Contact Dual Pairs

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Introduction

A *symplectic dual pair* consists of a pair of surjective Poisson maps

$$(M_1, \{-,-\}_1) \xleftarrow{\varphi_1} (M, \omega) \xrightarrow{\varphi_2} (M_2, \{-,-\}_2)$$

satisfying the orthogonality condition: $\ker T\varphi_1 = (\ker T\varphi_2)^{\perp\omega}$.

Tracing back to [LIE, 1890], with their modern origin in the study of the local structure of Poisson manifolds [WEINSTEIN, 1983], they play a central role in:

- Morita equivalence of Poisson manifolds,
- superintegrable Hamiltonian systems,
- moment maps and reduction theory.

The close analogy between symplectic/Poisson & contact/Jacobi geometry $\rightsquigarrow$

what, if any, is the contact analogue of symplectic dual pairs?

Outline

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Locally conformal symplectic (lcs) manifolds

A *lcs structure* on a manifold M consists of a representation ∇ of TM on a line bundle $L \to M$ and non-degenerate d_∇-closed $\omega \in \Omega^2(M; L)$.

- In the *coorientable case* ($L = M \times \mathbb{R}$), a lcs structure reduces to a closed $\eta \in \Omega^1(M)$ and a non-degenerate $\omega \in \Omega^2(M)$, such that $d\omega + \omega \wedge \eta = 0$,
- $\Gamma(L) \to \mathfrak{X}(M)$, $\lambda \mapsto \mathcal{X}_\lambda := \omega^\sharp(d_\nabla \lambda)$, (linear 1st order) DO from L to TM,
- (skew-symmetric) bi-DO from L to L satisfying the Jacobi identity

$$\{-,-\}_M : \Gamma(L) \times \Gamma(L) \to \Gamma(L), \ (\lambda, \mu) \mapsto \{\lambda, \mu\}_M := \omega(\mathcal{X}_\lambda, \mathcal{X}_\mu),$$

it is the *associated Jacobi structure* and fully encodes the lcs structure.

Contact Structures

A *contact structure* on a manifold M is equivalently described by:

- a *contact distribution* $\mathcal{H} \subset TM$, or
- a *contact form* $\vartheta \in \Omega^1(M; L)$, for some line bundle $L \to M$,

with $L = TM/\mathcal{H}$ and $\vartheta(X) = X \bmod \mathcal{H}$, or in the opposite direction $\mathcal{H} = \ker \vartheta$.

- the *maximally non-integrability* of $\mathcal{H}$ means that the associated *curvature* 2-*form* $\omega_{\mathcal{H}} \in \Gamma(\wedge^2 \mathcal{H}^* \otimes L)$, with $\omega_{\mathcal{H}}(X, Y) := \vartheta[X, Y]$, is non-degenerate,
- in the *coorientable case* ($L = M \times \mathbb{R}$), one gets that $\vartheta \in \Omega^1(M)$, and the maximal non-integrability of $\mathcal{H}$ reduces to $(\mathrm{d}\vartheta)^n \wedge \vartheta$ is a volume form on M^{2n+1},
- $\Gamma(L) \to \mathfrak{X}(M),\ \lambda \mapsto \mathcal{X}_\lambda$, (1st order linear) DO from L to TM,
- (skew-symmetric) bi-DO from L to L satisfying the Jacobi identity

$$\{-,-\}_\vartheta : \Gamma(L) \times \Gamma(L) \to \Gamma(L),\ (\lambda, \mu) \mapsto \{\lambda, \mu\}_\vartheta := \vartheta[\mathcal{X}_\lambda, \mathcal{X}_\mu],$$

it is the *associated Jacobi structure* and fully encodes the contact structure.

Outline

The gauge (or Atiyah) algebroid of a line bundle

The DOs from L to L can be seen as sections of $DL := (J^1L)^* \otimes L \to M$, the *gauge algebroid*, with Lie algebroid structure on DL and DL-representation on L

$$[\Delta, \Delta'] := \Delta \circ \Delta' - \Delta' \circ \Delta, \quad \rho(\Delta) := \sigma_\Delta, \quad \nabla_\Delta := \Delta, \qquad \forall \Delta, \Delta' \in \Gamma(DL).$$

Each *regular* line bundle morphism $\Phi : L_1 \to L_2$, over $\varphi : M_1 \to M_2$, determines a morphism of gauge algebroids $D\Phi : DL_1 \to DL_2$.

- *Schouten–Jacobi bracket*, i.e. graded Lie bracket $[\![-,-]\!]_{SJ}$ on $(\mathcal{D}^\bullet L)[1]$, with

$$\mathcal{D}^\bullet L := \Gamma(\wedge^\bullet (J^1L)^* \otimes L) = \{\text{multi-DOs from } L \text{ to } L\},$$

- *der-complex* $(\Omega_D^\bullet(L), \mathrm{d}_D)$, with $\Omega_D^\bullet(L) := \Gamma(\wedge^\bullet (DL)^* \otimes L) = \{\textit{Atiyah forms}\}$.

Jacobi manifolds I

A *Jacobi bundle* [MARLE] is a line bundle $L \to M$ with a *Jacobi structure*, i.e. a Lie bracket $\{-,-\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ which is a (linear first order) bi-DO. A *Jacobi manifold* is a manifold with a Jacobi bundle over it.

$$\{-,-\} \text{ a Jacobi structure on } L \Longleftrightarrow \begin{cases} \mathcal{J} \in \mathrm{MC}((\mathcal{D}^\bullet L)[1], [\![-,-]\!]_{SJ}) \\ \text{i.e. } \mathcal{J} \in \mathcal{D}^2 L \text{ and } [\![\mathcal{J},\mathcal{J}]\!]_{SJ} = 0. \end{cases}$$

A *Jacobi morphism* $(M_1, L_1, \{-,-\}_1) \to (M_2, L_2, \{-,-\}_2)$ is a *regular* line bundle morphism $\Phi : L_1 \to L_2$ s.t. the pull-back of sections $\Phi^* : \Gamma(L_2) \to \Gamma(L_1)$ satisfies

$$\Phi^*\{\lambda,\mu\}_2 = \{\Phi^*\lambda, \Phi^*\mu\}_1, \quad \forall \lambda,\mu \in \Gamma(L_2).$$

Jacobi manifolds II

- *coorientable case* ($L = M \times \mathbb{R}$), a Jacobi structure reduces to a *Jacobi pair* (Π, E), i.e. a bi-vector $\Pi \in \mathfrak{X}^2(M)$ and a vector $E \in \mathfrak{X}(M)$ such that:

$$[\![\Pi, \Pi]\!] = \Pi \wedge E, \qquad [\![E, \Pi]\!] = 0.$$

- In particular, any Poisson structure is a Jacobi structure.

Proposition For any $L \to M$, there is a canonical 1-1 correspondence between:

- L-valued contact forms $\vartheta \in \Omega^1(M; L)$,
- *symplectic Atiyah forms* ϖ, i.e. non-deg. $\varpi \in \Omega^2_D(L)$ with $\mathrm{d}_D \varpi = 0$,
- *non-degenerate* Jacobi structures $\mathcal{J} \in \mathcal{D}^2 L$, i.e. such that $\mathcal{J}^\sharp : J^1 L \to DL$ iso.

Outline

Contact Groupoids

Let $\mathcal{G} \rightrightarrows \mathcal{G}_0$ be a Lie groupoid, with structure maps s, t, m, u, i. A *multiplicative contact structure* on $\mathcal{G} \rightrightarrows \mathcal{G}_0$ is equivalently described by:

- contact distributions $\mathcal{H}$ on $\mathcal{G}$ which are *multiplicative*, i.e. $\mathcal{H}$ is a *wide* subgroupoid of the tangent groupoid $T\mathcal{G} \rightrightarrows T\mathcal{G}_0$,
- contact forms $\vartheta \in \Omega^1(\mathcal{G}; t^*L_0)$, with values in a representation $L_0 \to \mathcal{G}_0$ of $\mathcal{G}$, which are *multiplicative*, i.e. for all $(g, h) \in \mathcal{G}^{(2)}$

  $$(m^*\vartheta)_{(g,h)} = (\mathrm{pr}_1^*\vartheta)_{(g,h)} + g \cdot (\mathrm{pr}_2^*\vartheta)_{(g,h)},$$

  where $\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{G}^{(2)} := \mathcal{G}_s \times_t \mathcal{G} \to \mathcal{G}$ are the projections.

A *contact groupoid* is a Lie groupoid with a multiplicative contact structure.

From Contact Groupoids to Contact Dual Pairs

Let $\mathcal{G} \rightrightarrows \mathcal{G}_0$ be a contact groupoid, $\vartheta \in \Omega^1(\mathcal{G}; t^*L_0)$, $\mathcal{H} = \ker\vartheta$ and $L = T\mathcal{G}/\mathcal{H}$.

- The left and right translations induce regular line bundle morphisms $S : L \to L_0$ and $T : L \to L_0$ covering respectively $s : \mathcal{G} \to \mathcal{G}_0$ and $t : \mathcal{G} \to \mathcal{G}_0$.
- There is a unique Jacobi structure $\mathcal{J}_0 = \{-,-\}_0$ on $L_0 \to \mathcal{G}_0$ such that

$$(\mathcal{G}_0, L_0, \mathcal{J}_0) \xleftarrow{S} (\mathcal{G}, \mathcal{H}) \xrightarrow{T} (\mathcal{G}_0, L_0, -\mathcal{J}_0) \text{ are Jacobi morphisms.}$$

1. $\mathcal{H}$ is transverse to both $T^s\mathcal{G}$ and $T^t\mathcal{G}$,
2. $S^*\lambda$ and $T^*\mu$ commute wrt $\{-,-\}_{\mathcal{H}}$, for all $\lambda, \mu \in \Gamma(L)$,
3. $\mathcal{H}^s := \mathcal{H} \cap T^s\mathcal{G}$ is the orthogonal complement of $\mathcal{H}^t := \mathcal{H} \cap T^t\mathcal{G}$ wrt $\omega_{\mathcal{H}}$.

So contact groupoids lead us to single out the notion of contact dual pair.

Contact Dual Pairs

Consider a contact manifold (M, L, ϑ) and a pair of Jacobi morphisms

$$(M_1, L_1, \{-,-\}_1) \xleftarrow{\Phi_1} (M, L, \vartheta) \xrightarrow{\Phi_2} (M_2, L_2, \{-,-\}_2), \qquad (\star)$$

covering surjective submersions $M_1 \xleftarrow{\varphi_1} M \xrightarrow{\varphi_2} M_2$.

Definition The above diagram $(\star)$ forms a *contact dual pair* if:

1. $\mathcal{H} = \ker \vartheta$ is transverse to the fibers of $\varphi_1 : M \to M_1$ and $\varphi_2 : M \to M_2$,
2. $\Phi_1^* \lambda_1$ and $\Phi_2^* \lambda_2$ commute wrt $\{-,-\}_{\mathcal{H}}$, for all $\lambda_1 \in \Gamma(L_1)$ and $\lambda_2 \in \Gamma(L_2)$,
3. $\mathcal{H} \cap \ker T\varphi_1$ is the orthogonal complement of $\mathcal{H} \cap \ker T\varphi_2$ wrt $\omega_{\mathcal{H}}$.

Proposition Diagram $(\star)$ is a contact dual pair iff $\ker D\Phi_1 = (\ker D\Phi_2)^{\perp \varpi}$, where $\varpi \in \Omega^2_D(L)$ is the symplectic Atiyah form corresponding to ϑ.

Outline

Contact Actions

A *contact action* of a contact groupoid $\mathcal{G} \rightrightarrows \mathcal{G}_0$ on a contact manifold M, with *moment* $\mu : M \to \mathcal{G}_0$, is a map $\Phi : \mathcal{G}\,{}_s\!\times_\mu M \to M$, $(g, x) \mapsto g \cdot x$, s.t.

$$\mu(g \cdot x) = t(g), \quad \mu(x) \cdot x = x, \quad (gh) \cdot x = g \cdot (hx),$$

and these two equivalent *compatibility properties* hold:

1. $(T\Phi)(u, v) \in \mathcal{H}_M \iff u \in \mathcal{H}_{\mathcal{G}}$; for all $(u, v) \in T(\mathcal{G}\,{}_s\!\times_\mu M)$, with $v \in \mathcal{H}_M$,
2. μ lifts to a regular LB morphism $\widehat{\mu} : L_M \to L_0$ (so that $L_M \simeq \mu^* L_0$) and

$$(\Phi^* \vartheta_M)_{(g,x)} = (\mathrm{pr}_1^* \vartheta_{\mathcal{G}})_{(g,x)} + g \cdot (\mathrm{pr}_2^* \vartheta_M)_{(g,x)}.$$

- $\mathcal{G}^{(2)} \xrightarrow{m} \mathcal{G}$ is a contact action of $\mathcal{G}$ on $\mathcal{G}$ with moment $\mathcal{G} \xrightarrow{t} \mathcal{G}_0$.
- Lie group action $G \curvearrowright M$ by contactomorphisms + *transversality* give rise to a moment $\mu : M \to \mathbb{P}(\mathfrak{g}^*)$ and a contact action of $\mathbb{P}(T^*G) \rightrightarrows \mathbb{P}(\mathfrak{g}^*)$ on M.

CDPs from Contact Reduction

Fix a *free and proper* contact action of a *s-connected* $\mathcal{G} \rightrightarrows \mathcal{G}_0$ on M with moment $\mu : M \to \mathcal{G}_0$. Assume that μ is *surjective* and has *connected fibers*. Then one gets:

- μ is a submersion, and its lift $\widehat{\mu} : L_M \to L_0$ is a Jacobi morphism,
- there exists a unique Jacobi structure $\{-,-\}_{M/\mathcal{G}}$ on $L_M/\mathcal{G} \to M/\mathcal{G}$ s.t. quotient map $\widehat{q} : L_M \to L_M/\mathcal{G}$ is a Jacobi morphism covering $q : M \to M/\mathcal{G}$.

Proposition The following is a contact dual pair, with connected fibers,

$$(M/\mathcal{G}, L_{M/\mathcal{G}}, \{-,-\}_{M/\mathcal{G}}) \xleftarrow{\widehat{q}} (M, \mathcal{H}_M) \xrightarrow{\widehat{\mu}} (\mathcal{G}_0, L_0, \{-,-\}_0).$$

Corollary (Groupoid Contact Reduction [Zambon & Zhu]) The *reduced orbit spaces* $\mu^{-1}(\mathcal{O})/\mathcal{G}$ are *transitive Jacobi manifolds*, for all orbits $\mathcal{O}$ of $\mathcal{G}$ in $\mathcal{G}_0$, and they form the characteristic foliation of $(M/\mathcal{G}, L_M/\mathcal{G}, \{-,-\}_{M/\mathcal{G}})$.

Outline

A Glimpse at the Local Structure of Jacobi Manifolds

Let $(M, L, \mathcal{J} = \{-,-\})$ be a Jacobi manifold.

- The *characteristic foliation* $\mathcal{F}$ of M s.t. $T\mathcal{F} := \operatorname{im}\{J^1L \xrightarrow{\mathcal{J}^\sharp} DL \xrightarrow{\sigma} TM\} \subset TM$.
- If $T\mathcal{F} = TM$, then the Jacobi manifold is said to be *transitive*.

Characteristic Foliation Theorem [Kirillov '76]

1. Each characteristic leaf inherits a transitive Jacobi structure, with
   - odd-dim transitive Jacobi manifolds $\xleftrightarrow{\text{1-1}}$ contact manifolds,
   - even-dim transitive Jacobi manifolds $\xleftrightarrow{\text{1-1}}$ lcs manifolds.

Transverse Structure [Dazord, Lichnerowicz, Marle '91]

1. Each characteristic leaf inherits a transverse structure, specifically:
   - contact leaf $\rightsquigarrow$ transverse homogeneous Poisson structure,
   - lcs leaf $\rightsquigarrow$ transverse Jacobi structure.

CDPs & Local Structure of Jacobi Manifolds

Consider a contact dual pair

$$(M_1, L_1, \{-,-\}_1) \xleftarrow{\Phi_1} (M, L, \vartheta) \xrightarrow{\Phi_2} (M_2, L_2, \{-,-\}_2),$$

with additionally *connected fibers* of the underlying maps $M_1 \xleftarrow{\varphi_1} M \xrightarrow{\varphi_2} M_2$.

The Characteristic Leaf Correspondence Theorem

1. The relation $\varphi_1^{-1}(\mathcal{S}_1) = \varphi_2^{-1}(\mathcal{S}_2)$ establishes a 1-1 correspondence between the characteristic leaves $\mathcal{S}_1$ of M_1 and the characteristic leaves $\mathcal{S}_2$ of M_2.
2. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be in correspondence. Then $\dim \mathcal{S}_1 \equiv \dim \mathcal{S}_2 \bmod 2$, and
   - there exists a close relation between:

     contact (resp. lcs) structure of $\mathcal{S}_1 \rightleftarrows$ contact (resp. lcs) structure of $\mathcal{S}_2$,

   - there exists a canonical anti-isomorphism between:

     transverse structure to $\mathcal{S}_1 \simeq$ transverse structure to $\mathcal{S}_2$.

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Thank you!