Homological Reduction in Contact Geometry

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Outline

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Coisotropic Submanifolds

Coisotropic submanifolds play a distinguished role in Poisson geometry:

- *Lagrangian submanifolds* of symplectic manifolds,
- *level sets* of moment maps,
- *morphisms* of Poisson manifolds,
- *branes* in Poisson σ-models.

In Hamiltonian Mechanics

coisotropic submanifolds = first class contraints

The Reduced Phase Space

Characteristic Distribution

Let (P, ω) be a symplectic manifold, $C \subset P$ a coisotropic submanifold. The distribution

$$K := \left\langle X_f|_C : f \textit{ vanishes on } C \right\rangle$$

is an involutive distribution on C.

Reduced Phase Space

Physically, the degrees of freedom along K are *gauges* and should be quotiented out:

$$P_{\mathrm{red}} := C/K$$

is the true *physical phase space*. But P_{red} might be *singular*.

Singular Reduction

A way out non-smoothness is *thinking of* P_{red} *algebraically*, putting

$$C^\infty(P_{\mathrm{red}}) := \{f \in C^\infty(C) : X(f) = 0 \textit{ for all } X \in \Gamma(K)\}\,.$$

Proposition

$C^\infty(P_{\mathrm{red}})$ *is a Poisson algebra in a natural way.*

$C^\infty(P_{\mathrm{red}})$ might be too small to be useful!

Remark

The commutative algebra $C^\infty(P_{\mathrm{red}})$ is “cohomologically resolved” by the *leaf-wise de Rham complex* $\Omega^\bullet(K) := \Gamma(\wedge^\bullet K^*)$:

$$0 \to C^\infty(P_{\mathrm{red}}) \hookrightarrow C^\infty(C) \to \Omega^1(K) \to \cdots \to \Omega^i(K) \to \cdots.$$

Homological Reduction

$\Omega^\bullet(K)$ can be thought of as a *desingularization* of the commutative algebra $C^\infty(P_{\text{red}})$. How about the Poisson algebra structure?

Theorem [Oh & Park 2005], [Cattaneo & Felder 2007]

1. $\Omega^\bullet(K)$ *is an* L_∞*-algebra in a canonical way (up to* L_∞*-isomorphisms);*
2. *the binary bracket induces the Poisson bracket on* $C^\infty(P_{\text{red}})$*;*
3. *the multibrackets are multiderivations (*P_∞*-algebra).*

Quantization of Coisotropic Submanifolds

[Cattaneo & Felder 2007] use

- the P_∞-algebra of C, and
- an adaptation of Formality,

to quantize $C^\infty(P_{\text{red}})$ *up to anomalies*.

Outline

1. Coisotropic Reduction
2. Contact Geometry and Mechanics
3. Contact Homological Reduction
4. Proof of the Main Theorem

Contact Manifolds

The Aim of the Talk

is discussing the *Contact Geometry* analogue of the P_∞-algebra of a coisotropic submanifold in a symplectic manifold.

Definition

A *contact manifold* is a manifold P + a *contact distribution*, i.e. a codimension 1 distribution H satisfying a *maximal non-integrability condition*:

$R_H : H \times H \to TP/H, \quad (X, Y) \mapsto [X, Y] \operatorname{mod} H$ is non-degenerate.

Darboux Theorem

Locally, there are coordinates (x^i, u, p_i) such that H is spanned by

$$D_i := \frac{\partial}{\partial x^i} + p_i \frac{\partial}{\partial u}, \quad \frac{\partial}{\partial p_i}.$$

The Jacobi Bracket

Let (P, H) be a contact manifold. Then $L := TP/H$ is a line bundle.

Remark

1. There is a canonical vector space isomorphism

$$\Gamma(L) \xrightarrow{\cong} \{\textit{infinitesimal symmetries of } H\}, \quad \lambda \mapsto X_\lambda;$$

2. this induces a Lie bracket $\{-,-\}$ on $\Gamma(L)$, the *Jacobi bracket*:

$$X_{\{\lambda,\mu\}} = [X_\lambda, X_\mu];$$

3. locally $\Gamma(L) \cong C^\infty(P)$ and

$$X_f = \frac{\partial f}{\partial p_i} D_i - D_i f \frac{\partial}{\partial p_i} - f \frac{\partial}{\partial u},$$
$$\{f, g\} = D_i f \frac{\partial g}{\partial p_i} - D_i g \frac{\partial f}{\partial p_i} + f \frac{\partial g}{\partial u} - g \frac{\partial f}{\partial u}.$$

Contact Geometry and Mechanics

Contact Geometry is a language for *equilibrium thermodynamics* and *thermodynamic processes* are described by the flow of *contact vector fields*.

Contact Geometry is also a language for *dissipative systems*.

An Example: Damped Newton Mechanics

On a contact manifold M^3 with Darboux coordinates (x, u, p), and

$$f = \frac{p^2}{2m} + V(q) + \gamma u,$$

the flow of the *contact vector field* X_f is

$$\dot{x} = \frac{p}{m}, \quad \dot{u} = \frac{p^2}{2m} - V(q) - \gamma u, \quad \dot{p} = -\frac{\partial V(q)}{\partial q} - \gamma p,$$

which reproduces the *Damped Newton Equation*: $\ddot{x} = -\gamma \dot{x} - \frac{1}{m}\frac{\partial V(q)}{\partial q}$.

Outline

Coisotropic Submanifolds

Let (P, H) be a contact manifold, and let $L := TP/H$.

Definition

A submanifold $C \subset P$ is *coisotropic* if the contact vector field X_λ is tangent to C for every $\lambda \in \Gamma(L)$ vanishing on C.

Characteristic Distribution

Let $C \subset P$ be a coisotropic submanifold.

1. The distribution

$$K := \langle X_\lambda|_C : \lambda \textit{ vanishes on } C \rangle$$

is an involutive distribution on C.

2. The restricted line bundle $L|_C$ is equipped with a canonical flat *partial connection* ∇ along K:

$$\nabla_{X_\lambda|_C} \mu|_C := \{\lambda, \mu\}|_C.$$

The Reduced Contact Phase Space

We would like to quotient out the degrees of freedom along K, *keeping the information on the line bundle*!

$$
\begin{tikzcd}
L \arrow[d] & L|_C \arrow[l] \arrow[r, rightsquigarrow] \arrow[d] & L_{\mathrm{red}} \arrow[d] := L|_C/\nabla \\
P & C \arrow[l] \arrow[r, rightsquigarrow] & P_{\mathrm{red}} := C/K
\end{tikzcd}
$$

Remark

Sometimes, P_{red} is a smooth manifold and there is a well-defined line bundle $L_{\mathrm{red}} \to P_{\mathrm{red}}$ such that $L|_C$ is the pull-back, and flat sections are pull-back sections. But, in general, *this is not the case*!

Singular Contact Reduction

A way out non-smoothness is *thinking of* $L_{\mathrm{red}} \to P_{\mathrm{red}}$ *algebraically*:

$$C^\infty(P_{\mathrm{red}}) := \{f \in C^\infty(C) : X(f) = 0 \text{ for all } X \in \Gamma(K)\},$$
$$\Gamma(L_{\mathrm{red}}) := \{\lambda \in \Gamma(L|_C) : \nabla_X \lambda = 0 \text{ for all } X \in \Gamma(K)\}.$$

Proposition

$(C^\infty(P_{\mathrm{red}}), \Gamma(L_{\mathrm{red}}))$ is a Jacobi algebra in a natural way:

- *$\Gamma(L_{\mathrm{red}})$ is a $C^\infty(P_{\mathrm{red}})$-module;*
- *$\Gamma(L_{\mathrm{red}})$ is a Lie algebra with (induced) brackets $\{-,-\}_{\mathrm{red}}$;*
- *$\{-,-\}_{\mathrm{red}}$ is a(n algebraic) bi-derivation, i.e.*

$$\{\lambda, f\mu\}_{\mathrm{red}} = f\{\lambda,\mu\}_{\mathrm{red}} + X_\lambda(f)\mu, \quad \lambda,\mu \in \Gamma(L_{\mathrm{red}}),\ f \in C^\infty(P_{\mathrm{red}}),$$

for some derivation X_λ of $C^\infty(P_{\mathrm{red}})$.

As in the symplectic case, $(C^\infty(P_{\mathrm{red}}), \Gamma(L_{\mathrm{red}}))$ might be too small!

Contact Homological Reduction

Remark

The $C^\infty(P_{\mathrm{red}})$-module $\Gamma(L_{\mathrm{red}})$ is "resolved" by the *leaf-wise de Rham complex with coefficients in* $L|_C$: $\Omega^\bullet(K, L|_C) := \Gamma(\wedge^\bullet K^* \otimes L|_C)$:

$$0 \to \Gamma(L_{\mathrm{red}}) \hookrightarrow \Gamma(L|_C) \to \Omega^1(K, L|_C) \to \cdots \to \Omega^i(K, L|_C) \to \cdots.$$

$\Omega^\bullet(K, L)$ can be thought of as a *desingularization* of the module $\Gamma(L_{\mathrm{red}})$.
How about the Jacobi algebra structure?

Theorem [Lê, Oh, Tortorella & V 2014]

1. $\Omega^\bullet(K, L|_C)$ *is an* L_∞*-algebra in a canonical way;*
2. *the binary bracket induces the bracket* $\{-,-\}_{\mathrm{red}}$ *on* $\Gamma(L_{\mathrm{red}})$*;*
3. *the multibrackets are multiderivations (*J_∞*-algebra).*

The L_∞*-algebra* $\Omega^\bullet(K, L|_C)$ *controls deformations of* C.

Outline

L_∞-Algebras

An L_∞-algebra is a *Lie algebra up to homotopy*. Let V be a graded space.

Definition

An L_∞-algebra structure in V is a sequence of operations:

$$\mathfrak{l}_n : \wedge^n V \longrightarrow V[2-n]$$

satisfying the following *coherence conditions*

1. $\mathfrak{l}_1^2(u) = 0$
2. $\mathfrak{l}_1 \mathfrak{l}_2(u, v) = \mathfrak{l}_2(\mathfrak{l}_1 u, v) \pm \mathfrak{l}_2(u, \mathfrak{l}_1 v)$
3. $\mathfrak{l}_2(u, \mathfrak{l}_2(v, w)) + \circlearrowleft \; = \mathfrak{l}_1 \mathfrak{l}_3(u, v, w) - \mathfrak{l}_3(\mathfrak{l}_1 u, v, w) \mp \mathfrak{l}_3(u, \mathfrak{l}_1 v, w) \mp \mathfrak{l}_3(u, v, \mathfrak{l}_1 w)$
4. $\ldots$

$H(V, \mathfrak{l}_1)$ *is a honest (graded) Lie algebra!*

T. Voronov's Construction

V-data

- A graded Lie algebra $(\mathcal{L}, [-,-])$,
- a Maurer-Cartan element $\Delta \in \mathcal{L}$,
- an abelian subalgebra $\mathfrak{a} \subset \mathcal{L}$,
- a linear projector $\mathfrak{P} : \mathcal{L} \to \mathfrak{a}$, such that $\ker \mathfrak{P}$ is a subalgebra.

Theorem [T. Voronov 2005]

Let $(\mathcal{L}, \Delta, \mathfrak{a}, \mathfrak{P})$ be V-data. The higher derived brackets:

$$\mathfrak{l}_n(u_1, u_2, \ldots, u_n) := \pm \mathfrak{P}[[\cdots[[\Delta, u_1], u_2], \cdots], u_n]$$

give $\mathfrak{a}$ the structure of an L_∞-algebra.

Remark [Cattaneo & Felder 2007]

One can construct V-data from a coisotropic submanifold.

The Schouten-Jacobi Algebra of Multiderivations

Let (P, H) be a contact manifold and $\{-,-\}$ the *Jacobi bracket*.

The Schouten-Jacobi Algebra

Let $L = TP/H$. Skew-symmetric multiderivations

$$\Gamma(L) \times \cdots \times \Gamma(L) \to \Gamma(L)$$

form a graded Lie algebra $\mathcal{D}^\bullet(L)$ with the *Gerstenhaber bracket* $[-,-]$.

$\mathcal{D}^\bullet(L)$ is a *line bundle version* of the Schouten algebra of multivectors.

Remark

$\{-,-\}$ can be seen as a Maurer-Cartan element Δ in $\mathcal{D}^\bullet(L)$.

We are half-way from a set of V-data!

V-Data from Coisotropic Submanifolds

Let (P, H) be a contact manifold, $C \subset P$ a coisotropic submanifold.

Proposition

There is a canonical projection $\mathfrak{P} : \mathcal{D}^\bullet(L) \to \Omega^\bullet(K, L|_C)$.

We need *additional data* to complete our set of V-data:

- a tubular neighborhood $P \supset U \xrightarrow{\tau} C$,
- an identification $L \supset L|_U \xrightarrow{\cong} \tau^* L|_C$.

Proposition

- *The additional data determine an inclusion*

$$\mathfrak{I} : \Omega^\bullet(K, L|_C) \to \mathcal{D}^\bullet(L|_U);$$

- *the tuple* $(\mathcal{D}^\bullet(L|_U), \Delta, \operatorname{im}\mathfrak{I}, \mathfrak{P})$ *is a set of* V*-data.*

J_∞-Algebras from Coisotropic Submanifolds

Theorem

1. *The multibrackets*

$$\mathfrak{l}_n(\omega_1, \omega_2, \ldots, \omega_n) := \pm \mathfrak{P}[[\cdots[[\Delta, \mathfrak{I}\omega_1], \mathfrak{I}\omega_2], \cdots], \mathfrak{I}\omega_n]$$

   give $\Omega^\bullet(C, L|_C)$ the structure of an L_∞-algebra;
2. $\mathfrak{l}_1$ *is the leaf-wise differential;*
3. $\mathfrak{l}_2$ *induces $\{-,-\}_{\mathrm{red}}$ in $\Gamma(L_{\mathrm{red}}) = H^0(C, L|_C)$;*
4. *the $\mathfrak{l}_n$ are multiderivations (J_∞-algebra).*

The L_∞-algebra $(\Omega^\bullet(C, L|_C), \{\mathfrak{l}_n\})$ is independent of the additional data up to L_∞-isomorphisms.

Coisotropic Neighborhood Theorem

The L_∞-algebra of a coisotropic submanifold C does only depend on the *intrinsic geometry* of C.

Assume: C is not Legendrian, and $H_C := H \cap TC$ has constant rank.

Theorem

H_C knows everything about an appropriate neighborhood of C.

Corollary

The L_∞-algebra $(\Omega^\bullet(C, L|_C), \{\mathfrak{l}_n\})$ does only depend on H_C.

Choose a distribution W on C complementary to K. Then

$$\mathfrak{l}_n = \mathcal{O}\left(R_W^{n-2}\right) \quad \text{for } n > 1,$$

where $R_W : W \times W \to TC/W = K$ is the *curvature* of W.

Perspectives

- *Fedosov quantization* of contact manifolds has been achieved by [Boutet de Monvel 1995] via the *symplectization trick*: contact manifolds are equivalent to homogeneous symplectic manifolds.
- It would be natural to look at the quantization of a coisotropic submanifold in a contact manifold as in [Cattaneo & Felder 2007] (Quantization of *dissipative systems with constraints?!*).
- More generally, quantizing *Jacobi manifolds* and their coisotropic submanifolds might have an interest.

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Thank you!