text{ left ideal} \\ \mathrm{N}(\boldsymbol{\mathfrak{J}}) &:= \{ f \in \mathscr{C}^\infty(M)[\![\lambda]\!] \mid [\boldsymbol{\mathfrak{J}}, f]_\star \subseteq \boldsymbol{\mathfrak{J}} \} \end{aligned}$$

$\rightsquigarrow$ Reduced algebra: $\boldsymbol{\mathscr{A}}_{\mathrm{red}} = \mathrm{N}(\boldsymbol{\mathfrak{J}})/\boldsymbol{\mathfrak{J}} \simeq \mathscr{C}^\infty(M_{\mathrm{red}})[\![\lambda]\!]$

Coisotropic Algebras & Bimodules

Coisotropic Algebras

Definition (Coisotropic triple of algebras)

Triple $\mathscr{A} = (\mathscr{A}_{\mathrm{tot}}, \mathscr{A}_{\mathrm{N}}, \mathscr{A}_0)$ with

- $\mathscr{A}_{\mathrm{tot}}$ unital algebra
- $\mathscr{A}_{\mathrm{N}} \subseteq \mathscr{A}_{\mathrm{tot}}$ unital subalgebra
- $\mathscr{A}_0 \subseteq \mathscr{A}_{\mathrm{N}}$ two-sided ideal

$\rightsquigarrow$ Category: $\mathsf{C_3Alg}$.

Proposition (Reduction)

Functor $\mathrm{red}\colon \mathsf{C_3Alg} \longrightarrow \mathsf{Alg}$:

$$
\begin{aligned}
\mathscr{A} = (\mathscr{A}_{\mathrm{tot}}, \mathscr{A}_{\mathrm{N}}, \mathscr{A}_0) \quad &\longmapsto \quad \mathscr{A}_{\mathrm{red}} = \mathscr{A}_{\mathrm{N}} / \mathscr{A}_0 \\
\Phi\colon \mathscr{A} \longrightarrow \mathscr{B} \quad &\longmapsto \quad \Phi_{\mathrm{red}}\colon \mathscr{A}_{\mathrm{red}} \longrightarrow \mathscr{B}_{\mathrm{red}}
\end{aligned}
$$

Coisotropic Bimodules

What are representations of coisotropic algebras?

Definition (Bimodule over coisotropic triple of algebras)

Let $\mathscr{A}, \mathscr{B} \in \mathsf{C_3Alg}$. A $(\mathscr{B}, \mathscr{A})$-bimodule is a triple $\mathscr{E} = (\mathscr{E}_{\mathrm{tot}}, \mathscr{E}_{\mathrm{N}}, \mathscr{E}_0)$ with

- $\mathscr{E}_{\mathrm{tot}}$ is $(\mathscr{B}_{\mathrm{tot}}, \mathscr{A}_{\mathrm{tot}})$-bimodule
- $\mathscr{E}_{\mathrm{N}}$ is $(\mathscr{B}_{\mathrm{N}}, \mathscr{A}_{\mathrm{N}})$-bimodule together with $\iota_{\mathscr{E}} : \mathscr{E}_{\mathrm{N}} \longrightarrow \mathscr{E}_{\mathrm{tot}}$
- $\mathscr{E}_0 \subseteq \mathscr{E}_{\mathrm{N}}$ is $(\mathscr{B}_{\mathrm{N}}, \mathscr{A}_{\mathrm{N}})$-submodule
  - $\mathscr{B}_0 \cdot \mathscr{E}_{\mathrm{N}} \subseteq \mathscr{E}_0$ and $\mathscr{E}_{\mathrm{N}} \cdot \mathscr{A}_0 \subseteq \mathscr{E}_0$

$\rightsquigarrow$ Category: $\mathsf{C_3Bimod}(\mathscr{B}, \mathscr{A})$

Reduction of Coisotropic Bimodules

Functor $\mathrm{red}\colon \mathsf{C_3Bimod}(\mathscr{B}, \mathscr{A}) \longrightarrow \mathsf{Bimod}(\mathscr{B}_{\mathrm{red}}, \mathscr{A}_{\mathrm{red}})$:

$$\begin{aligned} \mathscr{E} = (\mathscr{E}_{\mathrm{tot}}, \mathscr{E}_{\mathrm{N}}, \mathscr{E}_0) \quad &\longmapsto \quad \mathscr{E}_{\mathrm{red}} = \mathscr{E}_{\mathrm{N}} / \mathscr{E}_0 \\ \Phi\colon \mathscr{E} \longrightarrow \mathscr{F} \quad &\longmapsto \quad \Phi_{\mathrm{red}}\colon \mathscr{E}_{\mathrm{red}} \longrightarrow \mathscr{F}_{\mathrm{red}} \end{aligned}$$

Example

$\mathscr{A}$ commutative coisotropic triple

$$\begin{aligned} \mathrm{Der}(\mathscr{A})_{\mathrm{tot}} &= \mathrm{Der}(\mathscr{A}_{\mathrm{tot}}) \\ \mathrm{Der}(\mathscr{A})_{\mathrm{N}} &= \{\Phi \in \mathrm{Der}(\mathscr{A}_{\mathrm{tot}}) \mid \Phi(\mathscr{A}_{\mathrm{N}}) \subseteq \mathscr{A}_{\mathrm{N}}, \Phi(\mathscr{A}_0) \subseteq \mathscr{A}_0\} \\ \mathrm{Der}(\mathscr{A})_0 &= \{\Phi \in \mathrm{Der}(\mathscr{A}_{\mathrm{tot}}) \mid \Phi(\mathscr{A}_{\mathrm{N}}) \subseteq \mathscr{A}_0\} \end{aligned}$$

$\implies \mathrm{Der}(\mathscr{A})_{\mathrm{red}} \hookrightarrow \mathrm{Der}(\mathscr{A}_{\mathrm{red}})$

Nice geometry: $(TM)_{\mathrm{red}} \simeq T(M_{\mathrm{red}})$

Morita Theory

Reminder: Bicategory of Rings and Morita Theory

Theorem

For two rings R *and* S *the following statements are equivalent*

- R-Mod $\simeq$ S-Mod *(Morita equivalence)*
- R *and* S *are equivalent in the bicategory* Bimod
- $\exists e \in \mathrm{M}_n(\mathsf{R})$ *s.t.* $e^2 = e$, $\mathsf{R}e\mathsf{R} = \mathsf{R}$ *and* $\mathsf{S} \simeq \mathrm{End}_{\mathsf{R}}(e\mathsf{R}^n)$

Bicategory Bimod:

- **Objects:** rings R, S
- **1-Morphisms:** bimodules ${}_{\mathsf{S}}\mathscr{E}_{\mathsf{R}}$
- **2-Morphisms:** bimodules homomorphisms
- **Tensor product:** ${}_{\mathsf{T}}\mathscr{F}_{\mathsf{S}} \otimes_{\mathsf{S}} {}_{\mathsf{S}}\mathscr{E}_{\mathsf{R}}$
- Left unit, right unit, associativity, ...

Bicategory for Coisotropic Algebras

- **Objects:** coisotropic algebras $\mathscr{A} = (\mathscr{A}_{\mathrm{tot}}, \mathscr{A}_{\mathrm{N}}, \mathscr{A}_0)$
- **1-Morphisms:** coisotropic bimodules ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}} = (\mathscr{E}_{\mathrm{tot}}, \mathscr{E}_{\mathrm{N}}, \mathscr{E}_0)$
- **2-Morphisms:** morphisms of coisotropic bimodules
- **Tensor product:** ${}_{\mathscr{C}}\mathscr{F}_{\mathscr{B}}$, ${}_{\mathscr{B}}\mathscr{E}_{\mathscr{A}}$
  - $(\mathscr{F} \otimes_{\mathscr{B}} \mathscr{E})_{\mathrm{tot}} = \mathscr{F}_{\mathrm{tot}} \otimes_{\mathscr{B}_{\mathrm{tot}}} \mathscr{E}_{\mathrm{tot}}$
  - $(\mathscr{F} \otimes_{\mathscr{B}} \mathscr{E})_{\mathrm{N}} = \mathscr{F}_{\mathrm{N}} \otimes_{\mathscr{B}_{\mathrm{N}}} \mathscr{E}_{\mathrm{N}}$
  - $(\mathscr{F} \otimes_{\mathscr{B}} \mathscr{E})_0 = \mathscr{F}_{\mathrm{N}} \otimes_{\mathscr{B}_{\mathrm{N}}} \mathscr{E}_0 + \mathscr{F}_0 \otimes_{\mathscr{B}_{\mathrm{N}}} \mathscr{E}_{\mathrm{N}}$
- Left unit, right unit, associativity, ...

$\rightsquigarrow$ **Bicategory**: $\mathsf{C_3Bimod}$

Definition (Morita equivalence)

$\mathscr{A}, \mathscr{B} \in \mathsf{C_3Alg}$ are *Morita equivalent* iff they are isomorphic in $\mathsf{C_3Bimod}$.

Characterization of Morita Equivalence

Theorem (Characterization of Equivalence Bimodules)

$\mathscr{A}, \mathscr{B} \in \mathsf{C_3Alg}$ *are Morita equivalent iff there exists idempotent* $e \in \mathrm{M}_n(\mathscr{A}_{\mathrm{N}})$ *s.t.* $\mathscr{A}_{\mathrm{N}} e \mathscr{A}_{\mathrm{N}} = \mathscr{A}_{\mathrm{N}}$ *and*

$$
\begin{aligned}
\mathscr{B}_{\mathrm{tot}} &\simeq \mathrm{End}_{\mathscr{A}_{\mathrm{tot}}}(e\mathscr{A}^n_{\mathrm{tot}}) \\
\mathscr{B}_{\mathrm{N}} &\simeq \mathrm{End}_{\mathscr{A}_{\mathrm{N}}}(e\mathscr{A}^n_{\mathrm{N}}) \\
\mathscr{B}_0 &\simeq \mathrm{Hom}_{\mathscr{A}_{\mathrm{N}}}(e\mathscr{A}^n_{\mathrm{N}}, e\mathscr{A}^n_0) \subseteq \mathrm{End}_{\mathscr{A}_{\mathrm{N}}}(e\mathscr{A}^n_{\mathrm{N}})
\end{aligned}
$$

Reduction & Classical Limit

Reduction of Bimodules

Reduction preserves Morita equivalence

2-Functor of reduction $\mathrm{red}\colon \mathsf{C_3Bimod} \longrightarrow \mathsf{Bimod}$:

$$\begin{array}{rlcl} \textbf{Objects:} & \mathscr{A} = (\mathscr{A}_{\mathrm{tot}}, \mathscr{A}_{\mathrm{N}}, \mathscr{A}_0) & \longmapsto & \mathscr{A}_{\mathrm{red}} = \mathscr{A}_{\mathrm{N}} / \mathscr{A}_0 \\ \textbf{1-morphisms:} & \mathscr{E} = (\mathscr{E}_{\mathrm{tot}}, \mathscr{E}_{\mathrm{N}}, \mathscr{E}_0) & \longmapsto & \mathscr{E}_{\mathrm{red}} = \mathscr{E}_{\mathrm{N}} / \mathscr{E}_0 \\ \textbf{2-morphisms:} & \Phi\colon \mathscr{E} \longrightarrow \mathscr{F} & \longmapsto & \Phi_{\mathrm{red}}\colon \mathscr{E}_{\mathrm{red}} \longrightarrow \mathscr{F}_{\mathrm{red}} \end{array}$$

red is a functor of bicategories!

Classical Limit for Coisotropic Bimodules

Classical limit commutes with reduction

Classical limit 2-functor $\mathrm{cl}\colon \mathsf{C_3Bimod}_{\mathsf{R}[\![\lambda]\!]} \longrightarrow \mathsf{C_3Bimod}_{\mathsf{R}}$:

$$\begin{aligned} \mathrm{cl}(\mathscr{A})_{\mathrm{tot}} &= \mathscr{A}_{\mathrm{tot}} / \lambda \mathscr{A}_{\mathrm{tot}} \\ \mathrm{cl}(\mathscr{A})_{\mathrm{N}} &= \mathscr{A}_{\mathrm{N}} / (\lambda \mathscr{A}_{\mathrm{tot}} \cap \mathscr{A}_{\mathrm{N}}) \\ \mathrm{cl}(\mathscr{A})_{0} &= \mathscr{A}_{0} / (\lambda \mathscr{A}_{\mathrm{tot}} \cap \mathscr{A}_{0}) \end{aligned}$$

$$\begin{aligned} \mathrm{cl}(\mathscr{E})_{\mathrm{tot}} &= \mathscr{E}_{\mathrm{tot}} / \lambda \mathscr{E}_{\mathrm{tot}} \\ \mathrm{cl}(\mathscr{E})_{\mathrm{N}} &= \mathscr{E}_{\mathrm{N}} / \lambda \mathscr{E}_{\mathrm{N}} \\ \mathrm{cl}(\mathscr{E})_{0} &= \mathscr{E}_{0} / (\lambda \mathscr{E}_{\mathrm{N}} \cap \mathscr{E}_{0}) \end{aligned}$$