Graded Geometry and Gravity

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Outline

- Interaction via deformation
- Graded spacetime mechanics
- Generalized geometry
- Graded geometry and gravity

Interaction via deformation

example: relativistic particle in einbein formalism

$$S = \int d\tau \left(\frac{1}{2e} g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu - \frac{1}{2} e m^2 + A_\mu(x)\dot{x}^\mu \right) \rightsquigarrow p_\mu = \frac{1}{e} g_{\mu\nu}\dot{x}^\nu + A_\mu$$

$$S_H = \int p_\mu dx^\mu - \frac{1}{2} e \left((p_\mu - A_\mu)^2 + m^2 \right) d\tau$$

p_μ: canonical momentum

$$\boxed{S_H = \int (p_\mu + A_\mu)\, dx^\mu - \frac{1}{2} e \left(p_\mu^2 + m^2 \right) d\tau}$$

p_μ: physical momentum

$$\omega' = d\big(p_\mu + A_\mu\big) \wedge dx^\mu \quad \rightsquigarrow$$

$$\boxed{\{p_\mu, p_\nu\}' = F_{\mu\nu},\ \{x^\mu, p_\nu\}' = \delta^\mu_\nu,\ \{x^\mu, x^\nu\}' = 0}$$

$$\boxed{\{p_\lambda, \{p_\mu, p_\nu\}'\}' + \text{cycl.} = (dF)_{\lambda\mu\nu} = (*j_m)_{\lambda\mu\nu}}$$

$\leftarrow$ magnetic 4-current

magnetic sources $\Leftrightarrow$ non-associativity

Interaction via deformation

gauge field $A \rightsquigarrow$ deformation ω' ... *vice versa* $\omega' \rightsquigarrow A$:

Moser's lemma

Let $\omega_s = \omega + sF$, with ω_s symplectic for $s \in [0,1]$.

note: $d\omega_s = 0 \;\Rightarrow dF = 0 \;\Rightarrow$ locally $F = dA$

$\omega' \equiv \omega_1$ and ω are related by a change of phase space coordinates generated by the flow of a vector field V_s defined up to gauge transformations by the gauge field $i_{V_s}\omega_s = A$, i.e. $V_s = \theta_s(A, -)$.

Proof: $\mathcal{L}_{V_s}\omega_s = i_{V_s} d\omega_s + d\, i_{V_s}\omega_s = 0 + dA = \frac{d}{ds}\omega_s$. Moser 1965

Example: For $\omega = dp_i \wedge dx^i$ and $A = A_i(x)dx^i$: $V_s = A_i(x)\frac{\partial}{\partial p_i}$

Quantum version of the lemma: Jurco, PS, Wess 2000-2002

Interaction via deformation

Quantization

- canonical? depends... (✓)
- deformation quantization ✓
- path integral ✓

Deformed CCR:

$$[p_\mu, p_\nu] = i\hbar F_{\mu\nu}, \quad [x^\mu, p_\nu] = i\hbar\delta^\mu_\nu, \quad [x^\mu, x^\nu] = 0$$

Let $\mathbf{p} = \gamma^\mu p_\mu$ and $H = \frac{1}{2}\mathbf{p}^2 \rightsquigarrow$ correct coupling of fields to spin

$$H = \frac{1}{8}\big(\{\gamma^\mu, \gamma^\nu\}\{p_\mu, p_\nu\} + [\gamma^\mu, \gamma^\nu][p_\mu, p_\nu]\big) = \frac{1}{2}p^2 - \frac{i\hbar}{2}S^{\mu\nu}F_{\mu\nu}$$

Lorentz-Heisenberg equations of motion (ignoring spin)

$$\dot{p}_\mu = \frac{i}{\hbar}[H, p_\mu] = \frac{1}{2}(F_{\mu\nu}\dot{x}^\nu + \dot{x}^\nu F_{\mu\nu}) \quad \text{with} \quad \dot{x}^\nu = \frac{i}{\hbar}[H, x^\nu] = p^\nu$$

this formalism allows $dF \neq 0$: magnetic sources, non-associativity

Interaction via deformation

local non-associativity: $\frac{1}{3}[p_\lambda,[p_\mu,p_\nu]]\,dx^\lambda dx^\mu dx^\nu = \hbar^2 dF = \hbar^2 *j_m$

$j_m \neq 0 \Leftrightarrow$ no operator representation of the p_μ!

spacetime translations are still generated by p_μ, but magnetic flux Φ_m leads to path-dependence with phase $e^{i\phi}$; where $\phi = iq_e\Phi_m/\hbar$

globally:

$$\Phi_m = \int_S F = \int_{\partial S} A \qquad \leftrightarrow \text{ non-commutativity}$$

$$\Phi_m = \int_{\partial V} F = \int_V dF = \int_V *j_m = q_m \qquad \leftrightarrow \text{ non-associativity}$$

global associativity requires $\phi \in 2\pi\mathbb{Z} \;\Rightarrow\; \boxed{\frac{q_e q_m}{2\pi\hbar} \in \mathbb{Z}}$ Dirac quantization

non-relativistic version: Jackiw 1985, 2002

Graded spacetime mechanics

Now try to do the same for gravity! Deformation maybe fine for curvature $R_{\mu\nu}$, however, the metric $g_{\mu\nu}$ is symmetric but $\{,\}$ is not.

- consider derived brackets

$$g^{\mu\nu} \sim \{\{x^\mu, H\}, x^\nu\}\ , \qquad \{H, H\} = 0$$

- use graded geometry, i.e. odd variables and/or odd brackets
- $\rightsquigarrow$ algebraic approach to the geodesic equation, connections, curvature, etc. Properties like metricity follow from associativity. Local inertial coordinates are reinterpreted as Darboux charts
- the classical formulation requires graded variables ($\sim$ differentials), quantization leads to γ-matrices and Clifford algebras

$$\begin{aligned}
\text{classical} \quad &\leftrightarrow \quad \text{quantum} \\
\theta^\mu \quad &\leftrightarrow \quad \gamma^\mu \\
\theta^\mu\theta^\nu = -\theta^\nu\theta^\mu \quad &\leftrightarrow \quad \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]_- \\
\tfrac{1}{2}\{\theta^\mu, \theta^\nu\} = g^{\mu\nu} \quad &\leftrightarrow \quad \tfrac{1}{2}[\gamma^\mu, \gamma^\nu]_+ = g^{\mu\nu}
\end{aligned}$$

Graded spacetime mechanics

Graded Poisson algebra

$$\{\underset{a}{\theta^\mu}, \underset{a}{\theta^\nu}\} = 2\underset{0}{g^{\mu\nu}}(x) \qquad \{\underset{c}{p_\mu}, \underset{0}{x^\nu}\} = \underset{0}{\delta^\nu_\mu} \qquad \{p_\mu, f(x)\} = \partial_\mu f(x)$$

Since $g^{\mu\nu}(x)$ has degree 0, the Poisson bracket must have degree $b = -2a$ for θ^μ of degree a, i.e. it is an even bracket.

Since $g^{\mu\nu}(x)$ is symmetric, we must have $-(-1)^{b+a^2} \overset{!}{=} +1$, i.e. a is odd.

wlog: $\{,\}$ is of degree $b = -2$, θ^μ are Grassmann variables of degree 1, $\theta^\mu\theta^\nu = -\theta^\nu\theta^\mu$, and the momenta p_μ have degree $c = -b = 2$

$\Leftrightarrow$ a metric structur on TM and natural symplectic structure on T^*M, shifted in degree and combined into a graded Poisson structure on

$$\underset{p_\mu}{T^*[2]} \oplus \underset{\theta^\mu}{T[1]} \underset{x^\mu}{M}$$

Graded spacetime mechanics

Graded Poisson algebra on $T^*[2] \oplus T[1]M$

$$\{\underset{1}{\theta^\mu}, \underset{1}{\theta^\nu}\} = 2\underset{0}{g^{\mu\nu}}(x) \qquad \{\underset{2}{p_\mu}, \underset{0}{x^\nu}\} = \underset{0}{\delta^\nu_\mu} \qquad \{p_\mu, f(x)\} = \partial_\mu f(x)$$

associativity/Jacobi identity $\Leftrightarrow$ metric connection

$$\{p_\mu, \{\theta^\alpha, \theta^\beta\}\} = 2\partial_\mu g^{\alpha\beta} = \{\{p_\mu, \theta^\alpha\}, \theta^\beta\} + \{\theta^\alpha, \{p_\mu, \theta^\beta\}\}$$

$$\{\underset{2}{p_\mu}, \underset{1}{\theta^\alpha}\} = \nabla_\mu \theta^\alpha = \Gamma^\alpha_{\mu\beta} \underset{1}{\theta^\beta}$$

and curvature

$$\{\{p_\mu, p_\nu\}, \theta^\alpha\} = [\nabla_\mu, \nabla_\nu]\theta^\alpha = \theta^\beta R_\beta{}^\alpha{}_{\mu\nu}$$

$$\Rightarrow \quad \{\underset{2}{p_\mu}, \underset{2}{p_\nu}\} = \tfrac{1}{4} \underset{1}{\theta^\beta} \underset{1}{\theta^\alpha} R_{\beta\alpha\mu\nu}$$

Graded spacetime mechanics

symmetries = canonical transformations

- generator of degree 2 (degree-preserving):

$$v^\alpha(x)p_\alpha + \tfrac{1}{2}\Omega_{\alpha\beta}(x)\theta^\alpha\theta^\beta \quad \rightsquigarrow \quad \text{local Poincare algebra}$$

- generators of degree 1:

$$V = V_\alpha(x)\theta^\alpha \quad \rightsquigarrow \quad \{V, W\} = 2g(V, W)$$

- generators of degree 3:

$$\Theta = \theta^\alpha p_\alpha \quad (+\tfrac{1}{6}C_{\alpha\beta\gamma}\theta^\alpha\theta^\beta\theta^\gamma)$$

- generators of degree 4:

$$H = \tfrac{1}{2}g^{\mu\nu}(x)p_\mu p_\nu + \tfrac{1}{2}\Gamma_{\mu\nu}^{\beta}(x)\theta^\mu\theta^\nu p_\beta + \tfrac{1}{16}R_{\alpha\beta\mu\nu}(x)\theta^\alpha\theta^\beta\theta^\mu\theta^\nu$$

$$\rightsquigarrow \quad \text{SUSY algebra} \quad \tfrac{1}{4}\{\Theta, \Theta\} = H$$

Graded spacetime mechanics

Graded Poisson algebra on $T^*[2]M \oplus T[1]M$: $\{p_\mu, x^\nu\} = \delta_\mu^\nu$

$$\{\theta^\mu, \theta^\nu\} = 2g^{\mu\nu}(x) \qquad \{p_\mu, \theta^\alpha\} = \Gamma^\alpha_{\mu\beta}\theta^\beta \qquad \{p_\mu, p_\nu\} = \tfrac{1}{2}\theta^\beta\theta^\alpha R_{\beta\alpha\mu\nu}$$

Equations of motion with Hamiltonian (Dirac op.) $\Theta = \theta^\mu p_\mu$

$$\frac{dA}{d\tau} = \tfrac{1}{2}\{\Theta, \{\Theta, A\}\} = \tfrac{1}{2}\{\{\Theta, \Theta\}, A\} - \tfrac{1}{2}\{\Theta, \{\Theta, A\}\} =: \{H, A\}$$

and derived Hamiltonian

$$H = \tfrac{1}{4}\{\Theta, \Theta\} = \tfrac{1}{2}g^{\mu\nu}p_\mu p_\nu + \tfrac{1}{2}\theta^\mu\theta^\nu\Gamma^\beta_{\mu\nu}p_\beta + \tfrac{1}{16}\theta^\alpha\theta^\beta\theta^\mu\theta^\nu R_{\alpha\beta\mu\nu}$$

For a torsion-less connection, only the first term is non-zero.

Derived anchor map applied to $V = V_\alpha(x)\theta^\alpha$:

$$h(V)f = \{\{V, \Theta\}, f\} = V_\alpha(x)g^{\alpha\beta}\partial_\beta f$$

Graded spacetime mechanics

Equations of motion (cont'd)

$$\frac{dx^\mu}{d\tau} = \tfrac{1}{2}\{\Theta, \{\Theta, x^\mu\}\} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, x^\mu\} = g^{\mu\nu} p_\nu$$

$$\frac{dp_\nu}{d\tau} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, p_\nu\} = \tfrac{1}{2}(\partial_\mu g^{\alpha\beta}) p_\alpha p_\beta = {}^{g}\Gamma_\mu{}^{\alpha\beta} p_\alpha p_\beta$$

with *any* metric-compatible connection ${}^{g}\Gamma$.

Geodesic equation:

$$\frac{d^2 x^\mu}{d\tau^2} = \{\tfrac{1}{2} g^{\alpha\beta} p_\alpha p_\beta, g^{\mu\nu} p_\nu\} = -\frac{dx^\alpha}{d\tau} {}^{LC}\Gamma_{\alpha\beta}{}^\mu \frac{dx^\beta}{d\tau}$$

Nice warmup example and useful in its own right. For the grander scheme with B-field, dilaton, stringy symmetries, we need to double up . . .

Graded geometry

Courant sigma model

standard Courant algebroid $C = TM \oplus T^*M$
TFT with 3-dimensional membrane world volume Σ_3

$$S^{(2)}_{\mathrm{AKSZ}} = \int_{\Sigma_3} \Big(\phi_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\, G_{IJ}\, \alpha^I \wedge \mathrm{d}\alpha^J - h_I{}^i(X)\, \phi_i \wedge \alpha^I + \tfrac{1}{6}\, C_{IJK}(X)\, \alpha^I \wedge \alpha^J \wedge \alpha^K \Big)$$

embedding maps $X : \Sigma_3 \to M$, 1-form α, aux. 2-form ϕ, fiber metric G, anchor h, 3-form C (e.g. H-flux, f-flux, Q-flux, R-flux).

Graded geometry

Graded Poisson manifold $T^*[2]T[1]M$

- degree 0: x^i "coordinates"
- degree 1: $\xi^\alpha = (\theta^i, \chi_i)$
- degree 2: p_i "momenta"

symplectic 2-form

$$\omega = dp_i \wedge dx^i + \frac{1}{2} G_{\alpha\beta} d\xi^\alpha \wedge d\xi^\beta = dp_i \wedge dx^i + d\chi_i \wedge d\theta^i$$

even (degree -2) Poisson bracket on functions $f(x, \xi, p)$

$$\{x^i, x^j\} = 0, \quad \{p_i, x^j\} = \delta_i^j, \quad \{\xi^\alpha, \xi^\beta\} = G^{\alpha\beta}$$

metric $G^{\alpha\beta}$: natural pairing of TM, T^*M:

$$\{\chi_i, \theta^j\} = \delta_i^j \ , \quad \{\chi_i, \chi_j\} = 0 \ , \quad \{\theta^i, \theta^j\} = 0$$

Graded geometry

degree-preserving canonical transformations

- infinitesimal, generators of degree 2:

$$v^\alpha(x)p_\alpha + \tfrac{1}{2}M^{\alpha\beta}(x)\xi_\alpha\xi_\beta \quad \rightsquigarrow \quad \text{diffeos and } o(d,d)$$

- finite, idempotent ("coordinate flip"): $(\tilde{\chi},\tilde{\theta}) = \tau(\chi,\theta)$ with $\tau^2 = \mathrm{id}$

$\rightsquigarrow$ generating function F of type 1 with $F(\theta,\tilde{\theta}) = -F(\tilde{\theta},\theta)$:

$$F = \theta \cdot g \cdot \tilde{\theta} - \tfrac{1}{2}\,\theta \cdot B \cdot \theta + \tfrac{1}{2}\,\tilde{\theta} \cdot B \cdot \tilde{\theta}$$

$$\chi = \frac{\partial F}{\partial \theta} = \tilde{\theta} \cdot g + \theta \cdot B\ , \qquad \tilde{\chi} = -\frac{\partial F}{\partial \tilde{\theta}} = \theta \cdot g + \tilde{\theta} \cdot B$$

$$\Rightarrow \quad \tau(\chi,\theta) = (\chi,\theta) \cdot \begin{pmatrix} g^{-1}B & g^{-1} \\ g - Bg^{-1}B & -Bg^{-1} \end{pmatrix}$$

$\rightsquigarrow$ generalized metric

Generalized geometry as a derived structure

Cartan's magic identy:

$$\mathcal{L}_X = [i_X, d] \equiv i_X d + d\, i_X$$

Lie bracket $[X, Y]_{\text{Lie}}$ of vector fields as a derived bracket:

$$[[i_X, d], i_Y] = [\mathcal{L}_X, i_Y] = i_{[X,Y]_{\text{Lie}}} \quad \text{with } [d, d] = d^2 = 0$$

Graded geometry

Generalized geometry as a derived structure

degree 3 "Hamiltonian": Dirac operator

$$\Theta = \xi^\alpha h^i_\alpha(x) p_i \;+\; \underbrace{\tfrac{1}{6} C_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma}_{\text{twisting/flux terms}}$$

For $e = e_\alpha(x)\xi^\alpha \in \Gamma(TM \oplus T^*M)$ (degree 1, odd):

- pairing: $\langle e, e' \rangle = \{e, e'\}$
- anchor: $h(e)f = \{\{e, \Theta\}, f\}$
- bracket: $[e, e']_D = \{\{e, \Theta\}, e'\}$

Generalized geometry as a derived structure

$$\begin{aligned} h(\xi_1)\langle\xi_2,\xi_2\rangle &= \{\{\Theta,\xi_1\},\{\xi_2,\xi_2\}\} \\ &= 2\{\{\{\Theta,\xi_1\},\xi_2\},\xi_2\} = 2\left\langle[\xi_1,\xi_2],\xi_2\right\rangle \\ &= 2\{\xi_1,\{\{\Theta,\xi_2\},\xi_2\}\} = 2\left\langle\xi_1,[\xi_2,\xi_2]\right\rangle \end{aligned}$$

$$\begin{aligned} [\xi_1,[\xi_2,\xi_3]] &= \{\{\Theta,\xi_1\},\{\{\Theta,\xi_2\},\xi_3\}\} \\ &= [[\xi_1,\xi_2],\xi_3] + [\xi_2,[\xi_1,\xi_3]] + \frac{1}{2}\{\{\{\{\Theta,\Theta\},\xi_1\},\xi_2\},\xi_3\}. \end{aligned}$$

$\{\Theta,\Theta\} = 0 \quad \Leftrightarrow \quad$ axioms of a Courant algebroid

Courant algebroid

vector bundle $E \xrightarrow{\pi} M$, anchor $h : E \to TM$,
bracket $[-,-]$, pairing $\langle -,- \rangle$, s.t. for $e, e', e'' \in \Gamma E$:

$$[e,[e',e'']] = [[e,e'],e''] + [e',[e,e'']] \tag{1}$$

$$2\langle [e,e'],e' \rangle \overset{(2a)}{=} h(e)\langle e',e' \rangle \overset{(2b)}{=} 2\langle [e',e'],e \rangle \tag{2}$$

Consequences:

$$[e,fe'] = h(e).f\, e' + f[e,e'] \tag{3}$$

$$h([e,e']) = [h(e),h(e')]_{\mathrm{Lie}} \tag{4}$$

(2a+b) can be polarized, (1) and (3) define a Leibniz algebroid

Standard Courant algebroid

Idea: Extension of the tangent bundle; unify symplectic, complex and Riemannian structures; string symmetries.

Treat vector fields and forms on equal footing:

$$E = TM \oplus T^*M \text{ "generalized tangent bundle"}$$
$$V = X + \xi = X^i(x)\partial_i + \xi_i(x)dx^i \in \Gamma E$$

With the Dorfman bracket

$$[X+\xi, Y+\eta]_D = [X,Y] + \mathcal{L}_X\eta - \iota_Y d\xi \quad (+\text{twisting/flux terms}),$$

the natural pairing $\langle -, - \rangle$ of TM and T^*M and the projection $h : E \to TM$ (anchor) we obtain a Courant algebroid.

Symmetries: diffeomorphisms, B-transform, θ-transform

Generalized Geometry and Gravity

Generalized Metric

The pairing $\langle -, - \rangle$ has signature (n, n). An idempotent self-adjoint homomorphism τ can turn it into a positive definite generalized metric

$$\mathbb{G}(V, W) := \langle \tau(V), W \rangle \qquad (\mathbb{G}_{\alpha\beta}) = \begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}$$

Generalized Geometry and (super)gravity

established approach: choose Courant algebroid and follow the scheme

Generalized metric $\rightarrow$ Bismut connection $\rightarrow$ set torsion zero (add further conditions as needed) $\rightarrow$ curvature $\rightarrow$ equations of motion $\leftrightarrow$ action

DFT approach: double spacetime $\rightarrow$ universal action $\rightarrow$ strong constraint

new alternative approach: graded geometry, deformation (this talk)

Graded Geometry and Gravity

setup: $T^*[2]T[1]M$ ("step 2") even bracket with odd variables
deformation by a non-symmetric metric $\mathcal{G} = j \circ (g + B) \circ h$

$$\{\chi_i, \chi_j\} = 0 \quad \to \quad \{\chi_i, \chi_j\}' = 2g_{ij}(x)$$

$\Rightarrow$ for $X = X^i(x)\chi_i$ and $v = v^i(x)p_i$, the Poisson structure implies

$$\{v, X\}' = \nabla^{\mathcal{G}}_v X \ , \quad \{v, v'\}' = [v, v']_{\mathrm{Lie}} + R(v, v')$$

$\{\Theta, \Theta\} = 0 \Leftrightarrow R(v, v') = 0$ (no curvature!) Weitzenböck connection

$$\nabla^{\mathcal{G}}_i \chi_j = -(\partial_i \mathcal{G}_{jl})\, \theta^l$$

the derived bracket involves the Levi-Civita connection ∇^{LC} (no torsion!)

$$[X, Y]' = [X, Y]_D + 2g(\nabla^{\mathrm{LC}} X, Y) + H(-, X, Y)$$

plus skew symmetric torsion $H = dB$.

Khoo, Boffo; PS

Graded Geometry and Gravity

generalized Koszul formula for nonsymmetric $\mathcal{G} = g + B$

$$
\begin{aligned}
2g(\nabla_Z X, Y) &= \langle Z, [X, Y]' \rangle' \\
&= X\mathcal{G}(Y, Z) - Y\mathcal{G}(X, Z) + Z\mathcal{G}(X, Y) \\
&\quad -\mathcal{G}(Y, [X, Z]_{\text{Lie}}) - \mathcal{G}([X, Y]_{\text{Lie}}, Z) + \mathcal{G}(X, [Y, Z]_{\text{Lie}}) \\
&= 2g(\nabla^{LC}_X Y, Z) + H(X, Y, Z)
\end{aligned}
$$

$\Rightarrow$ non-symmetric Ricci tensor

$$
R_{jl} = R^{LC}_{jl} - \frac{1}{2}\nabla^{LC}_i H_{jl}{}^i - \frac{1}{4} H_{lm}{}^i H_{ij}{}^m \qquad R = \mathcal{G}_{ij} g^{ik} g^{jl} R_{kl}
$$

$\Rightarrow$ gravity action (closed string effective action) after partial integration:

$$
S_{\mathcal{G}} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} \left(R^{LC} - \frac{1}{12} H_{ijk} H^{ijk} \right)
$$

Khoo, Vysoky, Jurco, Boffo; PS

This formulation consistently combines all approaches of Einstein: Non-symmetric metric, Weitzenböck and Levi-Civita connections, without any of the usual drawbacks.

The dilaton $\phi(x)$ rescales the generalized tangent bundle. The deformation can be formulated in terms of vielbeins

$$E = e^{-\frac{\phi}{3}} \begin{pmatrix} 1 & 0 \\ g+B & 1 \end{pmatrix} \qquad E^{-1}\partial_i E = \begin{pmatrix} -\frac{1}{3}\partial_i\phi & 0 \\ \partial_i(g+B) & -\frac{1}{3}\partial_i\phi \end{pmatrix}$$

Going through the same steps as before we find in $d = 10$

$$S = \frac{1}{2\kappa}\int d^{10}x\, e^{-2\phi}\sqrt{-g}\Big(R^{\mathrm{LC}} - \tfrac{1}{12}H^2 + 4(\nabla\phi)^2\Big)$$

Boffo, PS

Fully deformed Poisson structure on $T^*[2]T[1]M$

$$\{v, f\} = v.f$$
$$\{V, W\} = G(V, W) \equiv \langle V, W \rangle$$
$$\{v, V\} = \nabla_v V \quad \leftarrow \text{connection metric wrt. } G$$
$$\{v, w\} = [v.w]_{\text{Lie}} + R(v, w) \quad \leftarrow \text{curvature of } \nabla$$

with

- degree 0: $f(x)$
- degree 1: $V = V^\alpha(x)\xi_\alpha$ "generalized vectors" $\in \Gamma(TM \oplus T^*M)$
- degree 2: $v = v^i(x)p_i$ "vector fields" $\in \Gamma(TM)$

general Hamiltonian

$$\Theta = \tilde{\xi}^\alpha h(\xi_\alpha) + \frac{1}{6} C_{\alpha\beta\gamma}\tilde{\xi}^\alpha\tilde{\xi}^\beta\tilde{\xi}^\gamma \quad \leftarrow \text{general flux (H,f,Q,R)}$$

derived bracket

$$\{\{\{V,\Theta\},W\},X\} = \langle\nabla_V W, X\rangle - \langle\nabla_W V, X\rangle + \langle\nabla_X V, W\rangle + C(V,W,X)$$

$$\{\{\{\xi_\alpha,\Theta\},\xi_\beta\},\xi_\gamma\} = \underbrace{\Gamma_{\alpha\beta\gamma} - \Gamma_{\beta\alpha\gamma}}_{\text{torsion}} + \Gamma_{\gamma\alpha\beta} + C_{\alpha\beta\gamma} =: \Gamma^{\text{new}}_{\gamma\alpha\beta}$$

"mother of all brackets"

$$\begin{aligned}[V,W] &= \nabla_V W - \nabla_W V + \langle\nabla V, W\rangle + C(V,W,-) \\ &= [V,W]_{\text{Lie}} + T(V,W) + \langle\nabla V, W\rangle + C(V,W,-)\end{aligned}$$

In order to obtain a regular Courant algebroid, impose

$$\{\Theta,\Theta\} = 0 \quad\Leftrightarrow\quad \nabla C + \frac{1}{2}\{C,C\} = 0\,,\quad G^{-1}|_h = 0\,,\ldots$$

generalized connection (involves anchor $h : E \to TM$)

$$\nabla_V(fW) = (h(V)f)W + f\nabla_V W\,, \qquad \nabla_{fV} W = f\nabla_V W$$

generalized Lie-bracket

$$[V,W]_{\mathrm{Lie}} = h(V)W - h(W)V \equiv \big(V^i\partial_i W_\alpha(x) - W^i\partial_i V_\alpha(x)\big)\xi^\alpha$$

generalized torsion tensor

$$T(V,W) = \nabla_V W - \nabla_W V - [V,W]_{\mathrm{Lie}}$$

generalized curvature tensor

$$R(V,W) = \nabla_V\nabla_W - \nabla_W\nabla_V - \nabla_{[V,W]_{\mathrm{Lie}}}$$

Summary + Discussion

- deformation: combines best aspects of Lagrange and Hamilton
- generalized geometry provides a perfect setting for the formulation of theories of gravity
- our approach is based on deformed graded geometry and is algebraic in nature: almost everything follows from associativity as unifying principle (which can be generalized)
- more traditional approaches are based on the generalized metric (with occasional covariance and uniqueness problems)
- string effective action without string theory; target space approach.

Thanks for listening!