Double Field Theory on Para-Hermitian Manifolds

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Action MP 1405
Quantum Structure of Spacetime

Generalized Geometry and Symmetries
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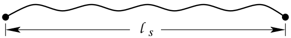
String Geometry



- Strings see geometry in different ways than particles do

- T-duality: $\mathsf{T} : R \longrightarrow R' = 1/R$

- String theory on S^1 of radius R is physically equivalent to string theory on S^1 of radius $1/R$ (automorphism of CFT)

- Exchanges discrete momentum p and winding w

- Exchanges S^1 coordinate x with dual S^1 coordinate $\tilde{x}$

- Acts on a "doubled circle" with coordinates $(x, \tilde{x})$:

Strings "see" a doubled geometry

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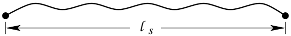

- For a d-torus T^d with background fields (g, B), worldsheet theory is

$$S = \int \mathrm{d}^2\sigma \ E_{ij}(x)\,\partial_+ x^i\,\partial_- x^j \qquad , \qquad E = g + B$$

- T-duality symmetry $O(d, d; \mathbb{Z})$:

$$E' = (a\,E + b)\,\frac{1}{c\,E + d} \qquad , \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d, d; \mathbb{Z})$$

- Acts on d discrete momenta and d winding numbers:
  String theory "sees" doubled torus T^{2d}

- More generally, if M is a T^d-bundle, then string theory "sees" torus bundle with doubled torus fibres T^{2d}:

T-duality $O(d, d; \mathbb{Z}) \subset GL(2d, \mathbb{Z})$ acts geometrically

Generalized Geometry

(Hitchin '02; Gualtieri '04)

- (g, B) satisfy field equations that determine a CFT

- Reproduced from target space theory ($d = 10$):

$$S_{\mathrm{SUGRA}}[g, B] = \int \mathrm{d}^d x \, \sqrt{g} \, \Big(R(g) - \frac{1}{12} H^2 \Big) \quad , \quad H = \mathrm{d}B$$

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- (g, B) and (g', B') give same CFT if related by:

  S1. Diffeomorphisms and B-field gauge transformations

  S2. T-duality

- S1. captured as transition functions in Generalized Geometry

Generalized Geometry

- String Hamiltonian $h = \frac{1}{2}\,\mathcal{H}_{IJ}\,P^I\,P^J$ with:

$$\mathcal{H}(g,B) = \begin{pmatrix} g - B\,g^{-1}\,B & B\,g^{-1} \\ -g^{-1}\,B & g^{-1} \end{pmatrix} \quad , \quad P = \begin{pmatrix} w^i \\ p_i \end{pmatrix}$$

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- Generalized Geometry doubles tangent bundle

$$TM \longrightarrow \mathbb{T}M = TM \oplus T^*M$$

  with structure of Courant algebroid, twisted by B-field

- $\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ $O(d,d)$-structure (fibre metric of $\mathbb{T}M$), bracket of sections is the Courant bracket

- $\mathcal{H}(g,B) \in O(d,d)/O(d)\times O(d)$ Generalized metric on $\mathbb{T}M$, P section of $\mathbb{T}M$

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- S2. not a manifest symmetry: T-duality is an isomorphism between (twisted) Courant algebroids of T^d-bundles (Cavalcanti & Gualtieri '10)

Non-Geometric Backgrounds

- New features of T-duality when $H = \mathrm{d}B \neq 0$
- Prototypical examples come from torus bundles $M \xrightarrow{T^d} W$ (with H-flux $[H] \in H^3(M, \mathbb{Z})$)

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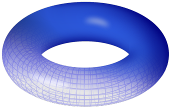
- E.g. $W = S^1$, M = twisted torus, $H = 0$:

Twisted torus

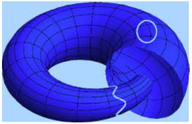


$\downarrow$

S^1

$\xrightarrow{\mathrm{T}_z}$

T-fold



$\downarrow$

S^1

Patching: Diffeos

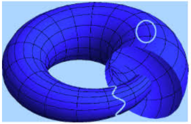
Patching: T-duality

Generalized Flux Backgrounds

$M = T^3$ with H-flux $H = m\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, $B = m\,x\,\mathrm{d}y \wedge \mathrm{d}z$ gives geometric and non-geometric fluxes (Hull '05; Shelton, Taylor & Wecht '05; . . .)

$$H_{ijk} \xrightarrow{\mathsf{T}_i} f^i{}_{jk} \xrightarrow{\mathsf{T}_j} Q^{ij}{}_k \xrightarrow{\mathsf{T}_k} R^{ijk}$$

$$\begin{tikzcd}
(T^3, H\text{-flux}):\ [H] = m & \\
\text{Nilfold}\ \ (f) \arrow[u, "\mathsf{T}_y"] \arrow[d, "S^1", "m"'] \arrow[r, "\mathsf{T}_z"] & \text{T-fold}\ \ (Q) \arrow[d] \\
T^2 & S^1
\end{tikzcd}$$



$$\Big\downarrow \mathsf{T}_x$$

$\widetilde{\mathrm{T}}$-fold (R)

Double Field Theory

(Siegel '93; Hull & Zwiebach '09; Hohm, Hull & Zwiebach '10)

- Duality-covariantization of supergravity:

  $O(d,d)$ symmetry is manifest

- Consequence of string field theory on torus T^d:

$$\psi(p,w) \xrightarrow{\text{Fourier}} \psi(x,\tilde{x})$$

- Strings see doubled spacetime $M \longrightarrow \mathcal{M} = M \times \widetilde{M}$:

$$\mathbb{X}^I = (x^i, \tilde{x}_i) \quad , \quad \partial_I = (\partial_i, \tilde{\partial}^i)$$

- Needed to describe non-geometric backgrounds and generalized T-duality; doubled geometry is physical and dynamical

- $O(d,d)$-structure η / generalized metric $\mathcal{H}(g,B)$

Double Field Theory

- Einstein-Hilbert type action from generalized Ricci scalar $\mathcal{R}(\mathcal{H})$:

$$S_{\rm DFT}[\mathcal{H}] = \int {\rm d}^{2d}\mathbb{X}\ \mathcal{R}(\mathcal{H})$$

- Invariance under generalized Lie derivative: $\delta_\epsilon \mathcal{H}^{IJ} \ = \ \mathsf{L}_\epsilon \mathcal{H}^{IJ}$

- Strong constraint: $\partial^I \otimes \partial_I \ = \ 0$ (worldsheet level matching)
  Solutions select polarisations defining d-dimensional 'physical' null submanifolds of doubled space, DFT reduces to supergravity in different duality frames related by $O(d,d)$-transformations

- Supergravity frame: $\tilde{\partial}_i = 0$ $(w^i = 0)$, $S_{\rm DFT}[\mathcal{H}] \longrightarrow S_{\rm SUGRA}[g,B]$

- C-bracket: Closure $[\mathsf{L}_{\epsilon_1}, \mathsf{L}_{\epsilon_2}] = \mathsf{L}_{[\![\epsilon_1,\epsilon_2]\!]}$ after strong constraint:

$$[\![\epsilon_1,\epsilon_2]\!]^J \ = \ \epsilon_1^K\,\partial_K\epsilon_2^J - \tfrac{1}{2}\,\epsilon_1^K\,\partial^J\epsilon_{2\,K} - (\epsilon_1 \leftrightarrow \epsilon_2)$$

  Reduces to Courant bracket after polarisation

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- Problems with Double Field Theory:
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- Para-Hermitian Geometry: A "real version" of complex Hermitian geometry, addresses these issues (Hull '04; Vaisman '12; Freidel, Rudolph & Svoboda '17; Chatzistavrakidis, Jonke, Khoo & Sz '18; Svoboda '18; Marotta & Sz '18; Mori, Sasaki & Shiozawa '19; ...)

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- Other applications of para-Hermitian geometry:
  - Formulation of $\mathcal{N} = 2$ vector multiplets in Euclidean spacetimes (Cortés, Mayer, Mohaupt & Saueressig '03; Cortés & Mohaupt '09)
  - Lagrangian and non-Lagrangian dynamical systems (Marotta & Sz '18)

Para-Hermitian Manifolds

- Para-complex structure $K : T\mathcal{M} \longrightarrow T\mathcal{M}$ on $2d$-dim manifold $\mathcal{M}$ with $K^2 = +\mathbb{1}$, whose ± 1 eigenbundles $L_\pm$ have same rank d: $K|_{L_\pm} = \pm\,\mathbb{1}$ with projections $P_\pm = \frac{1}{2}\,(\mathbb{1} \pm K)$

- Splits $T\mathcal{M} = L_+ \oplus L_-$, integrability of L_+ and L_- independent

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- Fundamental 2-form $\omega = \eta\,K$ (almost symplectic); if symplectic ($\mathrm{d}\omega = 0$) then (K,η) para-Kähler structure

- $L_\pm$ maximally isotropic with respect to η and ω

Example: Phase Spaces

- Cotangent bundle: $\mathcal{M} = T^*M$, Darboux coordinates $\mathbb{X}^I = (x^i, p_i)$, $\partial_I = (\partial_i, \tilde{\partial}^i)$, canonical symplectic 2-form $\omega_0 = \mathrm{d}p_i \wedge \mathrm{d}x^i$

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- ω_0-compatible para-Kähler structure on T^*M: $\eta_C = \omega_0\,K_C$ is an $O(d,d)$-metric iff C is symmetric

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  - Flat metric: $\eta_B(H_B, H_B) = \eta_B(V, V) = 0$

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- Fundamental 2-form: $\omega_B = \eta_B\, K_B = \mathrm{d}p_i \wedge \mathrm{d}x^i + 2\, \varepsilon_{ijk}\, B^k\, \mathrm{d}x^i \wedge \mathrm{d}x^j$ gives magnetic Poisson brackets:

$$\{x^i, x^j\}_B = 0 \ , \quad \{x^i, p_j\}_B = \delta^i{}_j \ , \quad \{p_i, p_j\}_B = 2\, \varepsilon_{ijk}\, B^k$$

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- Para-Kähler iff Maxwell's equations: $\partial_i B^i = 0$

Para-Hermitian Connections

- Para-Hermitian connection: Connection ∇ on a para-Hermitian manifold $(\mathcal{M}, K, \eta)$ preserving eigenbundles $L_\pm$: $\nabla K = \nabla \eta = 0$

- E.g. Levi-Civita connection of η: ∇^{LC} para-Hermitian iff $(\mathcal{M}, K, \eta)$ is para-Kähler ($\omega = \eta\, K$ symplectic)

- Canonical para-Hermitian connection on any para-Hermitian manifold:

$$\nabla^{\mathsf{can}} = P_+\, \nabla^{\mathsf{LC}}\, P_+ + P_-\, \nabla^{\mathsf{LC}}\, P_-$$

D-Bracket

- Canonical D-bracket on $T\mathcal{M}$ compatible with K:

$$\eta(\llbracket X,Y\rrbracket^{\mathsf{D}}_K,Z)=\eta(\nabla^{\mathsf{can}}_X Y-\nabla^{\mathsf{can}}_Y X,Z)+\eta(\nabla^{\mathsf{can}}_Z X,Y)$$

  with $\llbracket L_\pm,L_\pm\rrbracket^{\mathsf{D}}_K\subseteq L_\pm$ (Dirac structures), metric-compatible, ...

- $\big(T\mathcal{M}\,,\,\eta\,,\,\llbracket\,\cdot\,,\,\cdot\,\rrbracket^{\mathsf{D}}_K\big)$ is a metric algebroid

- Canonical because projection of Lie bracket of vector fields:

$$\llbracket P_\pm(X),P_\pm(Y)\rrbracket^{\mathsf{D}}_K=P_\pm\big([P_\pm(X),P_\pm(Y)]\big)$$

- C-bracket: $\llbracket X,Y\rrbracket^{\mathsf{C}}_K=\frac{1}{2}\big(\llbracket X,Y\rrbracket^{\mathsf{D}}_K-\llbracket Y,X\rrbracket^{\mathsf{D}}_K\big)$

- Reduces to C-bracket of DFT in flat limit: $\eta=\begin{pmatrix}0&\mathbb{1}\\ \mathbb{1}&0\end{pmatrix}$, $\nabla^{\mathsf{LC}}=\mathrm{d}$

Weak Integrability and Fluxes

- If (K,η) and (K',η) are para-Hermitian structures on $\mathcal{M}$, then K' is D-integrable with respect to K if $[\![L'_\pm, L'_\pm]\!]^{\mathrm{D}}_K \subseteq L'_\pm$
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- Fluxes measure (lack of) D-integrability of para-Hermitian structures
- B_+-transformation of (K, η) on $T\mathcal{M} = L_+ \oplus L_-$:

$$e^{B_+} = \begin{pmatrix} \mathbb{1} & 0 \\ B_+ & \mathbb{1} \end{pmatrix} \in O(d,d) \quad \text{where} \quad B_+ : L_+ \longrightarrow L_- \quad \text{with}$$

$$\eta\big(B_+(X), Y\big) = -\eta\big(X, B_+(Y)\big) =: b_+(X, Y)$$

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- $K \longrightarrow K_{B_+} = e^{B_+}\, K\, e^{-B_+}$ where (K_{B_+}, η) is another para-Hermitian structure with fundamental 2-form $\omega_{B_+} = \eta\, K_{B_+} = \omega + 2\, b_+$
- D-integrability controlled by covariant H-flux (Lie algebroid 3-form)

Example: Phase Space Dynamics II

- Canonical para-Kähler structure on $\mathcal{M} = T^*M$ $(C = 0)$:

$$K_0 = \partial_i \otimes \mathrm{d}x^i - \tilde{\partial}^i \otimes \mathrm{d}p_i \quad , \quad \omega_0 = \mathrm{d}p_i \wedge \mathrm{d}x^i$$

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- D-bracket and H-flux:

$$[\![h_i, h_j]\!]^{\mathrm{D}}_{K_0} = \partial_i(\varepsilon_{jkl}\, B^l)\, \tilde{\partial}^k = \eta^{-1}\big(\mathrm{d}b_+(h_i, h_j)\big)$$

Para-Quaternionic Manifolds

- Generalized metric on a para-Hermitian manifold $(\mathcal{M}, K, \eta)$: Riemannian metric $\mathcal{H}$ on $\mathcal{M}$ satisfying compatibility

$$\eta^{-1}\,\mathcal{H} = \mathcal{H}^{-1}\,\eta \quad , \quad \omega^{-1}\,\mathcal{H} = -\mathcal{H}^{-1}\,\omega$$

- $I = \mathcal{H}^{-1}\,\omega \quad , \quad J = \eta^{-1}\,\mathcal{H} \quad , \quad K = \eta^{-1}\,\omega$
  define para-Quaternionic structure on $\mathcal{M}$:

$$I\,J\,K = -\mathbb{1} \qquad\qquad \begin{array}{c} I = J\,K = -K\,J \\ J = I\,K = -K\,I \\ K = J\,I = -K\,I \end{array}$$

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- Generalized metric on a para-Hermitian manifold $(\mathcal{M}, K, \eta)$: Riemannian metric $\mathcal{H}$ on $\mathcal{M}$ satisfying compatibility

$$\eta^{-1}\,\mathcal{H} = \mathcal{H}^{-1}\,\eta \quad , \quad \omega^{-1}\,\mathcal{H} = -\mathcal{H}^{-1}\,\omega$$

- $I = \mathcal{H}^{-1}\,\omega \quad , \quad J = \eta^{-1}\,\mathcal{H} \quad , \quad K = \eta^{-1}\,\omega$
  define para-Quaternionic structure on $\mathcal{M}$:

$$I\,J\,K = -\mathbb{1} \qquad\qquad \begin{array}{c} I = J\,K = -K\,J \\ J = I\,K = -K\,I \\ K = J\,I = -K\,I \end{array}$$

- $(\eta, \omega, \mathcal{H})$ is a Born geometry, DFT is a limit of Born geometry:
  - Flat space limit: $\eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\mathcal{H}(g_+) = \begin{pmatrix} g_+ & 0 \\ 0 & g_+^{-1} \end{pmatrix}$
  - B_+-transformation gives DFT generalized metric:

$$\left(e^{-B_+}\right)^\top \mathcal{H}(g_+)\,e^{-B_+} = \mathcal{H}(g, B) \qquad (g = g_+ \ , \ B = b_+)$$

Recovering Physical Spacetime

- Polarization: Choice of para-Hermitian structure (K, η) on $\mathcal{M}$ (splitting $T\mathcal{M} = L_+ \oplus L_-$ into maximally isotropic sub-bundles)
- Strong constraint: Compatibility condition of Dirac structures (L_+, L_-) in metric algebroid, such that $T\mathcal{M}$ is Courant algebroid

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- Change of polarization:

$$(K, \eta) \longmapsto (K_\vartheta, \eta) \ , \quad K_\vartheta = \vartheta^{-1}\, K\, \vartheta \ , \quad \vartheta \in O(d, d)$$

Example: Doubled Twisted Torus

(Hull & Reid-Edwards '07; Dall'Agata, Prezas, Samtleben & Trigiante '07; Marotta & Sz '18)

- H = 3d Heisenberg group with Drinfel'd double $T^*\mathsf{H} = \mathsf{H} \ltimes \mathbb{R}^3$, basis $\{Z_i, \tilde{Z}^i\}_{i=x,y,z}$ of left-invariant vector fields on $T(T^*\mathsf{H})$

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- Nilfold polarization:

$$[Z_x, Z_y] = m\,Z_z \ , \ [Z_x, \tilde{Z}^y] = m\,\tilde{Z}^z \ , \ [Z_z, \tilde{Z}^y] = -m\,\tilde{Z}^x \quad (m \in \mathbb{Z})$$

  - $\mathcal{H}(g, B)$: $g_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -m\,x \\ 0 & -m\,x & 1+(m\,x)^2 \end{pmatrix} \ , \ b_+ = 0$
  - Weakly integrable: $[\![Z_x, Z_y]\!]_K^{\mathrm{D}} = m\,Z_z$ (no H-flux)

Example: Doubled Twisted Torus

- *H*-flux polarization: There is a B_+-transformation preserving η and mapping K to the splitting:

$$[Z'_x, Z'_y] = -m\,\tilde{Z}'^z \ , \ [Z'_x, Z'_z] = m\,\tilde{Z}'^y \ , \ [Z'_z, Z'_y] = m\,\tilde{Z}'^x$$

- In this new polarization:
  - $\mathcal{H}(g', B')$: $g'_+ = \mathbb{1}$, $b'_+ = m\,x\,\mathrm{d}y \wedge \mathrm{d}z$
  - H-flux: $[\![Z'_i, Z'_j]\!]^{\mathrm{D}}_K = \eta^{-1}\big(\mathrm{d}b_+(Z'_i, Z'_j)\big) = m\,\varepsilon_{ijk}\,\tilde{Z}'^k$

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- This change of polarization gives usual T-dual backgrounds — We can go on and obtain all geometric and non-geometric frames in the T-duality chain:

$$H_{ijk} \xrightarrow{\ \mathrm{T}_i\ } f^i{}_{jk} \xrightarrow{\ \mathrm{T}_j\ } Q^{ij}{}_k \xrightarrow{\ \mathrm{T}_k\ } R^{ijk}$$