M–theory Fluxes & Threebrane Sigma–Models

Athanasios Chatzistavrakidis

Rudjer Bošković Institute, Zagreb

1901.07775 with Larisa Jonke, Dieter Lüst & Richard Szabo

Bayrischzell Workshop 2019

Goal

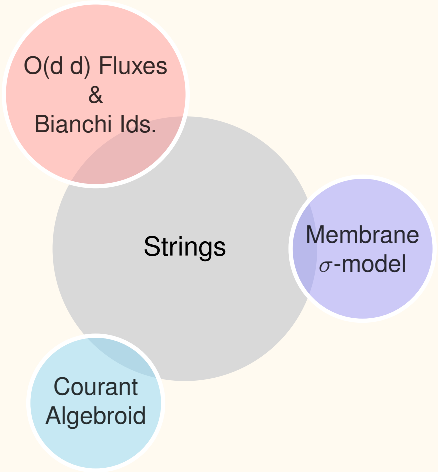
Understand the interrelation of:

- Flux compactification of M-theory, incl. non-geometric fluxes (here for 7+4 dims.)
- Gauge structure of 4D sigma-models.
- Higher algebroids and exceptional generalised geometry.

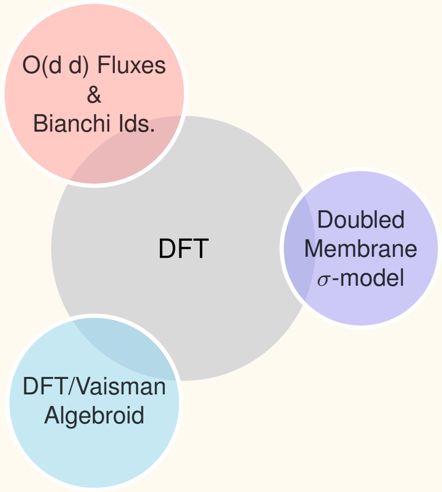
Why expect this?

For closed strings, a single set of equations describes:

- Geometric & non-geometric fluxes (and Bianchi Ids.) in string compactifications.
- Gauge structure for membrane sigma-models with generalised WZ-term.
- Axioms of a Courant algebroid, and $O(d,d)$ generalised geometry of $TM \oplus T^*M$.


O(d d) Fluxes
&
Bianchi Ids.
Strings
Membrane
σ-model
Courant
Algebroid

Also for double field theory



Motivation: Flux Compactifications

In string theory, T-duality reveals unconventional backgrounds w/ non-geometric fluxes.

$$\text{NSNS sector}: \quad H_{ijk}\,, \quad f_{ij}{}^{k}\,, \quad Q_i{}^{jk}\,, \quad R^{ijk}$$

$$\text{RR sector (IIB)}: \quad F_i\,, \quad F_{ijk}\,, \quad F_{ijklm}\,, \quad P_i{}^{jk}\,, \quad P_i{}^{jklm}\,, \quad \&c.$$

Potentially useful for de Sitter vacua, moduli stabilization and model building.

Sourced by extended, non-perturbative, dynamical objects: exotic branes.

Motivation: Flux Compactifications

In string theory, T-duality reveals unconventional backgrounds w/ non-geometric fluxes.

$$\text{NSNS sector}: \quad H_{ijk}\,, \quad f_{ij}{}^{k}\,, \quad Q_i{}^{jk}\,, \quad R^{ijk}$$

$$\text{RR sector (IIB)}: \quad F_i\,, \quad F_{ijk}\,, \quad F_{ijklm}\,, \quad P_i{}^{jk}\,, \quad P_i{}^{jklm}\,, \quad \&c.$$

Potentially useful for de Sitter vacua, moduli stabilization and model building.

Sourced by extended, non-perturbative, dynamical objects: exotic branes.

What about M-theory?

Motivation: Sigma-Models

From a worldsheet perspective, the 3-form flux appears as WZ-term.

In general, all types of fluxes appear as WZ-terms in Courant sigma-models.

1-1 correspondence between such membrane sigma-models and Courant algebroids.

Hofman, Park '02; Ikeda '02; Roytenberg '06

Axiomatic organisation of the properties of the (twisted) Courant bracket of $TM \oplus T^*M$

Courant '90; Liu, Weinstein, Xu '95

$$[X+\xi, Y+\eta] \quad = \quad [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \tfrac{1}{2}\,\mathrm{d}(\iota_X\eta - \iota_Y\xi) + H(X,Y)\ .$$

Which sigma-model could account for M-theory fluxes as WZ-terms in the same spirit?

Motivation: Geometry of Duality

The Courant bracket is "symmetric" under diffs and B-field gauge transformations.

A generalised geometry on $TM \oplus T^*M$ places g & B on equal footing. Hitchin '02; Gualtieri '04

In M-theory, higher Courant bracket. $O(d,d) \mapsto$ Cremmer-Julia groups (here $SL(5)$).

An exceptional generalised geometry for g and the C-fields. Hull '07; Pacheco, Waldram '08

$$TM \oplus \wedge^2 T^*M \oplus \wedge^5 T^*M \oplus \wedge^6 TM \,.$$

$\rightsquigarrow$ M-theory as generalised geometry / exceptional field theory where M is extended.

Coimbra, Strickland-Constable, Waldram '11; Hohm, Samtleben '13 . . .

Higher Courant Bracket

General idea: Extend the tangent bundle TM over a manifold M ($\dim M = d$) by p-forms
Hagiwara '02; Bi, Sheng '10; Zambon '10

$$E_p = TM \oplus \wedge^p T^*M \,,$$

$$\Gamma(E_p) \ni A = X + \eta \qquad \text{with} \quad X \in \Gamma(TM)\,,\ \eta \in \Gamma(\wedge^p T^*M)\,.$$

E_p is endowed with a non-degenerate symmetric fiber pairing given by contraction

$$\langle X+\eta, Y+\xi \rangle = \tfrac{1}{2}\,(\iota_X \xi + \iota_Y \eta) \in \bigwedge^{p-1} T^*M\,;$$

for $p = 1$ it defines an $O(d,d)$-invariant metric, used e.g. in double field theory.

One can also define binary operations. Higher Dorfman bracket (gen'd Lie derivative)

$$(X+\eta)\circ(Y+\xi) = [X,Y] + \mathcal{L}_X \xi - \iota_Y \eta\,,$$

or its antisymmetrization, a higher Courant bracket

$$[X+\eta, Y+\xi] = [X,Y] + \mathcal{L}_X \xi - \mathcal{L}_Y \eta - \tfrac{1}{2}\,\mathrm{d}(\iota_X \xi + \iota_Y \eta)\,.$$

Properties

Modified Jacobi identity: (where $\mathcal{N}(A, B, C) = \frac{1}{3}\langle[A, B], C\rangle + \text{cyclic}(A, B, C)$)

$$[[A, B], C] + \text{cyclic}(A, B, C) = \mathrm{d}\,\mathcal{N}(A, B, C)\ ;$$

Homomorphism and modified Leibniz rule w.r.t. a (anchor) map $\rho : E_p \to TM$

$$\rho[A, B] = [\rho(A), \rho(B)]\ ;$$

$$[A, f\,B] = f\,[A, B] + \big(\rho(A)f\big)\,B - \mathrm{d}f \wedge \langle A, B\rangle\ .$$

$$\mathcal{L}_{\rho(C)}\langle A, B\rangle = \langle [C, A] + \mathrm{d}\langle C, A\rangle, B\rangle + \langle A, [C, B] + \mathrm{d}\langle C, B\rangle\rangle\ .$$

The higher Courant bracket may be twisted by a $(p+2)$-form H,

$$[X + \eta, Y + \xi]_H = [X + \eta, Y + \xi] + \iota_X \iota_Y H\ .$$

In closed string theory, $p = 1$ and the twist is identified with the NS-NS 3-form flux.

Generalised Geometry

When $p = 1$, $O(d,d)$ trafos: automorphisms, B-transforms & β-transforms Gualtieri '04

When $p = 2$ & $d = 4$, $SL(5)$ trafos: $SL(4)$, C-transforms, Ω-transforms. Hull '07

$$X + \eta \overset{C}{\mapsto} X + \eta + \iota_X C \,, \qquad X + \eta \overset{\Omega}{\mapsto} X + \eta + \iota_\eta \Omega \,.$$

Generalised metrics $\mathcal{H}_1$ & $\mathcal{H}_2$ may be parametrized in terms of g and B & C

$$\mathcal{H}_1 = \begin{pmatrix} g - B\,g^{-1}\,B & -B\,g^{-1} \\ g^{-1}\,B & g^{-1} \end{pmatrix} \,, \quad \mathcal{H}_2 = \begin{pmatrix} g + \frac{1}{2}\,C\,g^{-1}{\wedge}g^{-1}\,C & -\frac{1}{2}\,C\,g^{-1}{\wedge}g^{-1} \\ -\frac{1}{2}\,g^{-1}{\wedge}g^{-1}\,C & \frac{1}{2}\,g^{-1}{\wedge}g^{-1} \end{pmatrix} \,.$$

where g is a Riemannian metric on M and B a (Kalb-Ramond) 2-form, or C a 3-form.

The main players in string/membrane duality rotations, DFT/ExFT, &c.

Shapere, Wilczek '88; Giveon, Rabinovici, Veneziano '88; Duff '89; Tseytlin '90; Maharana, Schwarz '92; ...

Duff '90; Hull '07; Berman, Perry '10; ...

General twists and M-theory

The most general bracket twists come in six types, denoted as (vector degree, form degree):

$$(0, p+2), \quad (1,2), \quad (p+1,1), \quad (p,p+1), \quad (2p,p), \quad (2p+1,0).$$

For string theory, $p=1$ gives four possibilities, identified with the corresponding fluxes

$$H_{ijk}, \quad F_{ij}{}^{k}, \quad Q_k{}^{ij}, \quad R^{ijk}.$$

What about M-theory? For M2-branes, the relevant structure is $p=2$ (when $d=4$):

$$(0,4), \quad (1,2), \quad (3,1), \quad (2,3), \quad (4,2), \quad (5,0).$$

The first one is a 4-form, the G-flux. What do the other five twists correspond to?

$SL(5)$ M-Theory Fluxes

Strategy: consider a general basis of the extended bundle (indices: i, j flat, a, b curved.)

As in the string cae, see Halmagyi '09; Blumenhagen, Deser, Plauschinn, Rennecke '12

$$e_a := e_a{}^i (\partial_i + \tfrac{1}{2} C_{ijk} \, \mathrm{d}x^j \wedge \mathrm{d}x^k) \ ,$$

$$e^{ab} := e^a{}_i e^b{}_j (\tfrac{1}{2} \, \mathrm{d}x^i \wedge \mathrm{d}x^j + \tfrac{1}{2} \, \Omega^{ijk} \, e_k) \ ,$$

and compute the higher Courant bracket, which is generally given as

$$\begin{aligned} [e_a, e_b] &= G_{abcd} \, e^{cd} + F_{ab}{}^c \, e_c \ , \\ [e_a, e^{bc}] &= \widetilde{F}_{ade}{}^{bc} \, e^{de} + Q_a{}^{bc,d} \, e_d \ , \\ [e^{ab}, e^{cd}] &= R^{ab,cd,e} \, e_e + \widetilde{Q}_{ef}{}^{ab,cd} \, e^{ef} \ . \end{aligned}$$

This may be done for any d. However, the physical case is $d = 4$, and it is very special.

$SL(5)$ M-Theory Fluxes

First, there are trace relations, indicating that $\widetilde{F}$ and $\widetilde{Q}$ are not independent fluxes in 4D,

$$\widetilde{F}_{ijl}{}^{lk} = F_{ij}{}^{k} \, , \qquad \widetilde{Q}_{im}{}^{jk,lm} = Q_i{}^{jkl} - \tfrac{1}{2}\,\delta_i^{[j}\, Q_n{}^{k]ln} + \tfrac{1}{4}\,\delta_i^{l}\, Q_n{}^{jkn} \qquad \text{(in holonomic basis)}$$

Second, the 5-vector R turns out to be a mixed-symmetry tensor of type (1,4) in 4D.

The general local coordinate expressions for the remaining fluxes are found to be

$$G_{abcd} = 4\,\nabla_{[a} C_{bcd]} \, ,$$

$$F_{ab}{}^{c} = f_{ab}{}^{c} - \tfrac{1}{2}\, G_{abde}\, \Omega^{dec} \, ,$$

$$Q_a{}^{bcd} = \tfrac{1}{2}\left(\partial_a \Omega^{bcd} + 3\,\Omega^{e[bc}\, f_{ae}{}^{d]} - \tfrac{1}{2}\,\Omega^{def}\,\delta_a^{[b}\, f_{ef}{}^{c]} - \tfrac{1}{2}\,\Omega^{e[bc}\, G_{aefg}\,\Omega^{d]fg}\right) ,$$

$$R^{ab,cd,e} = \tfrac{1}{2}\,\widehat{\nabla}^{a[b}\Omega^{cde]} - \tfrac{1}{2}\,\widehat{\nabla}^{b[a}\Omega^{cde]} - \tfrac{1}{2}\,\widehat{\nabla}^{c[d}\Omega^{abe]} + \tfrac{1}{2}\,\widehat{\nabla}^{d[c}\Omega^{abe]} \, ,$$

where $\widehat{\nabla}^{ab} = \Omega^{abc}\,\nabla_c$ and $f_{ab}{}^{c} = 2\, e^c{}_j\, e_{[a}{}^{i}\,\partial_i e_{b]}{}^{j} =: 2\,\Gamma_{[ab]}{}^{c}$ the purely geometric flux.

Agreement with results from $SL(5)$ group theory. Blair, Malek '14

Bianchi Identities

What is more, there is a systematic way to derive the Bianchi identities for all fluxes.

They follow from the (modified) Jacobi identity for the higher Courant bracket. E.g.

$$[[e_i, e_j], e_m] + \mathrm{cyclic}(i,j,m) - \tfrac{1}{3}\,\mathrm{d}\big(\langle [e_i, e_j], e_m\rangle + \mathrm{cyclic}(i,j,m)\big) = 0 \ ,$$

in a holonomic frame, gives directly two of the eight (for general d) Bianchi identities

$$\partial_{[m} G_{ijkl]} = -\tfrac{3}{5}\, G_{np[ij}\, \widetilde{F}_{m]kl}{}^{np} - \tfrac{3}{5}\, F_{[ij}{}^{n}\, G_{m]nkl} \quad \overset{4D}{\Rightarrow} \quad G_{n[lij}\, F_{mk]}{}^{n} = 0 \ ,$$

$$\partial_{[m} F_{ij]}{}^{l} - \tfrac{1}{3}\, \hat{\partial}^{lk} G_{ijmk} = -G_{nk[ij}\, Q_{m]}{}^{nkl} - F_{[ij}{}^{k}\, F_{m]k}{}^{l} \ .$$

This is the M-theory analog of the string results of Blumenhagen, Deser, Plauschinn, Rennecke '12

They reproduce consistency conditions of M-theory on twisted tori. cf. Hull, Reid-Edwards '06

Exceptional Field Theory

Strings: Momenta & Windings $\rightsquigarrow$ Coordinates x^i & Dual Coordinates $\tilde{x}_i$ $\rightsquigarrow$ Double FT

Membranes: Momenta & Wrappings $\rightsquigarrow$ $x^I = (x^i, \tilde{x}_{ij})$ in **10** of $SL(5)$ $\rightsquigarrow$ Exceptional FT

Local symmetries in ExFT are generated by a generalized Lie derivative

$$\mathcal{L}_\xi A^I = \xi^J \partial_J A^I - A^J \partial_J \xi^I + Y^{IJ}_{KL} A^K \partial_J \xi^L ,$$

where Y^{IJ}_{KL} is an invariant tensor of the U-duality group, controlling a section condition

$$Y^{IJ}_{KL} \partial_I \otimes \partial_J = 0 .$$

E.g. $Y^{IJ}_{KL} = \eta^{IJ} \eta_{KL}$ for $O(d,d)$, while $Y^{IJ}_{KL} = \epsilon^{\bar{a}IJ} \epsilon_{\bar{a}KL}$ for $SL(5)$, $\bar{a} = 1, \dots, 5$, $I = [\bar{a}\bar{b}]$.

Closure is associated to the $SL(5)$ covariantization of the higher Courant bracket

$$[\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{[\![\xi_1,\xi_2]\!]} , \qquad [\![\xi_1, \xi_2]\!] = \tfrac{1}{2} (\mathcal{L}_{\xi_1}\xi_2 - \mathcal{L}_{\xi_2}\xi_1) .$$

Fluxes in ExFT

The corresponding fluxes in ExFT are determined as the Lie brackets of two derivations

As for DFT in Blumenhagen, Gao, Herschmann, Shukla '13

$$D_i = \partial_i + \tfrac{1}{2}\, C_{ijk}\, \tilde{\partial}^{jk} \ ,$$

$$\tilde{D}^{jk} = \tfrac{1}{2}\, \tilde{\partial}^{jk} + \tfrac{1}{2}\, \Omega^{jkl}\, D_l \ .$$

The fluxes acquire new terms, as in DFT, for example

$$G_{abcd} = 4\, \nabla_{[a} C_{bcd]} + 2\, C_{ef[a}\, \widetilde{\nabla}^{ef} C_{bcd]} \ ,$$

$$F_{ab}{}^{c} = f_{ab}{}^{c} + C_{de[a}\, \tilde{\Gamma}^{de}{}_{b]}{}^{c} - \tfrac{1}{2}\, \Omega^{dec}\, G_{abde} + \tfrac{1}{2}\, \widetilde{\nabla}^{cd} C_{dab} \ ,$$

where we defined the dual connection (dual derivative acting on vielbein)

$$\tilde{\Gamma}^{ab}{}_{c}{}^{d} = e^{d}{}_{k}\, e^{[a}{}_{i}\, e^{b]}{}_{j}\, \tilde{\partial}^{ij} e^{k}{}_{c} \ ,$$

and

$$\widetilde{\nabla}^{ab} C_{cde} = \tilde{\partial}^{ab} C_{cde} - \tilde{\Gamma}^{ab}{}_{c}{}^{f}\, C_{fde} - \tilde{\Gamma}^{ab}{}_{d}{}^{f}\, C_{cfe} - \tilde{\Gamma}^{ab}{}_{e}{}^{f}\, C_{cdf} \ .$$

$\rightsquigarrow$ systematic derivation of ExFT fluxes modulo the section condition.

Flux Representations

The fluxes exhaust the $SL(5)$ representations $\overline{\mathbf{15}} \oplus \overline{\mathbf{40}} \oplus \overline{\mathbf{10}}$, based on the embedding tensor formalism for gaugings of 7D maximal supergravity. Samtleben, Weidner '05

Their decompositions under the embedding $SL(5) \supset SL(4)$, read as

$$\overline{\mathbf{10}} = \overline{\mathbf{4}} \oplus \mathbf{6}\ ,$$

$$\overline{\mathbf{15}} = \overline{\mathbf{10}} \oplus \overline{\mathbf{4}} \oplus \mathbf{1}\ ,$$

$$\overline{\mathbf{40}} = \overline{\mathbf{20}} \oplus \mathbf{10} \oplus \mathbf{6} \oplus \overline{\mathbf{4}}\ .$$

Flux Representations

The fluxes exhaust the $SL(5)$ representations $\overline{\mathbf{15}} \oplus \overline{\mathbf{40}} \oplus \overline{\mathbf{10}}$, based on the embedding tensor formalism for gaugings of 7D maximal supergravity. Samtleben, Weidner '05

Their decompositions under the embedding $SL(5) \supset SL(4)$, read as

$$\overline{\mathbf{10}} = \overline{\mathbf{4}} \oplus \mathbf{6} \,,$$

$$\overline{\mathbf{15}} = \overline{\mathbf{10}} \oplus \overline{\mathbf{4}} \oplus \mathbf{1} \,,$$

$$\overline{\mathbf{40}} = \overline{\mathbf{20}} \oplus \mathbf{10} \oplus \mathbf{6} \oplus \overline{\mathbf{4}} \,.$$

Correspondingly, the fluxes are identified as G_{abcd}, $F_{ab}{}^{c}$, $Q_{a}{}^{[bcd]}$, $\tilde{\Gamma}^{ab}{}_{b}{}^{c}$, $R^{abcd,e}$, $\partial_a g_7$

Blair, Malek '14; Lüst, Malek, Syväri '17

Three equations, three viewpoints

Let us go back to the closed string theory case. There are three interesting equations:

$$\eta^{IJ}\,\rho^i{}_I\,\rho^j{}_J = 0\;,$$

$$\rho^i{}_I\,\partial_i\rho^j{}_J - \rho^i{}_J\,\partial_i\rho^j{}_I - \eta^{KL}\,\rho^j{}_K\,T_{LIJ} = 0\;,$$

$$4\,\rho^i{}_{[L}\,\partial_i T_{IJK]} + 3\,\eta^{MN}\,T_{M[IJ}\,T_{KL]N} = 0\;,$$

where indices $i, j, \ldots$ run over $1, \ldots, d$ while $I, J, \ldots$ run through $1, \ldots, 2d$.

η_{IJ} is the $O(d,d)$-invariant metric & T_{IJK} corresponds to the 4 fluxes H_{ijk}, $F_{ij}{}^k$, $Q_i{}^{jk}$, R^{ijk}.

Three equations, three viewpoints

These three equations may be interpreted in three different but related ways:

- As fluxes and Bianchi identities in (duality twisted) string compactification.
- As the local form of the axioms of a Courant algebroid on E_1.
- As the gauge invariance & closure for the Courant sigma-model:

$$S[X,A,F]=\int_{\Sigma_3}\Big(F_i\wedge \mathrm{d}X^i+\tfrac{1}{2}\eta_{IJ}\,A^I\wedge \mathrm{d}A^J-\rho^i{}_J(X)\,A^J\wedge F_i+\tfrac{1}{6}\,T_{IJK}(X)\,A^I\wedge A^J\wedge A^K\Big)\ ,$$

with $\rho^i{}_J$ being the anchor components and T_{IJK} the X-pull-back of $\langle e_I,[e_J,e_K]\rangle$.

Three equations, three viewpoints; the DFT case

Similarly, three equations that may be found in the flux formulation of double field theory
Geissbühler, Marqués, Núñez, Penas '13

$$\begin{aligned} \eta^{IJ}\,\rho^K{}_I\,\rho^L{}_J &= \eta^{KL}\,, \\ 2\rho^L{}_{[I}\,\partial_{\underline{L}}\rho^K{}_{J]} - \eta^{MN}\,\rho^K{}_M\,\hat{T}_{NIJ} &= \rho_{L[I}\,\partial^K\rho^L{}_{J]}\,, \\ 4\,\rho^M{}_{[L}\,\partial_{\underline{M}}\hat{T}_{IJK]} + 3\,\eta^{MN}\,\hat{T}_{M[IJ}\,\hat{T}_{KL]N} &= \mathcal{Z}_{IJKL}\,, \end{aligned}$$

correspond to the gauge structure of a doubled membrane sigma-model

$$S_{\mathrm{DFT}}[\mathbb{X},A,F] = \int_{\Sigma_3}\Big(F_I\wedge \mathrm{d}\mathbb{X}^I + \eta_{IJ}A^I\wedge \mathrm{d}A^J - (\rho_+)^I{}_J A^J\wedge F_I + \tfrac{1}{3}\,\hat{T}_{IJK}A^I\wedge A^J\wedge A^K\Big)\ .$$

Gauge symmetries and their closure (or the classical master equation), obstructed.

Obstructions vanish when a world volume analog of the strong constraint is satisfied.

"Algebroid" structure (properties of the C-bracket): Metric or DFT algebroid.
Vaisman '12; A.Ch., Jonke, Khoo, Szabo '18; Mori, Sasaki, Shiozawa '19

Membranes?

Is there a set of equations that captures the following?

- The fluxes and Bianchi identities of $SL(5)$ M-theory compactifications.
- An algebroid structure related to the higher Courant bracket.
- The gauge structure of a sigma-model for higher WZ terms.

Threebrane Sigma-Models

The starting point is a topological threebrane sigma-model with action functional
Ikeda, Uchino '10

$$S[X,\alpha,A,F] = \int_{\Sigma_4} \Big(F_i \wedge \mathrm{d}X^i - \alpha_I \wedge \mathrm{d}A^I + \rho^i{}_I(X)\, F_i \wedge A^I + \tfrac{1}{2}\, S^{IJ}(X)\, \alpha_I \wedge \alpha_J$$
$$+ \tfrac{1}{2}\, T^I{}_{JK}(X)\, \alpha_I \wedge A^J \wedge A^K + \tfrac{1}{4!}\, G_{IJKL}(X)\, A^I \wedge A^J \wedge A^K \wedge A^L \Big) \ .$$

Ingredients:

- Scalars $X = (X^i) : \Sigma_4 \longrightarrow M$.
- Auxiliary 3-form $F \in \Omega^3(\Sigma_4, X^* T^* M)$.
- 1-form $A \in \Omega^1(\Sigma_4, X^* E)$ & 2-form $\alpha \in \Omega^2(\Sigma_4, X^* E^*)$;
  E being some vector bundle. E-indices $I, J, \dots$.
- Structure functions ρ, S, T, G of $X(\sigma)$.

Gauge symmetries

Invariance under gauge symmetries with 0-, 1- and 2-form parameters $(\epsilon^I, \zeta_I, t_i)$:

$$\delta X^i = -\rho^i{}_I\,\epsilon^I\ ,$$

$$\delta A^I = \mathrm{d}\epsilon^I + S^{IJ}\,\zeta_J - T^I{}_{JK}\,A^J\,\epsilon^K\ ,$$

$$\delta\alpha_I = \mathrm{d}\zeta_I + \rho^i{}_I\,t_i + T^J{}_{IK}\,\zeta_J \wedge A^K + T^J{}_{IK}\,\alpha_J\,\epsilon^K + \tfrac{1}{2}\,G_{IJKL}\,\epsilon^J\,A^K \wedge A^L\ ,$$

$$\begin{aligned}\delta F_i = {} & -\mathrm{d}t_i + \partial_i\rho^j{}_I\,(\epsilon^I\,F_j + t_j \wedge A^I) - \partial_i T^J{}_{LI}\,\epsilon^I\,\alpha_J \wedge A^L \\ & - \tfrac{1}{6}\,\partial_i G_{IJKL}\,\epsilon^I\,A^J \wedge A^K \wedge A^L + \tfrac{1}{2}\,\partial_i T^I{}_{JK}\,\zeta_I \wedge A^J \wedge A^K + \partial_i S^{IJ}\,\zeta_I \wedge \alpha_J\ ,\end{aligned}$$

provided that...

Five equations

...provided that five conditions hold for the structure functions:

$$\rho^i{}_I\, S^{IJ} = 0 \,,$$

$$\rho^i{}_I\, \partial_i S^{JK} + S^{LJ}\, T^K{}_{IL} + S^{LK}\, T^J{}_{IL} = 0 \,,$$

$$\rho^i{}_I\, \partial_i \rho^j{}_J - \rho^i{}_J\, \partial_i \rho^j{}_I - \rho^i{}_K\, T^K{}_{IJ} = 0 \,,$$

$$3\rho^i{}_{[I}\, \partial_i T^J{}_{KL]} + S^{JM}\, G_{KLIM} - 3\, T^J{}_{M[K}\, T^M{}_{LI]} = 0 \,,$$

$$\rho^i{}_{[I}\, \partial_i G_{JKLM]} + T^N{}_{[IJ}\, G_{KLM]N} = 0 \,.$$

Moreover, local form of the axioms for a structure called Lie algebroid up to homotopy.
Ikeda, Uchino ’10

A special “*H*”-twisted Lie algebroid on *E*, with anchor, sym. 2-vector, E4-form & bracket.
Grützmann ’10; Grützmann, Strobl “’14”

Some Examples

- For $E = TM$, anchor being projection $\rho^i{}_j = \delta^i{}_j$ and $S = T = 0 \;\rightsquigarrow\; \mathrm{d}G = 0$.

  On-shell (with boundary metric term) describes M2-branes coupled to 3-form C:
  Kökenyesi, Sinkovics, Szabo '18

$$S_\partial[X] = \oint_{\partial\Sigma_4} \left(\tfrac{1}{2}\, g_{ij}\, \mathrm{d}X^i \wedge * \,\mathrm{d}X^j + \tfrac{1}{3!}\, C_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \right) .$$

- For $\rho^i{}_j = \mathrm{E}^i{}_j(\mathrm{X})$ the "vielbein" for a twisted 4-torus, and T its geometric flux,

$$\mathrm{d}\mathrm{E}^i = -\tfrac{1}{2}\, T^i{}_{jk}\, \mathrm{E}^j \wedge \mathrm{E}^k \ ,$$

  M-theory background on twisted torus, Hull, Reid-Edwards '06

$$S_\partial[X] = \oint_{\partial\Sigma_4} \tfrac{1}{2}\, g_{ij}\, \mathrm{E}^i \wedge * \,\mathrm{E}^j \ .$$

  Turning on G too, one gets the expected algebraic identity $T^n{}_{[ij}\, G_{klm]n} = 0$.

A different example

- Recall: In the Courant algebroid, a consistent choice is cf. Besso, Heller, Ikeda, Watamura '15

$$\rho^i{}_J = (0, \Pi^{ij}) \ , \quad [\Pi, \Pi]_{\mathrm{S}} = 0 \ , \quad Q = \mathrm{d}\Pi \ , \quad [\Pi, R]_{\mathrm{S}} = 0 \ .$$

This leads to a *geometric* R-flux model

$$\begin{aligned} S_{R,\Pi}[X] &= \oint_{\partial\Sigma_3} \tfrac{1}{2} \left(g_{ij} - \mathit{\Pi}^{-1}_{ik}\, g^{kl}\, \mathit{\Pi}^{-1}_{lj} \right) \mathrm{d}X^i \wedge * \, \mathrm{d}X^j \\ &\quad + \int_{\Sigma_3} \tfrac{1}{3!}\, R^{pqr}\, \mathit{\Pi}^{-1}_{ip}\, \mathit{\Pi}^{-1}_{jq}\, \mathit{\Pi}^{-1}_{kr}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \ . \end{aligned}$$

- In the present setting, for $E = T^*M$, and the choices (with 4-vector G):

$$\rho^{ij} = \Pi^{ij} \ , \quad T_i^{jk} = \partial_i \Pi^{jk} , \quad [\Pi, G]_{\mathrm{S}} = 0 \ ,$$

M2-branes in 4-form flux controlled by a 4-vector G,

$$\begin{aligned} S_\partial[X] &= \oint_{\partial\Sigma_4} \tfrac{1}{2} \left(g_{ij} - \mathit{\Pi}^{-1}_{ik}\, g^{kl}\, \mathit{\Pi}^{-1}_{lj} \right) \mathrm{d}X^i \wedge * \, \mathrm{d}X^j \\ &\quad + \int_{\Sigma_4} \tfrac{1}{4!}\, G^{pqrs}\, \mathit{\Pi}^{-1}_{ip}\, \mathit{\Pi}^{-1}_{jq}\, \mathit{\Pi}^{-1}_{kr}\, \mathit{\Pi}^{-1}_{ls}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \wedge \mathrm{d}X^l \ . \end{aligned}$$

$SL(5)$ fluxes from the five conditions?

In the above setting, it is not possible to obtain the rest of $SL(5)$ fluxes.

$SL(5)$ fluxes from the five conditions?

In the above setting, it is not possible to obtain the rest of $SL(5)$ fluxes.

The structure of R indicates that $T^I{}_{JK}$ may contain it for enlarged I-index range.

The bundle index I should take 10 values; 4 of type i, j and six of paired type $[ij], [kl]$.

$SL(5)$ fluxes from the five conditions?

In the above setting, it is not possible to obtain the rest of $SL(5)$ fluxes.

The structure of R indicates that $T^I{}_{JK}$ may contain it for enlarged I-index range.

The bundle index I should take 10 values; 4 of type i, j and six of paired type $[ij], [kl]$.

A proposal is to consider the same model, but start with

$$E = E_2 = TM \oplus \wedge^2 T^* M \ .$$

For the anchor components we make a general choice, not just projection to TM,

$$(\rho^i{}_I) = (\rho^i{}_j, \rho^{ijk}) = (\delta^i{}_j, \tfrac{1}{2}\,\Omega^{ijk}) \ .$$

Setting S and G to zero, there is a non-trivial identification of $T_{IJ}{}^K$ with the $SL(5)$ fluxes.

This is special; it requires an additional projection to $SL(5)$ representations.

$SL(5)$ fluxes as Wess-Zumino terms

Going back to the threebrane sigma-model, the E-valued fields now become:

$$A^I = (A^i, A_{ij}) =: (q^i, p_{ij}) \ , \quad \text{(1-forms)}$$

$$\alpha_I = (\alpha_i, \alpha^{ij}) =: (p_i, q^{ij}) \ , \quad \text{(2-forms)}$$

Apparent overabundance; but the flux identification says two additional things:

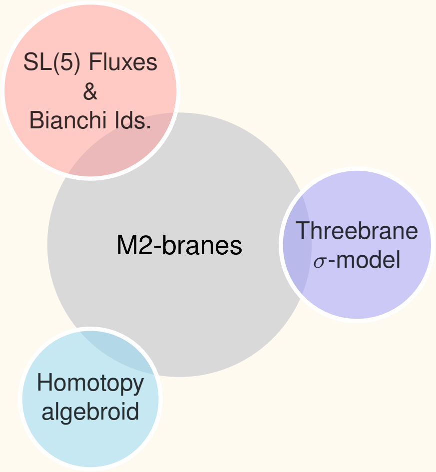
- q^{ij} is decomposable, i.e. $q^{ij} = q^i \wedge q^j$.
- $p_j = q^i \wedge p_{ij}$.

This leads to the sigma-model with WZ terms being the four types of M-theory fluxes

$$\begin{aligned} S = \int_{\Sigma_4} \Big(& F_i \wedge \mathrm{d}X^i - q^i \wedge p_{ij} \wedge \mathrm{d}q^j - q^i \wedge q^j \wedge \mathrm{d}p_{ij} + F_i \wedge q^i + \tfrac{1}{2}\, \Omega^{ijk} F_i \wedge p_{jk} \\ & + \tfrac{3}{2}\, F^m{}_{jk}\, q^i \wedge q^j \wedge q^k \wedge p_{im} + \tfrac{1}{2}\, G_{ijkl}\, q^i \wedge q^j \wedge q^k \wedge q^l \\ & + \tfrac{3}{2}\, Q_l{}^{ijk}\, q^l \wedge q^m \wedge p_{im} \wedge p_{jk} + \tfrac{1}{2}\, R^{jk,lm,i}\, q^n \wedge p_{ni} \wedge p_{jk} \wedge p_{lm} \Big) \ . \end{aligned}$$

Boundary metric terms are entries of $\mathcal{H}_2$, e.g. the characteristic $g^{ijkl} = g^{ik} g^{jl} - g^{il} g^{jk}$.

Epilogue



SL(5) Fluxes
&
Bianchi Ids.
M2-branes
Threebrane
σ-model
Homotopy
algebroid

Some open problems

- Does this hold for larger U-duality groups?

  M5-branes, extended bundles are not as simple as E_p. but known... Hull '07

- Is there a sigma-model with extended base, not only bundle?

  Manifest U-duality in the world volume is hard... Duff, Lu, Percacci, Pope, Samtleben, Sezgin '15

  Origin of the section condition?

- M-theory R-flux shows signs of non-associativity, how to capture it here?

  Günaydin, Lüst, Malek '16; Kupriyanov, Szabo '17

  As in the previous step, for the stringy R-flux. Mylonas, Schupp, Szabo '12

- M-theory exotic brane couplings?