Open-string non-associativity in an *R*-flux background

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Bayrischzell Workshop
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D-branes in flux backgrounds

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based on

This talk is (partially) based on the following papers ::

- *Open-string non-associativity in an R-flux background*
  w/ D. Lüst, E. Malek, M. Syväri
  arXiv:1903.05581

- *Open-string T-duality and applications to non-geometric backgrounds*
  w/ F. Cordonier-Tello, D. Lüst
  arXiv:1806.01308, JHEP 1808 198

- *Non-geometric backgrounds in string theory*
  arXiv:1811.11203, Phys.Rept. 798

outline

1. motivation
2. non-geometric backgrounds
3. d-branes & freed-witten anomaly
4. open-string non-associativity
5. conclusions

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motivation I

String theory describes the dynamics of strings in a background space.

String-theory backgrounds are specified by a choice of ::

- space — broadly defined,
- vacuum expectation values for the fields of the theory (metric, field strengths, dilaton),
- D-branes and orientifold planes.

motivation II

Properties ::

- A background is constrained by various consistency conditions (solution to eom, absence of anomalies, …), which relate its data in an intricate way.
- String-backgrounds can have properties very different from point-particle configurations, such as non-commutative or non-associative behaviour.

This talk ::

- Discuss the interplay between D-branes and (non-)geometric fluxes.
- Investigate non-associativity of open strings on D-branes.

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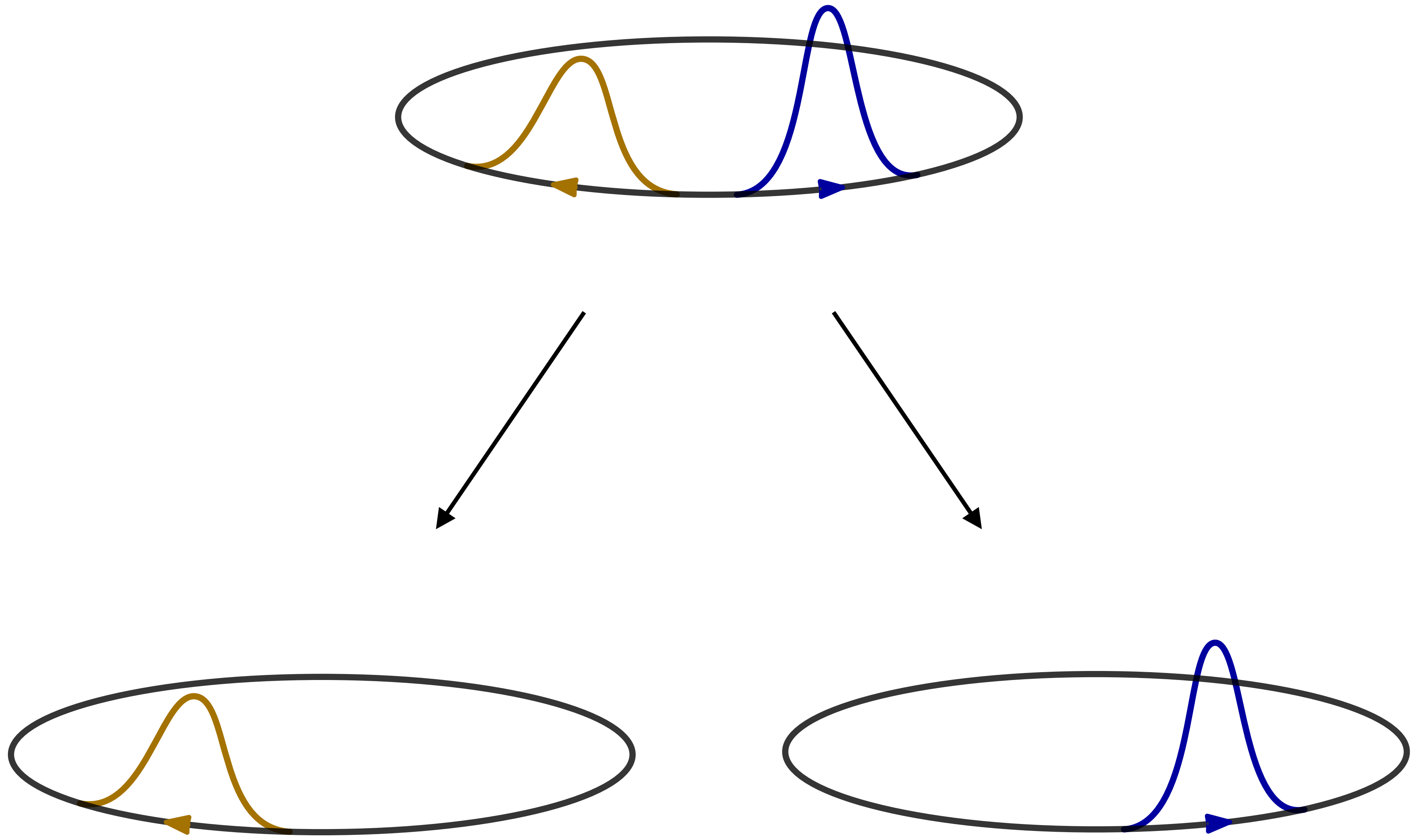
non-geometry :: generalities

The equation of motion for a closed string (in the simplest setting) is the wave equation in two dimensions

$$0 = \partial_\alpha \partial^\alpha X^\mu(\sigma)\,.$$

- The general solution splits into a left-moving and right-moving part

$$X^\mu(\sigma^0,\sigma^1) = X^\mu_L(\sigma^0+\sigma^1) + X^\mu_R(\sigma^0-\sigma^1)\,.$$



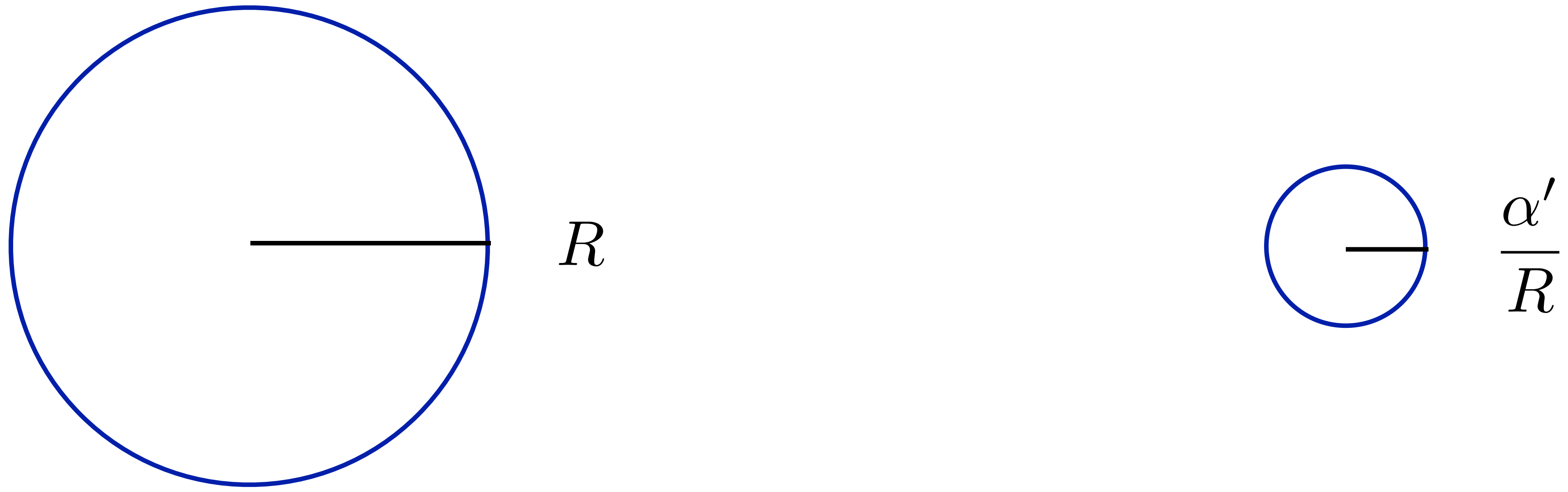
- If both parts *see* the same geometry, the space is geometric.
- If the two parts *see* different geometries, the space is non-geometric (but well-defined for a string).

outline

non-geometry :: t-duality I

T-duality ::

- Two string-theory compactifications on dual circles cannot be distinguished.



- The duality group for the circle is $\mathbb{Z}_2 \times \mathbb{Z}_2$.
- T-duality is a string-theory duality — not existing for point particles.

non-geometry :: t-duality II

A string-theory background (in the NS-NS sector) is characterized by a choice of

- metric $G_{\mu\nu}$,
- anti-symmetric two-form $B_{\mu\nu}$,
- dilaton Φ .

T-duality transformations act on (G, B, Φ) in a non-trivial way (Buscher rules).

For D-dimensional toroidal compactifications the duality group is $O(D, D; \mathbb{Z})$,

- which for $\mathcal{O} \in O(D, D; \mathbb{Z})$ is specified by $\mathcal{O}^T \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \mathcal{O} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$.

non-geometry :: duality group

The duality group $O(D, D; \mathbb{Z})$ contains the elements ::

- A-transformations ($\mathsf{A} \in GL(D, \mathbb{Z})$)

$$\mathcal{O}_{\mathsf{A}} = \begin{pmatrix} \mathsf{A}^{-1} & 0 \\ 0 & \mathsf{A}^{T} \end{pmatrix} \longrightarrow \text{diffeomorphisms}$$

- B-transformations (B_{ij} an anti-symmetric matrix)

$$\mathcal{O}_{\mathsf{B}} = \begin{pmatrix} \mathbb{1} & 0 \\ \mathsf{B} & \mathbb{1} \end{pmatrix} \longrightarrow \text{gauge transformations } B \to B + \alpha' \mathsf{B}$$

- β-transformations (β^{ij} an anti-symmetric matrix)

$$\mathcal{O}_{\beta} = \begin{pmatrix} \mathbb{1} & \beta \\ 0 & \mathbb{1} \end{pmatrix}$$

- factorized duality (E_{i} with only non-zero $E_{\mathsf{ii}} = 1$)

$$\mathcal{O}_{\pm\mathsf{i}} = \begin{pmatrix} \mathbb{1} - E_{\mathsf{i}} & \pm E_{\mathsf{i}} \\ \pm E_{\mathsf{i}} & \mathbb{1} - E_{\mathsf{i}} \end{pmatrix} \longrightarrow \text{T-duality transformations } G_{\mathsf{ii}} \to \frac{\alpha'^2}{G_{\mathsf{ii}}}$$

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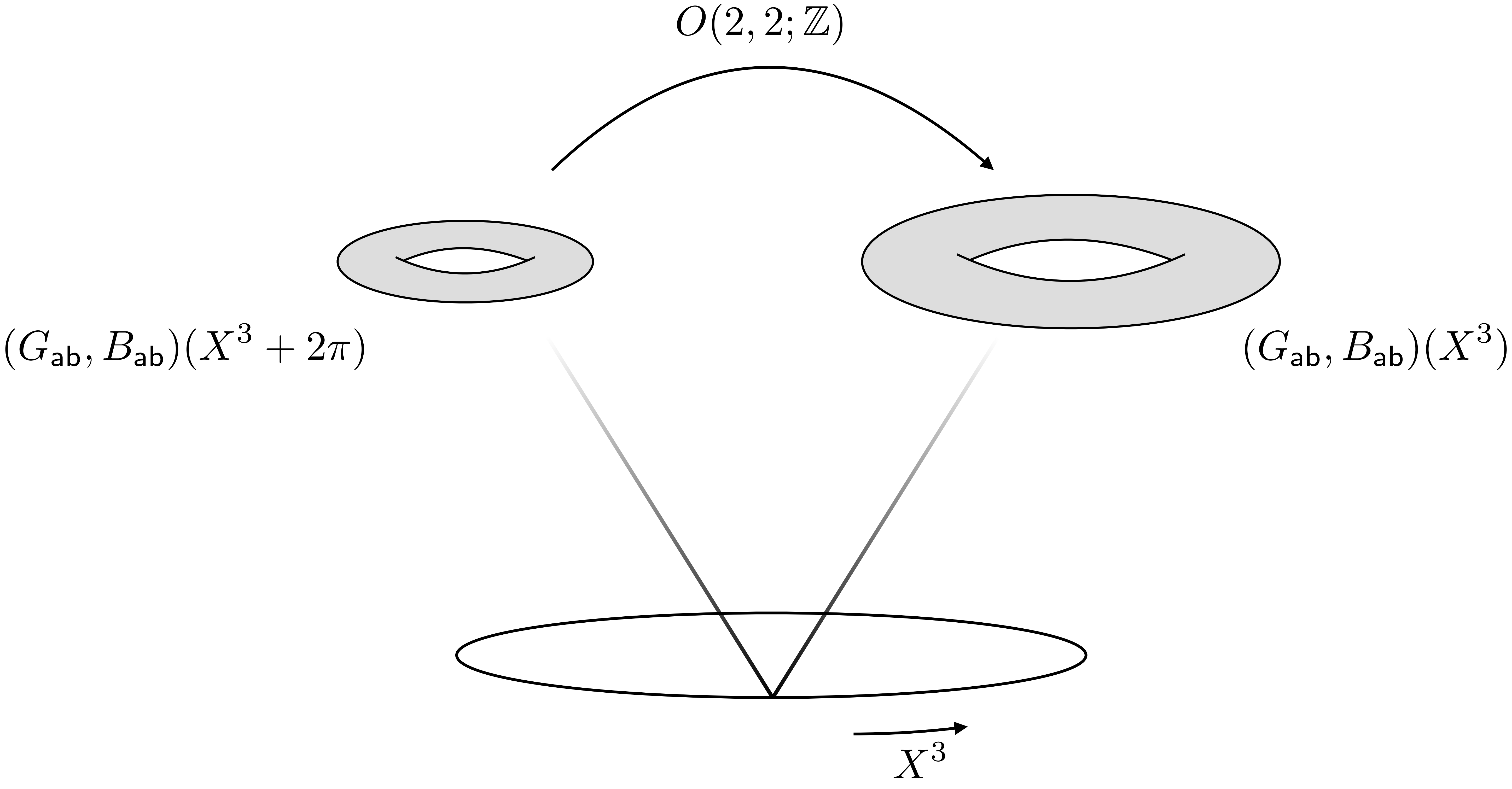
non-geometry :: fibrations I

The standard example for a non-geometric background is a $\mathbb{T}^2$-fibration over a circle.

$$G_{ij} = \begin{pmatrix} G_{\mathsf{ab}}(X^3) & 0 \\ 0 & R_3^2 \end{pmatrix}$$

$$B_{ij} = \begin{pmatrix} B_{\mathsf{ab}}(X^3) & 0 \\ 0 & 0 \end{pmatrix}$$



non-geometry :: fibrations II

The non-geometric background is part of a family of $\mathbb{T}^2$-fibrations ::

$$\begin{tikzcd}
\boxed{\mathbb{T}^3 \text{ with } H\text{-flux}} \arrow[loop above, "\mathcal{O}_{\mathsf{B}}"] \arrow[r, leftrightarrow, "\mathcal{O}_E"] & \boxed{\text{twisted } \mathbb{T}^3} \arrow[loop above, "\mathcal{O}_{\mathsf{A}}"] \arrow[r, leftrightarrow, "\mathcal{O}_E"] & \boxed{\text{T-fold}} \arrow[loop above, "\mathcal{O}_\beta"]
\end{tikzcd}$$

non-geometry :: three-torus with *h*-flux

A three-torus with *H*-flux is characterized as follows ::

1. Metric and *B*-field

$$G_{ij} = \begin{pmatrix} R_1^2 & 0 & 0 \\ 0 & R_2^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix} , \qquad B_{ij} = \begin{pmatrix} 0 & +\frac{\alpha'}{2\pi} h\, X^3 & 0 \\ -\frac{\alpha'}{2\pi} h\, X^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad h \in \mathbb{Z} \,.$$

2. The background is well-defined under $X^3 \to X^3 + 2\pi$ using a gauge transformation.

3. The *H*-flux $H = dB$ can be expressed in a vielbein basis as

$$H = \frac{\alpha'}{2\pi} \, \frac{h}{R_1 \, R_2 \, R_3} \, e^1 \wedge e^2 \wedge e^3 \, , \qquad H_{123} = \frac{\alpha'}{2\pi} \, \frac{h}{R_1 \, R_2 \, R_3} \, .$$

non-geometry :: twisted three-torus

After a T-duality along X^1 one obtains a twisted three-torus ::

1. Metric and *B*-field

$$G_{ij} = \begin{pmatrix} \frac{\alpha'^2}{R_1^2} & -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & 0 \\ -\frac{\alpha'^2}{R_1^2}\frac{h}{2\pi}X^3 & R_2^2 + \frac{\alpha'^2}{R_1^2}\left[\frac{h}{2\pi}X^3\right]^2 & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad B_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad h \in \mathbb{Z}\,.$$

2. The background is well-defined under $X^3 \to X^3 + 2\pi$ using a diffeomorphism.

3. A geometric *f*-flux is defined via a vielbein basis as

$$de^a = \frac{1}{2} f_{bc}{}^a \, e^b \wedge e^c \,, \qquad\qquad f_{23}{}^1 = \frac{\alpha'}{2\pi}\,\frac{h}{R_1\,R_2\,R_3}\,.$$

Kachru, Schulz, Tripathy, Trivedi - 2002

A second T-duality along X^2 gives the T-fold background ::

1. Metric and *B*-field

$$G_{ij} = \begin{pmatrix} \frac{R_2^2}{\rho} & 0 & 0 \\ 0 & \frac{R_1^2}{\rho} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad B_{ij} = \frac{1}{\rho} \begin{pmatrix} 0 & -\frac{\alpha'}{2\pi} h\, X^3 & 0 \\ +\frac{\alpha'}{2\pi} h\, X^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{aligned} &\rho = \tfrac{R_1^2 R_2^2}{\alpha'^2} + \left[\tfrac{h}{2\pi} X^3\right]^2 , \\ &h \in \mathbb{Z} . \end{aligned}$$

2. The background is well-defined under $X^3 \to X^3 + 2\pi$ using a β-transformation.

3. A non-geometric Q-flux is defined via a vielbein basis and $(G-B)^{-1} = g - \beta$ as

$$Q_i{}^{jk} = \partial_i \beta^{jk} , \qquad Q_3{}^{12} = \frac{\alpha'}{2\pi} \frac{h}{R_1 R_2 R_3} .$$

Hull - 2004

non-geometry :: fluxes

The above family of backgrounds is characterized by (non-)geometric **fluxes** ::

$$\begin{tikzcd}
\boxed{\mathbb{T}^3 \text{ with } H\text{-flux}} \arrow[r, leftrightarrow, "\mathsf{T}_1"] \arrow[d] & \boxed{\text{twisted } \mathbb{T}^3} \arrow[r, leftrightarrow, "\mathsf{T}_2"] \arrow[d] & \boxed{\text{T-fold}} \arrow[d] \\
H_{123} & f_{23}{}^{1} & Q_{3}{}^{12}
\end{tikzcd}$$

non-geometry :: fluxes

The above family of backgrounds is characterized by (non-)geometric **fluxes** ::

$$\begin{tikzcd}
\boxed{\mathbb{T}^3 \text{ with } H\text{-flux}} \arrow[r, leftrightarrow, "\mathsf{T}_1"] \arrow[d] & \boxed{\text{twisted } \mathbb{T}^3} \arrow[r, leftrightarrow, "\mathsf{T}_2"] \arrow[d] & \boxed{\text{T-fold}} \arrow[r, leftrightarrow, "\mathsf{T}_3"] \arrow[d] & \boxed{R\text{-space}} \arrow[d] \\
H_{123} & f_{23}{}^{1} & Q_{3}{}^{12} & R^{123}
\end{tikzcd}$$

Shelton, Taylor, Wecht - 2005

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non-geometry :: summary

Summary ::

- T-duality is a string-theory duality — not present for point-particle theories.
- Non-geometric backgrounds are well-defined using T-duality.
- Non-triviality of the background is encoded in (non-)geometric fluxes.

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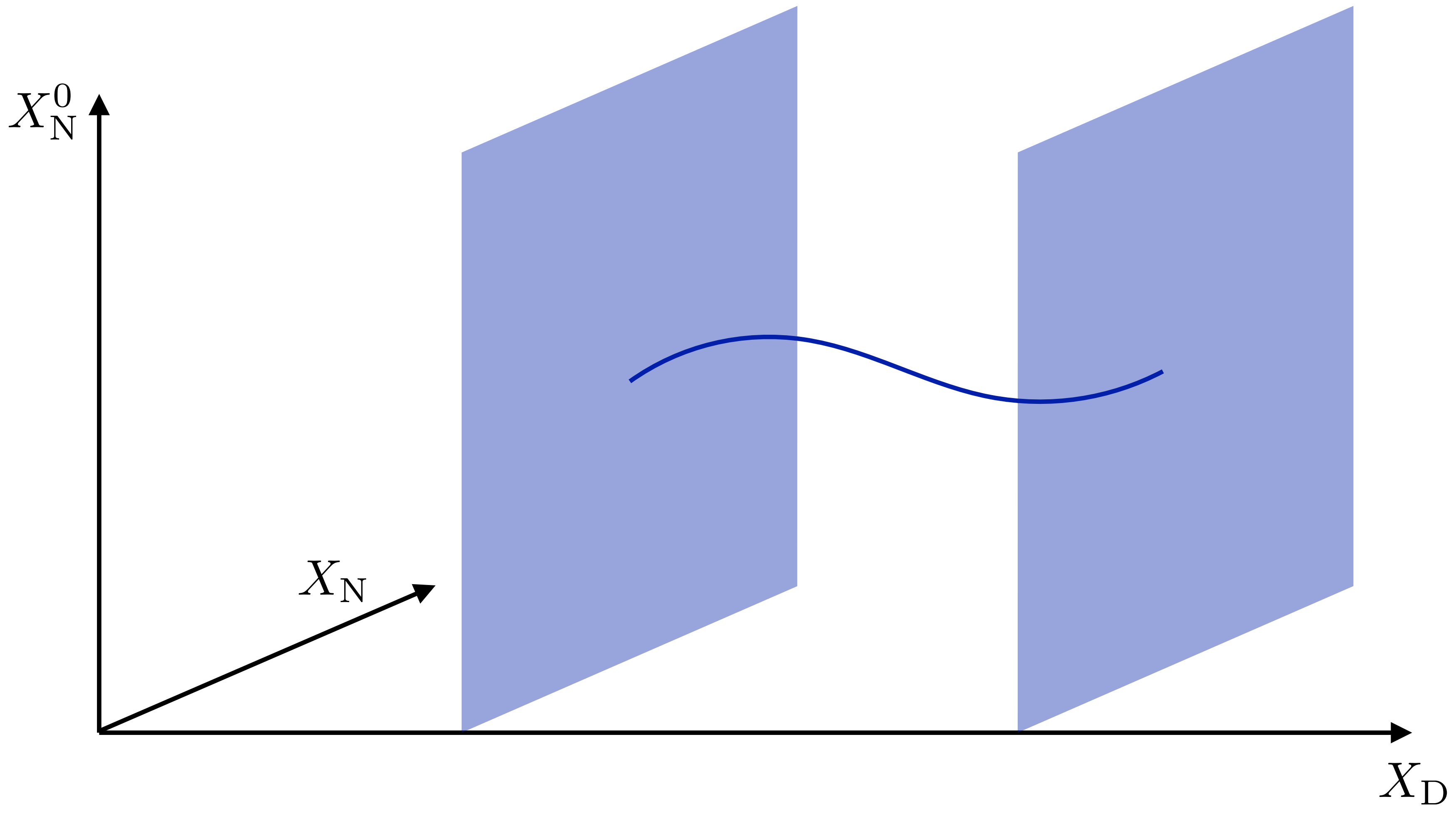
freed-witten :: d-branes I

Classically, D-branes are hyper-surfaces where open strings can end.

A D*p*-brane has *(p+1)* directions with Neumann boundary conditions and *(D-p-1)* directions with Dirichlet conditions.



A T-duality transformation interchanges Neumann and Dirichlet boundary conditions ::

- T-duality perpendicular to the D-brane :: D*p*-brane ⟶ D*(p+1)*-brane
- T-duality along the D-brane :: D*p*-brane ⟶ D*(p-1)* -brane

freed-witten :: d-branes II

In the quantized theory, D-branes are dynamical non-perturbative objects. Their action takes the form (bosonic part)

$$\mathcal{S}_{\mathrm{D}p} = -T_p \int_\Gamma d^{p+1}\xi \, e^{-\phi} \sqrt{-\det\big(G_{ab} + 2\pi\alpha' \mathcal{F}_{ab}\big)} - \mu_p \int_\Gamma \mathrm{ch}\left(2\pi\alpha'\mathcal{F}\right) \wedge \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} \wedge \bigoplus_q C_q \Bigg|_{p+1} ,$$

with

▪ submanifold wrapped by D*p*-brane	$\Gamma\,,$
▪ coordinates on the D*p*-brane	$\xi^a\,,\; a = 0, \dots p\,,$
▪ pulled-back metric	$G_{ab}\,,$
▪ open-string field strength	$2\pi\alpha'\mathcal{F} = B + 2\pi\alpha' F\,,$
▪ R-R *q*-form potentials	$C_q\,.$

D-branes couple to the R-R gauge potentials C_p and therefore carry the R-R charge

$$\mathcal{Q}_{\mathrm{D}p} = \mathrm{ch}\left(2\pi\alpha'\mathcal{F}\right) \wedge \sqrt{\frac{\hat{\mathcal{A}}(\mathcal{R}_T)}{\hat{\mathcal{A}}(\mathcal{R}_N)}} \wedge [\Gamma_{\mathrm{D}p}]\,.$$

Minasian, Moore - 1997

freed-witten :: orientifold planes

Orientifold p-planes are fixed loci of the orientifold projection in type I string theories. Their action reads (with $Q_p = -2^{p-4}$)

$$\mathcal{S}_{\mathrm{O}p} = -Q_p \, \mu_p \int_\Gamma \sqrt{\frac{\mathcal{L}(\mathcal{R}_T/4)}{\mathcal{L}(\mathcal{R}_N/4)}} \wedge \bigoplus_q C_q \, \bigg|_{p+1} \, .$$

Orientifold planes are charged under the R-R gauge symmetries and the charge is expressed as

$$\mathcal{Q}_{\mathrm{O}p} = Q_p \sqrt{\frac{\mathcal{L}(\mathcal{R}_T/4)}{\mathcal{L}(\mathcal{R}_N/4)}} \wedge [\Gamma_{\mathrm{O}p}] \, .$$

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freed-witten :: fluxes as operators

The components of the NS-NS fluxes (in a local basis) suggest that fluxes should be interpreted as operators acting on *p*-forms ::

components	operator	action on *p*-forms
H_{ijk}	$H = \frac{1}{6}\, H_{ijk}\, dx^i \wedge dx^j \wedge dx^k$	p-form $\rightarrow (p+3)$-form
$f_{ij}{}^k$	$f = \frac{1}{2}\, f_{ij}{}^k\, dx^i \wedge dx^j \wedge \iota_k$	p-form $\rightarrow (p+3)$-form
$Q_i{}^{jk}$	$Q = \frac{1}{2}\, Q_i{}^{jk}\, dx^i \wedge \iota_j \wedge \iota_k$	p-form $\rightarrow (p-1)$-form
R^{ijk}	$R = \frac{1}{6}\, R^{ijk}\, \iota_i \wedge \iota_j \wedge \iota_k$	p-form $\rightarrow (p-3)$-form

It is natural to combine these fluxes into a so-called twisted differential acting on multiforms

$$\mathcal{D} = d - H\wedge\, - f\circ\, - Q\bullet\, - R\llcorner\,.$$

Aldazabal, Camara, Font, Ibanez - 2006
Villadoro, Zwirner - 2006
Shelton, Taylor, Wecht - 2006

freed-witten :: bianchi identities I

In a background with D-branes and O-planes as sources, the Bianchi identities for the R-R potentials C_p take the form ::

$$\mathcal{D}F = \sum_{\mathrm{D}p} \mathcal{Q}_{\mathrm{D}p} + \sum_{\mathrm{O}p} \mathcal{Q}_{\mathrm{O}p} \,,$$

R-R field strength $F = \mathcal{D}C \,,$

R-R potentials $C = \sum_p C_p \,.$

The integrated expressions are known as the tadpole cancellation conditions ::

- These are important consistency conditions in string theory (cancellation of anomalies, absence of divergencies).
- They connect the open- and closed-string sectors, and are essential in string-theory model building.

freed-witten :: bianchi identities II

In the absence of NS-NS sources, the Bianchi identities in the NS-NS sector can be expressed as

$$\mathcal{D}^2 = 0\,.$$

- It would be desirable to develop a cohomology theory for $\mathcal{D}$.
- In the presence of NS-NS sources (NS5-branes, KK-monopoles, 5^2_2-branes), the Bianchi identities are modified.

In a local basis, the above conditions can be evaluated to be of the following form

$$\begin{aligned}
0 &= H_{m[\underline{ab}} f^m{}_{\underline{cd}]}\;, \\
0 &= f^m{}_{[\underline{bc}}\, f^d{}_{\underline{a}]m} + H_{m[\underline{ab}}\, Q_{\underline{c}]}{}^{md}\,, \\
0 &= f^m{}_{[\underline{ab}]}\, Q_m{}^{[\underline{cd}]} - 4\, f^{[\underline{c}}{}_{m[\underline{a}}\, Q_{\underline{b}]}{}^{\underline{d}]m} + H_{mab}\, R^{mcd}\,, \\
0 &= Q_m{}^{[\underline{bc}}\, Q_d{}^{\underline{a}]m} + R^{m[\underline{ab}}\, f^{\underline{c}]}{}_{md}\,, \\
0 &= R^{m[\underline{ab}}\, Q_m{}^{\underline{cd}]}\;, \\
&\ldots
\end{aligned}$$

Shelton, Taylor, Wecht - 2006
Robins, Wrase - 2007

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freed-witten :: anomaly I

The **Freed-Witten anomaly** arises for D-branes in backgrounds with *H*-flux. It is cancelled when

$$[H]\big|_{\text{D-brane}} = 0\,.$$

Freed, Witten - 1999

This condition can be re-written and generalized ::

- For a D-brane wrapping a submanifold $\Gamma_{\mathrm{D}p}$ one finds (always in cohomology)

  $$H \wedge [\Gamma_{\mathrm{D}p}] = 0\,.$$

- Applying T-duality transformations one obtains $f \circ [\Gamma_{\mathrm{D}p}] = 0$, which can be generalized as

  $$\mathcal{D}\,[\Gamma_{\mathrm{D}p}] = 0\,.$$

Camara, Font, Ibanez - 2005
Villadoro, Zwirner - 2006

- Including open-string fluxes $\mathcal{F}$ and requiring invariance of the superpotential under gauge transformations, one obtains

  $$\mathcal{D}\Big[\mathrm{ch}\,(2\pi\alpha'\mathcal{F}) \wedge [\Gamma_{\mathrm{D}p}]\Big] = 0\,.$$

Blumenhagen, Font, Fuchs, Herschmann, Plauschinn, Sekiguchi, Wolf - 2015

freed-witten :: anomaly II

The most natural generalization of the Freed-Witten condition is to include curvature terms as

$$0 = \mathcal{D}\,\mathcal{Q}_{\mathrm{D}p} = \mathcal{D}\left[\mathrm{ch}\,(2\pi\alpha'\mathcal{F}) \wedge \sqrt{\frac{\hat{A}(\mathcal{R}_T)}{\hat{A}(\mathcal{R}_N)}} \wedge [\Gamma_{\mathrm{D}p}]\right].$$

Since D-branes and orientifold planes are on similar footing, it is natural to require

$$0 = \mathcal{D}\,\mathcal{Q}_{\mathrm{O}p} = \mathcal{D}\left[\sqrt{\frac{\mathcal{L}(\mathcal{R}_T/4)}{\mathcal{L}(\mathcal{R}_N/4)}} \wedge [\Gamma_{\mathrm{O}p}]\right].$$

The tadpole cancellation condition then becomes a statement in $\mathcal{D}$ cohomology

$$\mathcal{D}F = \sum_{\mathrm{D}p} \mathcal{Q}_{\mathrm{D}p} + \sum_{\mathrm{O}p} \mathcal{Q}_{\mathrm{O}p}\,.$$

$$\uparrow \qquad\qquad\qquad\qquad \nwarrow \quad \nearrow$$

$\mathcal{D}$-exact $\qquad\qquad$ $\mathcal{D}$-closed

outline

Summary ::

- String-theory backgrounds with NS-NS fluxes can be described using a twisted differential $\mathcal{D}$.
- D-branes in flux backgrounds are subject to consistency conditions such as tadpole cancellation or the Freed-Witten anomaly cancellation.
- Generalizations of the Freed-Witten anomaly using $\mathcal{D}$ have been proposed — a microscopic understanding would be desirable.

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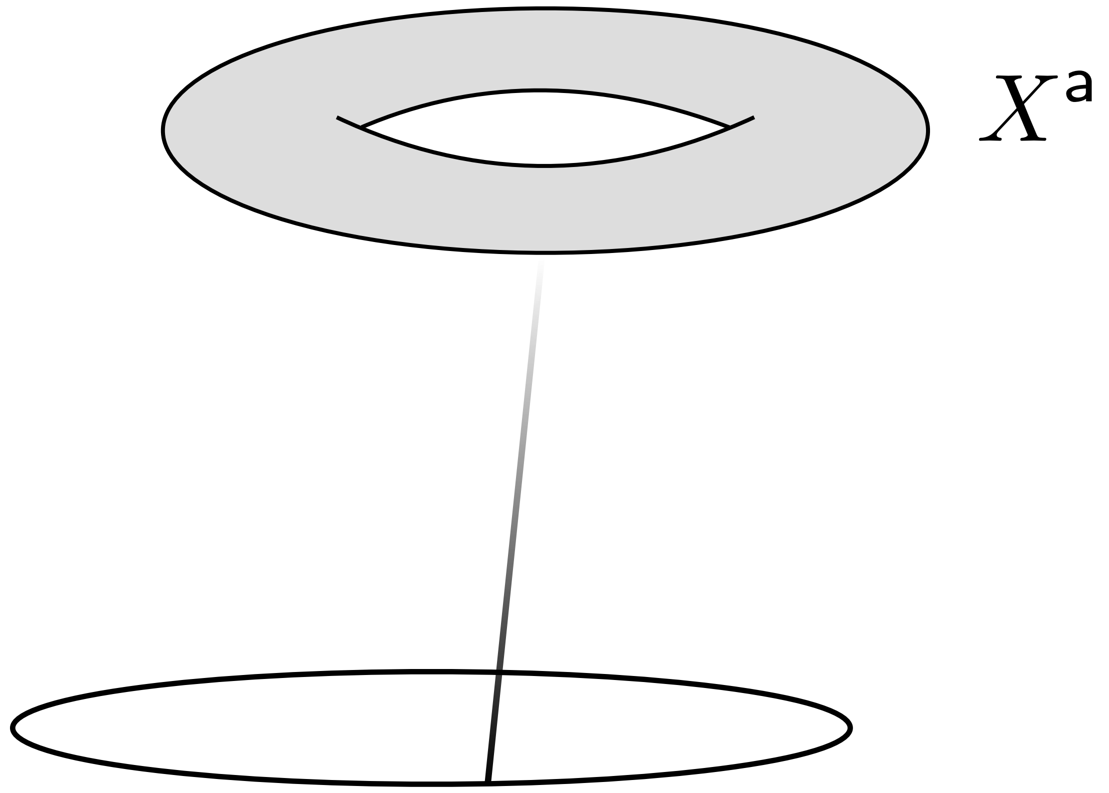
open-strings :: closed-string non-commutativity I



Consider a torus-fibration over a circle. For small fluxes $\Theta \ll 1$, the equations of motion for the closed-string fields read

$$0 = \partial_+\partial_- X^{\mathsf{a}}(\tau,\sigma) + \mathcal{O}(\Theta)\,.$$

The solutions split into left- and right-moving parts

$$X^{\mathsf{a}}(\tau,\sigma) = X^{\mathsf{a}}_L(\tau+\sigma) + X^{\mathsf{a}}_R(\tau-\sigma)\,.$$

In general, the commutation relations between the left- and right-moving sectors take the form

$$\big[X^{\mathsf{a}}_L, X^{\mathsf{b}}_L\big] = \tfrac{i}{2}\,\Theta^{\mathsf{ab}}_1\,, \qquad \big[X^{\mathsf{a}}_L, X^{\mathsf{b}}_R\big] = 0\,, \qquad \big[X^{\mathsf{a}}_R, X^{\mathsf{b}}_R\big] = \tfrac{i}{2}\,\Theta^{\mathsf{ab}}_2\,.$$

For a commuting background one has $\Theta^{\mathsf{ab}}_1 = -\Theta^{\mathsf{ab}}_2 = \Theta^{\mathsf{ab}}$, which leads to

$$\big[X^{\mathsf{a}}, X^{\mathsf{b}}\big] = \tfrac{i}{2}\big(\Theta^{\mathsf{ab}} - \Theta^{\mathsf{ab}}\big) = 0\,.$$

open-strings :: closed-string non-commutativity II

T-duality transformations change the overall sign in the right-moving sector

$$(\, X_L^{\hat{\mathsf{a}}} \,,\, X_R^{\hat{\mathsf{a}}} \,) \quad \longrightarrow \quad (\, +X_L^{\hat{\mathsf{a}}} \,,\, -X_R^{\hat{\mathsf{a}}} \,)\,.$$

The commutator then becomes

$$\big[\tilde{X}^{\hat{\mathsf{a}}}, X^{\mathsf{b}}\big] = \tfrac{i}{2}\big(\Theta^{\hat{\mathsf{a}}\mathsf{b}} + \Theta^{\hat{\mathsf{a}}\mathsf{b}}\big) = i\,\Theta^{\hat{\mathsf{a}}\mathsf{b}}\,.$$

A more careful analysis for torus fibrations over a circle leads to ::

$$\big[X^{\mathsf{a}}, X^{\mathsf{b}}\big] = i\,Q_k{}^{\mathsf{ab}}X^k\,.$$

Lüst - 2010
Andriot, Larfors, Lüst, Patalong - 2012

Non-commutative behaviour in *Q*-flux backgrounds has also been found in different ways.

Mathai, Rosenberg - 2004

open-strings :: closed-string non-associativity

Consider the equal-time Jacobiator of three closed-string fields

$$\left[X^i, X^j, X^k\right] := \lim_{\sigma_i \to \sigma} \left[[X^i(\tau,\sigma_1), Xj(\tau,\sigma_2)], X^k(\tau,\sigma_3)\right] + \text{cyclic}\,.$$

This Jacobiator has been evaluated for different settings ::

- For a three-sphere with flux (in a particular limit), one finds

  $$\left[X^i, X^j, X^k\right] = R^{ijk}\,.$$

Blumenhagen, Plauschinn - 2010

- For torus-fibrations ($\mathbb{T}^2_{(12)}$ over $S^1_{(3)}$) one obtains schematically

  Q-flux background $\quad \left[X^1, X^2, X^3\right] = \left[iQ_3{}^{12}X^3, X^3\right] = 0\,,$

  R-flux background $\quad \left[X^1, X^2, X^3\right] = \left[i\,R^{123}\tilde{X}_3, X^3\right] = R^{123}\,.$

Andriot, Hohm, Larfors, Lüst, Patalong - 2012

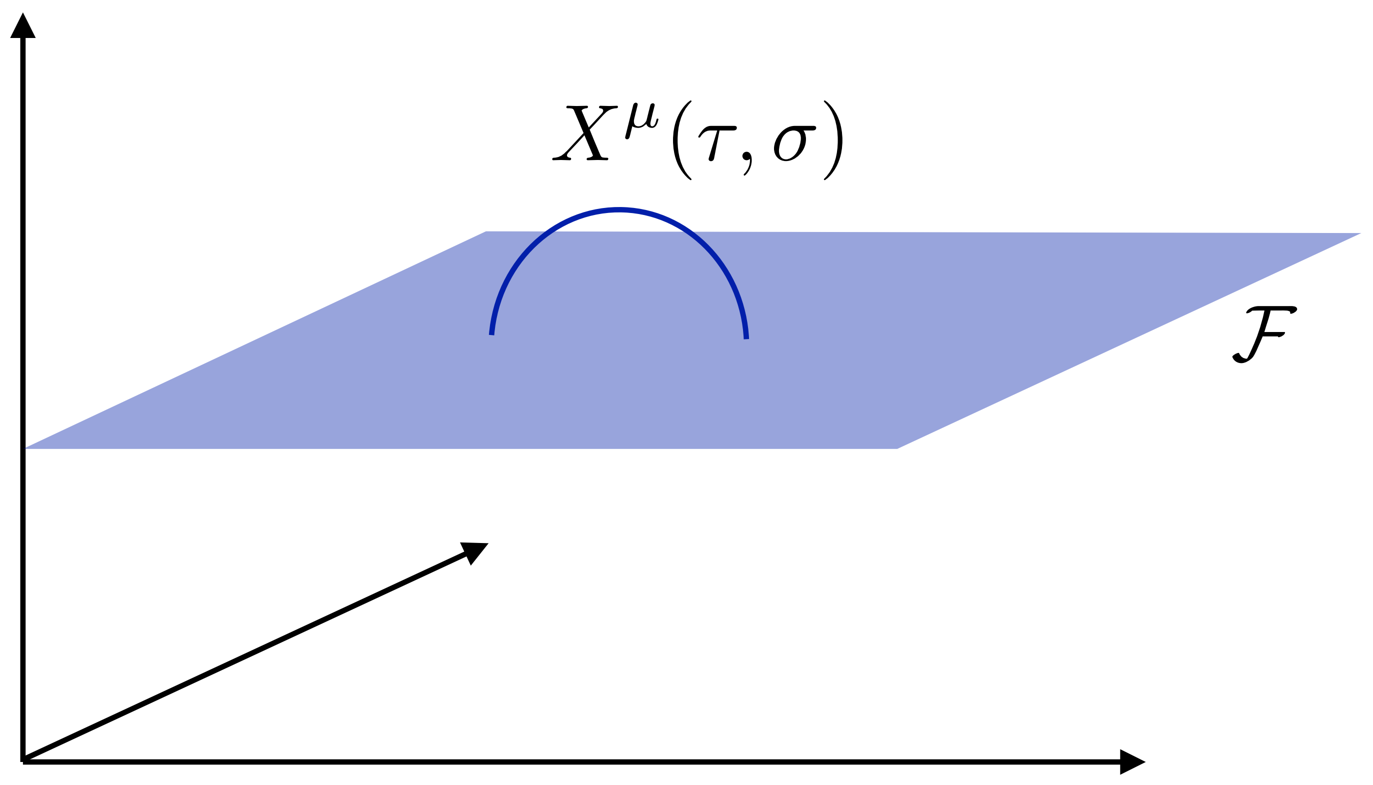
- Non-associative behaviour in *R*-flux backgrounds has also been found in different ways.

Bouwknegt, Hannabuss, Mathai - 2004

outline

open-strings :: seiberg-witten I

Consider a D-brane in a background with constant B-field.



The boundary conditions for open-string fields X^{μ} on the two-dimensional world-sheet are (with $\mathcal{F} = F - B$)

Dirichlet $$0 = \big(dX^{\mu}\big)_{\rm tan} \,,$$

Neumann $$0 = \big(dX^{\mu}\big)_{\rm norm} + \mathcal{F}^{\mu}{}_{\nu}\big(dX^{\nu}\big)_{\rm tan},$$

$$(dX^{\mu})_{\rm tan} \equiv t^{\alpha}\,\partial_{\alpha}X^{\mu}\,ds|_{\partial\Sigma}\,,$$

$$(dX^{\mu})_{\rm norm} \equiv n^{\alpha}\,\partial_{\alpha}X^{\mu}\,ds|_{\partial\Sigma}\,.$$

The mode expansion for an open string with Neumann-Neumann boundary conditions takes the form

$$\begin{aligned} X^{\mu}_{\rm NN}(\tau,\sigma) &= x^{\mu}_{0} + \frac{2\pi\alpha'}{\ell_{\rm s}}\big(p^{\mu}\,\tau - \mathcal{F}^{\mu}{}_{\nu}\,p^{\nu}\,\sigma\big) \\ &\quad + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,e^{-i\frac{n\,\pi\,\tau}{\ell_{\rm s}}}\left[\alpha^{\mu}_{n}\cos\left(\frac{n\,\pi\,\sigma}{\ell_{\rm s}}\right) - i\,\mathcal{F}^{\mu}{}_{\nu}\,\alpha^{\nu}_{n}\sin\left(\frac{n\,\pi\,\sigma}{\ell_{\rm s}}\right)\right]. \end{aligned}$$

open-strings :: seiberg-witten II

The commutator of two open-string coordinates is computed explicitly (with $M=(G-\mathcal{F})G^{-1}(G+\mathcal{F})$) ::

$$\bigl[X^i_{\mathrm{NN}}(\tau,\sigma),X^j_{\mathrm{NN}}(\tau,\sigma')\bigr]=\begin{cases}+2\pi i\,\alpha'\bigl(M^{-1}\mathcal{F}\bigr)^{ij} & \sigma=\sigma'=0\,,\\ -2\pi i\,\alpha'\bigl(M^{-1}\mathcal{F}\bigr)^{ij} & \sigma=\sigma'=\ell_{\mathrm{s}}\,,\\ 0 & \text{else}\,.\end{cases}$$

Chu, Ho - 1998

The endpoints of open strings realize a non-commutative gauge theory on the D-brane.

Seiberg, Witten - 1998

open-strings :: non-commutativity

For backgrounds with non-constant *B*-field, the Freed-Witten anomaly condition applies ::

- D3-brane with *H*-flux on its world-volume is forbidden.
- D2-brane in a *Q*-flux background is allowed.
- D3-brane in a *R*-flux background is allowed.

Consider now a D2-brane along directions (X^1, X^2) in the T-fold background specified by

$$G_{ij} = \begin{pmatrix} \frac{R_2^2}{\rho} & 0 & 0 \\ 0 & \frac{R_1^2}{\rho} & 0 \\ 0 & 0 & R_3^2 \end{pmatrix}, \qquad B_{ij} = \frac{1}{\rho} \begin{pmatrix} 0 & -\frac{\alpha'}{2\pi} N X^3 & 0 \\ +\frac{\alpha'}{2\pi} N X^3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\rho = \frac{R_1^2 R_2^2}{\alpha'^2} + \left[\frac{N}{2\pi} X^3\right]^2,$$

$$N \in \mathbb{Z}.$$

Quantizing the open string fiberwise and using the previous formula, the following commutator (on the D2-brane) can be determined

$$\left[X^1_{\mathrm{NN}}, X^2_{\mathrm{NN}}\right] = \mp i N X^3 .$$

outline

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2. non-geometric backgrounds
3. d-branes & freed-witten anomaly
4. open-string non-associativity
    a) review closed string
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open-strings :: open-string commutator

To compute a three-product for open strings, the following **commutators** are needed ::

$$\begin{aligned}\big[X^3_{\rm NN}\;,X^3_{\rm NN}\;\big] &= 0\,,\\ \big[X^3_{\rm NN}\;,X_{3\rm DD}\big] &= \pi\, i\,\alpha'\,,\\ \big[X_{3\rm DD}\,,X_{3\rm DD}\big] &= 0\,,\end{aligned}$$

$$\big[X^3_{\rm NN}\,,\tilde X_3\big] = \pi\, i\,\alpha'\,.$$

Performing a **T-duality** along X^3 for the D2-brane in the T-fold background gives

- a D3-brane in an *R*-flux background (satisfies the Freed-Witten condition),
- with commutator

$$\big[X^1_{\rm NN}, X^2_{\rm NN}\big] = i\,R^{123}\,\tilde X_3\,,$$

- and Jacobiator

$$\big[X^1_{\rm NN}, X^2_{\rm NN}, X^3_{\rm NN}\big] = \pi\,\alpha' R^{123}\,.$$

Lüst, Malek, Plauschinn, Syväri - 2019

outline

open-strings :: summary

Summary ::

- String-theory backgrounds with non-geometric fluxes give rise to non-commutative or non-associative behaviour in the closed-string sector.
- For open strings, non-commutativity arises from a constant B-field.
- Open strings show signs of non-associativity related to non-geometric R-flux.

outline

1. motivation
2. non-geometric backgrounds
3. d-branes & freed-witten anomaly
4. open-string non-associativity
5. conclusions

conclusions

Summary ::

- String-theory can be well-defined on non-geometric backgrounds.
- Strong consistency conditions have to be satisfied, such as the Freed-Witten anomaly cancellation condition in the presence of D-branes.
- Open strings in a non-geometric *R*-flux backgrounds show signs of non-associative behaviour.