M-theory Fluxes & Threebrane Sigma-Models

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Goal

Understand the interrelation of:

- Flux compactification of M-theory, incl. non-geometric fluxes (here for 7+4 dims.)
- Gauge structure of 4D sigma-models.
- Higher algebroids and exceptional generalised geometry.
For closed strings, a single set of equations describes:

- Geometric & non-geometric fluxes (and Bianchi Ids.) in string compactifications.
- Gauge structure for membrane sigma-models with generalised WZ-term.
- Axioms of a Courant algebroid, and $O(d, d)$ generalised geometry of $TM \oplus T^*M$. 
Strings

O(d d) Fluxes & Bianchi Ids.

Courant Algebroid

Membrane $\sigma$-model
Also for double field theory

- O(d d) Fluxes & Bianchi Ids.
- Doubled Membrane $\sigma$-model
- DFT/Vaisman Algebroid
Motivation: Flux Compactifications

In string theory, T-duality reveals unconventional backgrounds w/ non-geometric fluxes.

NSNS sector: \( H_{ijk}, f_{ij}^k, Q_i^{jk}, R^{ijk} \)

RR sector (IIB): \( F_i, F_{ijk}, F_{ijklm}, P_i^{jk}, P_i^{jklm}, \&c. \)

Potentially useful for de Sitter vacua, moduli stabilization and model building.

Sourced by extended, non-perturbative, dynamical objects: exotic branes.
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**What about M-theory?**
Motivation: Sigma-Models

From a worldsheet perspective, the 3-form flux appears as WZ-term.

In general, all types of fluxes appear as WZ-terms in Courant sigma-models.

1-1 correspondence between such membrane sigma-models and Courant algebroids.

Hofman, Park '02; Ikeda '02; Roytenberg '06

Axiomatic organisation of the properties of the (twisted) Courant bracket of $TM \oplus T^*M$

Courant '90; Liu, Weinstein, Xu '95

\[
[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi) + H(X, Y).
\]

Which sigma-model could account for M-theory fluxes as WZ-terms in the same spirit?
Motivation: Geometry of Duality

The Courant bracket is “symmetric” under diffs and $B$-field gauge transformations.

A generalised geometry on $TM \oplus T^*M$ places $g$ & $B$ on equal footing. Hitchin ’02; Gualtieri ’04

In M-theory, higher Courant bracket. $O(d, d) \leftrightarrow$ Cremmer-Julia groups (here $SL(5)$).

An exceptional generalised geometry for $g$ and the $C$-fields. Hull ’07; Pacheco, Waldram ’08

\[ TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \oplus \wedge^6 TM. \]

$\rightsquigarrow$ M-theory as generalised geometry / exceptional field theory where $M$ is extended.

Coimbra, Strickland-Constable, Waldram ’11; Hohm, Samtleben ’13 . . .
1 Introduction and Motivation

2 Exceptional Generalized Geometry and SL(5) M-theory Fluxes

3 SL(5) Fluxes in Exceptional Field Theory

4 Threebrane Sigma-Models, Homotopy Algebroids and M-theory

5 Comments and Conclusions
Higher Courant Bracket

General idea: Extend the tangent bundle $TM$ over a manifold $M$ (dim $M = d$) by $p$-forms

Hagiwara '02; Bi, Sheng '10; Zambon '10

$$E_p = TM \oplus \wedge^p T^* M ,$$

$$\Gamma(E_p) \ni A = X + \eta \quad \text{with} \quad X \in \Gamma(TM), \eta \in \Gamma(\wedge^p T^* M) .$$

$E_p$ is endowed with a non-degenerate symmetric fiber pairing given by contraction

$$\langle X + \eta, Y + \xi \rangle = \frac{1}{2} (\iota_X \xi + \iota_Y \eta) \in \wedge^{p-1} T^* M ;$$

for $p = 1$ it defines an $O(d, d)$-invariant metric, used e.g. in double field theory.

One can also define binary operations. Higher Dorfman bracket (gen’d Lie derivative)

$$(X + \eta) \circ (Y + \xi) = [X, Y] + \mathcal{L}_X \xi - \iota_Y \eta ,$$

or its antisymmetrization, a higher Courant bracket

$$[X + \eta, Y + \xi] = [X, Y] + \mathcal{L}_X \xi - \mathcal{L}_Y \eta - \frac{1}{2} d(\iota_X \xi + \iota_Y \eta) .$$
Properties

Modified Jacobi identity: (where $\mathcal{N}(A, B, C) = \frac{1}{3} \langle [A, B], C \rangle + \text{cyclic}(A, B, C)$)

$$[[A, B], C] + \text{cyclic}(A, B, C) = d \mathcal{N}(A, B, C);$$

Homomorphism and modified Leibniz rule w.r.t. a (anchor) map $\rho : E_\rho \to TM$

$$\rho[A, B] = [\rho(A), \rho(B)];$$

$$[A, f B] = f [A, B] + (\rho(A)f) B - df \wedge \langle A, B \rangle.$$ 

$$\mathcal{L}_{\rho(C)} \langle A, B \rangle = \langle [C, A] + d\langle C, A \rangle, B \rangle + \langle A, [C, B] + d\langle C, B \rangle \rangle.$$

The higher Courant bracket may be twisted by a $(\rho + 2)$-form $H$,

$$[X + \eta, Y + \xi]_H = [X + \eta, Y + \xi] + \iota_X \iota_Y H.$$

In closed string theory, $\rho = 1$ and the twist is identified with the NS-NS 3-form flux.
Generalised Geometry

When \( p = 1 \), \( O(d, d) \) trasfos: automorphisms, \( B \)-transforms & \( \beta \)-transforms \( \text{Gualtieri '04} \)

When \( p = 2 \) & \( d = 4 \), \( SL(5) \) trasfos: \( SL(4) \), \( C \)-transforms, \( \Omega \)-transforms. \( \text{Hull '07} \)

\[
X + \eta \mapsto X + \eta + \iota_X C , \quad X + \eta \mapsto X + \eta + \iota_\eta \Omega .
\]

Generalised metrics \( \mathcal{H}_1 \) & \( \mathcal{H}_2 \) may be parametrized in terms of \( g \) and \( B \) & \( C \)

\[
\mathcal{H}_1 = \begin{pmatrix}
g - B g^{-1} B & -B g^{-1} \\
g^{-1} B & g^{-1}
\end{pmatrix} , \quad \mathcal{H}_2 = \begin{pmatrix}
g + \frac{1}{2} C g^{-1} \wedge g^{-1} C & -\frac{1}{2} C g^{-1} \wedge g^{-1} \\
-\frac{1}{2} g^{-1} \wedge g^{-1} C & \frac{1}{2} g^{-1} \wedge g^{-1}
\end{pmatrix} .
\]

where \( g \) is a Riemannian metric on \( M \) and \( B \) a (Kalb-Ramond) 2-form, or \( C \) a 3-form.

The main players in string/membrane duality rotations, DFT/ExFT, &c.

Shapere, Wilczek '88; Giveon, Rabinovici, Veneziano '88; Duff '89; Tseytlin '90; Maharana, Schwarz '92; . . .
Duff '90; Hull '07; Berman, Perry '10; . . .
The most general bracket twists come in six types, denoted as (vector degree, form degree):

\[(0, p + 2), (1, 2), (p + 1, 1), (p, p + 1), (2p, p), (2p + 1, 0).\]

For string theory, \(p = 1\) gives four possibilities, identified with the corresponding fluxes

\[H_{ijk}, \quad F_{ij}^k, \quad Q_{k}^{ij}, \quad R^{ijk}.\]

What about M-theory? For M2-branes, the relevant structure is \(p = 2\) (when \(d = 4\)):

\[(0, 4), (1, 2), (3, 1), (2, 3), (4, 2), (5, 0).\]

The first one is a 4-form, the \(G\)-flux. What do the other five twists correspond to?
**SL(5) M-Theory Fluxes**

Strategy: consider a general basis of the extended bundle (indices: $i, j$ flat, $a, b$ curved.)

As in the string case, see Halmagyi '09; Blumenhagen, Deser, Plauschinn, Rennecke '12

\[
\begin{align*}
    e_a &:= e_a^i (\partial_i + \frac{1}{2} C_{ijk} \, dx^j \wedge dx^k), \\
    e^{ab} &:= e^a_i e^b_j \left( \frac{1}{2} \, dx^i \wedge dx^j + \frac{1}{2} \, \Omega^{ijk} \, e_k \right),
\end{align*}
\]

and compute the higher Courant bracket, which is generally given as

\[
\begin{align*}
    [e_a, e_b] &= G_{abcd} \, e^{cd} + F_{ab}^c \, e_c, \\
    [e_a, e^{bc}] &= \tilde{F}^{ade} \, e^{de} + Q_{a}^{bc,d} \, e_d, \\
    [e^{ab}, e^{cd}] &= R^{ab,cd,e} \, e_e + \tilde{Q}^{ab,cd,e} \, e^e.
\end{align*}
\]

This may be done for any $d$. However, the physical case is $d = 4$, and it is very special.
First, there are trace relations, indicating that $\widetilde{F}$ and $\widetilde{Q}$ are not independent fluxes in 4D,

$$\widetilde{F}_{ij}^{\, kl} = F_{ij}^{\, k} , \quad \widetilde{Q}_{im}^{\, jk, \, lm} = Q_{i}^{\, jkli} - \frac{1}{2} \delta_{i}^{\, [j} Q_{n}^{\, k]ln} + \frac{1}{4} \delta_{i}^{\, l} Q_{n}^{\, jkn} \quad \text{(in holonomic basis)}$$

Second, the 5-vector $R$ turns out to be a mixed-symmetry tensor of type $(1,4)$ in 4D.

The general local coordinate expressions for the remaining fluxes are found to be

$$G_{abcd} = 4 \nabla_{\lbrack a} C_{bcd \rbrack} ,$$

$$F_{ab}^{\, c} = f_{ab}^{\, c} - \frac{1}{2} G_{abde} \Omega^{dec} ,$$

$$Q_{a}^{\, bcd} = \frac{1}{2} \left( \partial_{a} \Omega^{bcd} + 3 \Omega^{e[bc} f_{ae}^{\, d]} - \frac{1}{2} \Omega^{def} \delta_{a}^{[b} f_{ef}^{\, c]} - \frac{1}{2} \Omega^{e[bc} G_{aefg} \Omega^{d]fg} \right) ,$$

$$R^{ab, cd, e} = \frac{1}{2} \hat{\nabla}^{a[b} \Omega^{cde]} - \frac{1}{2} \hat{\nabla}^{b[a} \Omega^{cde]} - \frac{1}{2} \hat{\nabla}^{c[d} \Omega^{abe]} + \frac{1}{2} \hat{\nabla}^{d[c} \Omega^{abe]} ,$$

where $\hat{\nabla}^{ab} = \Omega^{abc} \nabla_{c}$ and $f_{ab}^{\, c} = 2 \epsilon_{j}^{\, c} \epsilon_{[a}^{\, i} \partial_{i} e_{b]}^{\, j} =: 2 I_{[ab]}^{\, c}$ the purely geometric flux.

Agreement with results from $SL(5)$ group theory. Blair, Malek '14
What is more, there is a systematic way to derive the Bianchi identities for all fluxes. They follow from the (modified) Jacobi identity for the higher Courant bracket. E.g.

\[
[[e_i, e_j], e_m] + \text{cyclic}(i, j, m) - \frac{1}{3} d\left(\langle e_i, e_j \rangle, e_m \rangle + \text{cyclic}(i, j, m)\right) = 0,
\]
in a holonomic frame, gives directly two of the eight (for general \(d\)) Bianchi identities

\[
\partial_m G_{ijkl} = -\frac{3}{5} G_{np[ij} \tilde{F}_{m]kl} \eta^p - \frac{3}{5} F_{ij} \, G_m \eta^{nkl} \quad \Rightarrow \quad G_{n[ij} \, F_{mk]} \eta^n = 0,
\]

\[
\partial_m F_{ij}^l - \frac{1}{3} \hat{\partial}^{lk} G_{ijmk} = -G_{nk[ij} \, Q_m \eta^{nkl} - F_{ij}^k \, F_{m]k}^l.
\]

This is the M-theory analog of the string results of Blumenhagen, Deser, Plauschinn, Rennecke '12

They reproduce consistency conditions of M-theory on twisted tori. cf. Hull, Reid-Edwards '06
Exceptional Field Theory

Strings: Momenta & Windings $\rightsquigarrow$ Coordinates $x^i$ & Dual Coordinates $\tilde{x}_i$ $\rightsquigarrow$ Double FT

Membranes: Momenta & Wrappings $\rightsquigarrow$ $x^I = (x^i, \tilde{x}_{ij})$ in 10 of $SL(5)$ $\rightsquigarrow$ Exceptional FT

Local symmetries in ExFT are generated by a generalized Lie derivative

$$\mathcal{L}_\xi A^I = \xi^J \partial_J A^I - A^J \partial_J \xi^I + Y^{IJ}_{KL} A^K \partial_J \xi^L,$$

where $Y^{IJ}_{KL}$ is an invariant tensor of the U-duality group, controlling a section condition

$$Y^{IJ}_{KL} \partial_I \otimes \partial_J = 0.$$

E.g. $Y^{IJ}_{KL} = \eta^{IJ} \eta_{KL}$ for $O(d, d)$, while $Y^{IJ}_{KL} = \epsilon^{\bar{a}I\bar{b}J} \epsilon_{\bar{a}K\bar{L}}$ for $SL(5), \bar{a} = 1, \ldots, 5, \; I = [\bar{a}\bar{b}]$.

Closure is associated to the $SL(5)$ covariantization of the higher Courant bracket

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\xi_1, \xi_2]}, \quad [[\xi_1, \xi_2]] = \frac{1}{2} (\mathcal{L}_{\xi_1} \xi_2 - \mathcal{L}_{\xi_2} \xi_1).$$
Fluxes in ExFT

The corresponding fluxes in ExFT are determined as the Lie brackets of two derivations

\[ D_i = \partial_i + \frac{1}{2} C_{ijk} \tilde{\partial}^{jk}, \]

\[ \tilde{D}^{jk} = \frac{1}{2} \tilde{\partial}^{jk} + \frac{1}{2} \Omega^{jkl} D_l. \]

The fluxes acquire new terms, as in DFT, for example

\[ G_{abcd} = 4 \nabla_{[a} C_{bcd]} + 2 C_{ef[a} \tilde{\nabla}^{ef} C_{bcd]} , \]

\[ F_{ab}{}^c = f_{ab}{}^c + C_{de[a} \tilde{\Gamma}^{de}{}_{b]}{}^c - \frac{1}{2} \Omega^{dec} G_{abde} + \frac{1}{2} \tilde{\nabla}^{cd} C_{dab} , \]

where we defined the dual connection (dual derivative acting on vielbein)

\[ \tilde{\Gamma}^{ab}{}_{c}{}^{d} = e^{d}{}_{k} e^{[a}{}_{i} e^{b]}{}_{j} \tilde{\partial}^{ij} e^{k}{}_{c} , \]

and

\[ \tilde{\nabla}^{ab} C_{cde} = \tilde{\partial}^{ab} C_{cde} - \tilde{\Gamma}^{ab}{}_{c}{}^{f} C_{fde} - \tilde{\Gamma}^{ab}{}_{d}{}^{f} C_{cfe} - \tilde{\Gamma}^{ab}{}_{e}{}^{f} C_{cdf} . \]

\[ \leadsto \text{systematic derivation of ExFT fluxes modulo the section condition.} \]
Flux Representations

The fluxes exhaust the $SL(5)$ representations $\mathbf{15} \oplus \mathbf{40} \oplus \mathbf{10}$, based on the embedding tensor formalism for gaugings of 7D maximal supergravity. Samtleben, Weidner ’05

Their decompositions under the embedding $SL(5) \supset SL(4)$, read as

\[
\begin{align*}
\mathbf{10} &= \mathbf{4} \oplus \mathbf{6} , \\
\mathbf{15} &= \mathbf{10} \oplus \mathbf{4} \oplus \mathbf{1} , \\
\mathbf{40} &= \mathbf{20} \oplus \mathbf{10} \oplus \mathbf{6} \oplus \mathbf{4} .
\end{align*}
\]
The fluxes exhaust the $SL(5)$ representations $\overline{15} \oplus \overline{40} \oplus \overline{10}$, based on the embedding tensor formalism for gaugings of 7D maximal supergravity. Samtleben, Weidner '05

Their decompositions under the embedding $SL(5) \supset SL(4)$, read as

\[
\begin{align*}
\overline{10} &= \overline{4} \oplus 6 , \\
\overline{15} &= \overline{10} \oplus \overline{4} \oplus 1 , \\
\overline{40} &= \overline{20} \oplus 10 \oplus 6 \oplus \overline{4} .
\end{align*}
\]

Correspondingly, the fluxes are identified as $G_{abcd}$, $F_{ab}^{\ c}$, $Q_{a}^{[bcd]}$, $\tilde{\Pi}_{a}^{\ b\ c}$, $R_{abcd}^{e}$, $\partial_{a}g^{7}$

Blair, Malek '14; Lüst, Malek, Syväri '17
Three equations, three viewpoints

Let us go back to the closed string theory case. There are three interesting equations:

\[ \eta_{IJ} \rho^i_I \rho^j_J = 0, \]
\[ \rho^i_I \partial_i \rho^j_J - \rho^j_J \partial_i \rho^i_I - \eta^{KL} \rho^j_K T_{LIJ} = 0, \]
\[ 4 \rho^i_L \partial_i T_{IJK} + 3 \eta_{MN} T_{M[IJ} T_{KL]N} = 0, \]

where indices \( i, j, \ldots \) run over 1, \ldots, \( d \) while \( I, J, \ldots \) run through 1, \ldots, \( 2d \).

\( \eta_{IJ} \) is the \( O(d, d) \)-invariant metric & \( T_{IJK} \) corresponds to the 4 fluxes \( H_{ijk}, F_{ij}^k, Q_i^k, R_{ijk} \).
Three equations, three viewpoints

These three equations may be interpreted in three different but related ways:

- As fluxes and Bianchi identities in (duality twisted) string compactification.
- As the local form of the axioms of a Courant algebroid on $E_1$.
- As the gauge invariance & closure for the Courant sigma-model:

$$S[X, A, F] = \int_{\Sigma^3} \left( F_i \wedge dX^i + \frac{1}{2} \eta_{IJ} A^I \wedge dA^J - \rho^i_J(X) A^J \wedge F_i + \frac{1}{6} T_{IJK}(X) A^I \wedge A^J \wedge A^K \right),$$

with $\rho^i_J$ being the anchor components and $T_{IJK}$ the $X$-pull-back of $\langle e_I, [e_J, e_K] \rangle$. 
Similarly, three equations that may be found in the flux formulation of double field theory
Geissbühler, Marqués, Núñez, Penas '13

\[
\begin{align*}
\eta^{IJ} \rho^K_I \rho^L_J &= \eta^{KL}, \\
2 \rho^L_{[I} \partial_{L} \rho^K_{J]} - \eta^{MN} \rho^K_M \hat{T}_{NJ} &= \rho_{L[I} \partial^K \rho^L_{J]}, \\
4 \rho^M_{[L} \partial_{L} \hat{\mathcal{T}}_{IJK]} + 3 \eta^{MN} \hat{T}_{M[IJ} \hat{T}_{KL]} &= \mathcal{Z}_{IJKL},
\end{align*}
\]

correspond to the gauge structure of a doubled membrane sigma-model

\[
S_{DFT}[X, A, F] = \int_{\Sigma_3} \left( F_I \wedge dX^I + \eta_{IJ} A^I \wedge dA^J - (\rho_+)^I_{\phantom{I}J} A^J \wedge F_I + \frac{1}{3} \hat{T}_{IJK} A^I \wedge A^J \wedge A^K \right).
\]

Gauge symmetries and their closure (or the classical master equation), obstructed. Obstructions vanish when a world volume analog of the strong constraint is satisfied.

“Algebroid” structure (properties of the C-bracket): Metric or DFT algebroid.

Vaisman '12; A.Ch., Jonke, Khoo, Szabo '18; Mori, Sasaki, Shiozawa '19
Membranes?

Is there a set of equations that captures the following?

- The fluxes and Bianchi identities of $SL(5)$ M-theory compactifications.
- An algebroid structure related to the higher Courant bracket.
- The gauge structure of a sigma-model for higher WZ terms.
Threebrane Sigma-Models

The starting point is a topological threebrane sigma-model with action functional

Ikeda, Uchino '10

\[ S[X, \alpha, A, F] = \int_{\Sigma^4} \left( F_i \wedge dX^i - \alpha_I \wedge dA^I + \rho^i_I(X) F_i \wedge A^I + \frac{1}{2} S^{IJ}(X) \alpha_I \wedge \alpha_J \right. \]

\[ + \frac{1}{2} T^I_{JK}(X) \alpha_I \wedge A^J \wedge A^K + \frac{1}{4!} G_{IJKL}(X) A^I \wedge A^J \wedge A^K \wedge A^L \). \]

Ingredients:

- Scalars \( X = (X^i) : \Sigma^4 \rightarrow M \).
- Auxiliary 3-form \( F \in \Omega^3(\Sigma^4, X^* T^* M) \).
- 1-form \( A \in \Omega^1(\Sigma^4, X^* E) \) & 2-form \( \alpha \in \Omega^2(\Sigma^4, X^* E^*) \);
  \( E \) being some vector bundle. \( E \)-indices \( I, J, \ldots \).
- Structure functions \( \rho, S, T, G \) of \( X(\sigma) \).
Invariance under gauge symmetries with 0-, 1- and 2-form parameters \((\epsilon^I, \zeta_I, t_i)\):

\[
\delta X^i = -\rho^i \epsilon^I, \\
\delta A^I = d\epsilon^I + S^{IJ} \zeta_J - T^I_{JK} A^J \epsilon^K, \\
\delta \alpha_I = d\zeta_I + \rho^I t_i + T^J_{IK} \zeta_J \wedge A^K + T^J_{IK} \alpha_J \epsilon^K + \frac{1}{2} G_{IJKL} \epsilon^I A^K \wedge A^L, \\
\delta F_i = -dt_i + \partial_i \rho^I (\epsilon^I F_j + t_j \wedge A^I) - \partial_i T^I_{JL} \epsilon^J \alpha_J \wedge A^L \\
- \frac{1}{6} \partial_i G_{IJKL} \epsilon^I A^J \wedge A^K \wedge A^L + \frac{1}{2} \partial_i T^I_{JK} \zeta_I \wedge A^J \wedge A^K + \partial_i S^{IJ} \zeta_I \wedge \alpha_J,
\]

provided that...
Five equations

...provided that five conditions hold for the structure functions:

\[ \rho^i_I \ S^{IJ} = 0 , \]

\[ \rho^i_I \ \partial_i S^{JK} + S^{LJ} \ T^K_{IL} + S^{LK} \ T^J_{IL} = 0 , \]

\[ \rho^i_I \ \partial_i \rho^j_J - \rho^i_J \ \partial_i \rho^j_I - \rho^i_K \ T^K_{IJ} = 0 , \]

\[ 3 \rho^i_I \ [ \partial_i \ T^J_{KL} ] + S^{JM} \ G_{KLM} - 3 \ T^J_{M[K} \ T^M_{L]} = 0 , \]

\[ \rho^i_I \ [ \partial_i \ G_{JKLM} ] + T^N_{IJ} \ G_{KLMN} = 0 . \]

Moreover, local form of the axioms for a structure called Lie algebroid up to homotopy.

Ikeda, Uchino '10

A special “H”-twisted Lie algebroid on \( E \), with anchor, sym. 2-vector, E4-form & bracket.

Grützmann '10; Grützmann, Strobl “14”
Some Examples

- For $E = TM$, anchor being projection $\rho^i_j = \delta^i_j$ and $S = T = 0 \rightsquigarrow dG = 0$. On-shell (with boundary metric term) describes M2-branes coupled to 3-form $C$:

  Kőkenyesi, Sinkovics, Szabo ’18

  $$S_\partial[X] = \oint_{\partial \Sigma^4} \left( \frac{1}{2} g_{ij} dX^i \wedge * dX^j + \frac{1}{3!} C_{ijk} dX^i \wedge dX^j \wedge dX^k \right).$$

- For $\rho^i_j = E^i_j(X)$ the “vielbein” for a twisted 4-torus, and $T$ its geometric flux,

  $$dE^i = -\frac{1}{2} T^i_{jk} E^j \wedge E^k,$$

  M-theory background on twisted torus, Hull, Reid-Edwards ’06

  $$S_\partial[X] = \oint_{\partial \Sigma^4} \frac{1}{2} g_{ij} E^i \wedge * E^j.$$

  Turning on $G$ too, one gets the expected algebraic identity $T^n_{[ij} G_{klm]n} = 0$. 


A different example

Recall: In the Courant algebroid, a consistent choice is cf. Besso, Heller, Ikeda, Watamura '15

\[ \rho^i_j = (0, \Pi^{ij}) , \quad [\Pi, \Pi]_S = 0 , \quad Q = d\Pi , \quad [\Pi, R]_S = 0 . \]

This leads to a geometric R-flux model

\[
S_{R,\Pi}[X] = \int_{\partial \Sigma_3} \frac{1}{2} \left( g_{ij} - \Pi^{-1}_{ik} g^{kl} \Pi^{-1}_{lj} \right) dX^i \wedge * dX^j \\
+ \int_{\Sigma_3} \frac{1}{3!} R^{pqr} \Pi^{-1}_{ip} \Pi^{-1}_{jq} \Pi^{-1}_{kr} dX^i \wedge dX^j \wedge dX^k .
\]

In the present setting, for \( E = T^* M \), and the choices (with 4-vector \( G \)):

\[ \rho^{ij} = \Pi^{ij} , \quad T^{jk}_i = \partial_i \Pi^{jk} , \quad [\Pi, G]_S = 0 , \]

M2-branes in 4-form flux controlled by a 4-vector \( G \),

\[
S_{\partial}[X] = \int_{\partial \Sigma_4} \frac{1}{2} \left( g_{ij} - \Pi^{-1}_{ik} g^{kl} \Pi^{-1}_{lj} \right) dX^i \wedge * dX^j \\
+ \int_{\Sigma_4} \frac{1}{4!} G^{pqr} \Pi^{-1}_{ip} \Pi^{-1}_{jq} \Pi^{-1}_{kr} \Pi^{-1}_{ls} dX^i \wedge dX^j \wedge dX^k \wedge dX^l .
\]
SL(5) fluxes from the five conditions?

In the above setting, it is not possible to obtain the rest of SL(5) fluxes.
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The structure of $R$ indicates that $T^I_{JK}$ may contain it for enlarged $I$-index range.

The bundle index $I$ should take 10 values; 4 of type $i, j$ and six of paired type $[ij], [kl]$.
SL(5) fluxes from the five conditions?

In the above setting, it is not possible to obtain the rest of SL(5) fluxes.

The structure of $R$ indicates that $T^l_{jk}$ may contain it for enlarged $l$-index range.

The bundle index $l$ should take 10 values; 4 of type $i, j$ and six of paired type $[ij], [kl]$.

A proposal is to consider the same model, but start with

$$E = E_2 = TM \oplus \wedge^2 T^* M.$$  

For the anchor components we make a general choice, not just projection to $TM$,

$$(\rho^l_i) = (\rho^l_j, \rho^{ijk}) = (\delta^l_j, \frac{1}{2} \Omega^{ijk}).$$

Setting $S$ and $G$ to zero, there is a non-trivial identification of $T^l_{ij}^k$ with the SL(5) fluxes.

This is special; it requires an additional projection to SL(5) representations.
SL(5) fluxes as Wess-Zumino terms

Going back to the threebrane sigma-model, the $E$-valued fields now become:

$$A^I = (A^i, A_{ij}) =: (q^i, p_{ij}), \quad (1\text{-forms})$$

$$\alpha_I = (\alpha_i, \alpha^{ij}) =: (p_i, q^{ij}), \quad (2\text{-forms})$$

Apparent overabundance; but the flux identification says two additional things:

- $q^{ij}$ is decomposable, i.e. $q^{ij} = q^i \wedge q^j$.
- $p_j = q^i \wedge p_{ij}$.

This leads to the sigma-model with WZ terms being the four types of M-theory fluxes

$$S = \int_{\Sigma_4} \left( F_i \wedge dX^i - q^i \wedge p_{ij} \wedge dq^j - q^i \wedge q^j \wedge dp_{ij} + F_i \wedge q^i + \frac{1}{2} \Omega^{ijk} F_i \wedge p_{jk} ight.$$

$$\left. + \frac{3}{2} F^m_{\;\;jk} q^i \wedge q^j \wedge q^k \wedge p_{im} + \frac{1}{2} G_{ijkl} q^i \wedge q^j \wedge q^k \wedge q^l \right)$$

$$\left. + \frac{3}{2} Q^{ijk}_{\;\;il} q^i \wedge q^m \wedge p_{im} \wedge p_{jk} + \frac{1}{2} R_{jk,lm,i} q^n \wedge p_{ni} \wedge p_{jk} \wedge p_{lm} \right).$$

Boundary metric terms are entries of $\mathcal{H}_2$, e.g. the characteristic $g^{ijkl} = g^{ik} g^{jl} - g^{il} g^{jk}$. 


Epilogue

M2-branes

SL(5) Fluxes & Bianchi Ids.

Homotopy algebroid

Threebrane $\sigma$-model
Some open problems

- Does this hold for larger U-duality groups?
  M5-branes, extended bundles are not as simple as $E_p$. 
  Known... Hull '07

- Is there a sigma-model with extended base, not only bundle?
  Manifest U-duality in the world volume is hard... 
  Duff, Lu, Percacci, Pope, Samtleben, Sezgin '15

- Origin of the section condition?

- M-theory $R$-flux shows signs of non-associativity, how to capture it here?
  Günaydin, Lüst, Malek '16; Kupriyanov, Szabo '17
  As in the previous step, for the stringy $R$-flux. 
  Mylonas, Schupp, Szabo '12

- M-theory exotic brane couplings?