BRST Reduction of Quantum Algebras with *-Involution

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Based on joint work with Stefan Waldmann

Outline

Motivation

Classical point of view | **Quantized point of view**

$$(M, \omega) \xrightarrow{\text{DQ}} (M, \star)$$

- Observable algebra $\mathscr{C}^\infty(M)$
- Pointwise product $\cdot$
- Poisson bracket $\{\cdot, \cdot\}$

- Associative $\star$ on $\mathscr{C}^\infty(M)[[\hbar]]$
- $f \star g = f \cdot g + \hbar(\dots)$
- $[f, g]_\star = \mathrm{i}\hbar\{f, g\} + \hbar^2(\dots)$

Motivation

Classical point of view **Quantized point of view**

$$
\begin{tikzcd}
 & \mathrm{G} \circlearrowright (M, \omega) \arrow[rr, "\mathrm{DQ}"] & & (M, \star, \boldsymbol{J}) \arrow[dd, "\mathrm{BRST}"] \\
C = J^{-1}(\{0\}) \arrow[ur, "\iota"] \arrow[dr, "\pi"'] & & & \\
 & (M_{\mathrm{red}} = C/\mathrm{G}, \omega_{\mathrm{red}}) \arrow[rr, "\mathrm{DQ}"] & & (M_{\mathrm{red}}, \star_{\mathrm{red}})
\end{tikzcd}
$$

The Classical BRST Approach

- **Setting:** $\mathrm{G} \circlearrowright (M, \omega)$, equivariant momentum map $J \colon M \to \mathfrak{g}^*$ with regular value $0 \in \mathfrak{g}^*$ and regular constraint surface $C = J^{-1}(\{0\})$
- Proper action on M and free action on C, reduced manifold $M_{\mathrm{red}} = C/\mathrm{G}$, prolongation map $\mathrm{prol} \colon \mathscr{C}^\infty(C) \to \mathscr{C}^\infty(M)$
- Cohomological description of the phase space reduction:

1. $M \longrightarrow C$: Koszul complex $(\Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M), \partial)$ with $\partial = J_i \, \mathrm{i}(\mathrm{e}^i)$ and

$$\mathrm{H}_0^{\mathrm{Kos}}(\mathscr{C}^\infty(M)) \cong \mathscr{C}^\infty(C)$$

2. $C \longrightarrow M_{\mathrm{red}}$: Chevalley-Eilenberg complex $(\Lambda^\bullet \mathfrak{g}^* \otimes \mathscr{C}^\infty(C), \delta_{\mathrm{CE}})$ with

$$\mathrm{H}^0_{\mathrm{CE}}(\mathscr{C}^\infty(C)) = \mathscr{C}^\infty(C)^{\mathfrak{g}} \cong \mathscr{C}^\infty(M_{\mathrm{red}})$$

The Classical BRST Approach II

Classical BRST algebra

BRST algebra $\mathcal{A} = \Lambda^\bullet \mathfrak{g}^* \otimes \Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M)$ with

- Ghost number grading: $\mathcal{A}^{(n)} = \bigoplus_{n=k-l} \Lambda^k \mathfrak{g}^* \otimes \Lambda^l \mathfrak{g} \otimes \mathscr{C}^\infty(M)$
- BRST operator $D = \delta_{\mathrm{CE}} + 2\,\partial \colon \mathcal{A}^{(\bullet)} \longrightarrow \mathcal{A}^{(\bullet+1)}$ with $D^2 = 0$
- super Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ induced by natural pairing of $\mathfrak{g}$ and $\mathfrak{g}^*$
- Ghost number grading induced by $\mathsf{Gh} = \{\gamma, \,\cdot\,\}$ with *ghost charge* $\gamma \in \mathcal{A}^{(0)}$
- $D = \{\Theta, \,\cdot\,\}$ with *BRST charge* $\Theta \in \mathcal{A}^{(1)}$

$$\Longrightarrow \qquad \mathrm{H}^{(0)}_{\mathrm{BRST}}(\mathcal{A}) = \frac{\ker D \cap \mathcal{A}^{(0)}}{\operatorname{im} D \cap \mathcal{A}^{(0)}} \cong \mathscr{C}^\infty(M_{\mathrm{red}})$$

BRST Reduction in Deformation Quantization

- Equivariant star product $(\star, \boldsymbol{J})$ on M, i.e. a quantum momentum map $\boldsymbol{J} = J + \hbar(\dots)\colon M \to \mathfrak{g}^*[[\hbar]]$ with

$$\boldsymbol{J}(\xi)\star\boldsymbol{J}(\eta)-\boldsymbol{J}(\eta)\star\boldsymbol{J}(\xi) = \mathrm{i}\hbar\mathbf{J}([\xi,\eta]), \quad \boldsymbol{J}(\xi)\star f - f\star\boldsymbol{J}(\xi) = \mathrm{i}\hbar\{J(\xi), f\}$$

- Construct star product on $\mathcal{A}[[\hbar]] = \Lambda^\bullet\mathfrak{g}^* \otimes \Lambda^\bullet\mathfrak{g} \otimes \mathscr{C}^\infty(M)[[\hbar]]$ and quantum BRST operator $\boldsymbol{D} = \frac{1}{\mathrm{i}\hbar}\operatorname{ad}(\boldsymbol{\Theta})$ deforming D, Θ.

Theorem (Bordemann, Herbig, Waldmann; 2000)

Let G act properly on M and freely on C, then one has in the above setting

$$\boldsymbol{\iota}^*\colon \mathbf{H}^{(0)}_{\mathrm{BRST}}(\mathcal{A}[[\hbar]]) \xrightarrow{\cong} \mathscr{C}^\infty(C)^{\mathrm{G}}[[\hbar]] = \pi^*\mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]],$$

inducing a star product $\star_{\mathrm{red}}$ on M_{red}.

Compatibility with *-Involution

- We are interested in *-algebras, so assume

$$\overline{f \star g} = \overline{g} \star \overline{f} \qquad \forall\, f, g \in \mathscr{C}^\infty(M)[[\hbar]]$$

- **AIM**: Involution on $\mathcal{A}[[\hbar]] = \Lambda^\bullet \mathfrak{g}^* \otimes \Lambda^\bullet \mathfrak{g} \otimes \mathscr{C}^\infty(M)[[\hbar]]$ inducing involution on $\mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]]$

First Try

Involution * such that $\mathbf{\Theta} = \mathbf{\Theta}^\star$

$\Longrightarrow$ Canonical involution on $\ker \boldsymbol{D}$ and on $\mathbf{H}^{(\bullet)}_{\mathrm{BRST}}(\mathcal{A}[[\hbar]])$

BUT: π representation of $\mathcal{A}[[\hbar]]$ on pre-Hilbert space $\mathcal{H}$, then for all $\psi \in \mathcal{H}$

$$\langle \pi(\mathbf{\Theta})\psi, \pi(\mathbf{\Theta})\psi \rangle = \langle \psi, \pi(\mathbf{\Theta}^\star \mathbf{\Theta})\psi \rangle = \langle \psi, \pi(\mathbf{\Theta}^2)\psi \rangle = 0$$

Compatibility with *-Involution

- Involution on $(\Lambda^\bullet \mathfrak{g}^* \otimes \Lambda^\bullet \mathfrak{g})[[\hbar]]$ induced by positive definite inner product g on $\mathfrak{g}$ via

$$\mathfrak{g} \underset{\sharp}{\overset{\flat}{\rightleftharpoons}} \mathfrak{g}^*, \qquad \xi^\flat = g(\xi, \cdot)$$

- Imaginary ghost, i.e. $\left(\mathcal{A}^{(k)}[[\hbar]]\right)^* = \mathcal{A}^{(-k)}[[\hbar]]$
- **PROBLEM**: No induced involution on $\ker \boldsymbol{D}$ and $\mathbf{H}^{(\bullet)}_{\mathrm{BRST}}(\mathcal{A}[[\hbar]])$
- Adjoint BRST operator $\boldsymbol{D}^* = \frac{1}{\mathrm{i}\hbar} \operatorname{ad}(\boldsymbol{\Theta}^*) \colon \mathcal{A}^{(\bullet)}[[\hbar]] \to \mathcal{A}^{(\bullet-1)}[[\hbar]]$ with

$$\boldsymbol{D}^* a = (-1)^k (\boldsymbol{D} a^*)^* \quad \text{for} \quad a \in \mathcal{A}^{(k)}[[\hbar]]$$

- Consider BRST quotient

$$\widetilde{\mathbf{H}}^{(\bullet)}_{\mathrm{BRST}}(\mathcal{A}[[\hbar]]) = \frac{\ker \boldsymbol{D} \cap \ker \boldsymbol{D}^*}{\operatorname{im} \boldsymbol{D} \cap \operatorname{im} \boldsymbol{D}^*}$$

Theorem (AK, S. Waldmann)

If G *is compact in the above setting, then*

$$\widetilde{\mathbf{H}}^{(0)}_{\mathrm{BRST}}(\mathcal{A}[[\hbar]]) \cong \mathbf{H}^{(0)}_{\mathrm{BRST}}(\mathcal{A}[[\hbar]]) \cong \mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]],$$

inducing the complex conjugation as involution for $\star_{\mathrm{red}}$.

Main idea of proof:

$$\boldsymbol{\iota}^* : \mathbf{H}^{(0)}_{\mathrm{BRST}}(\mathcal{A}[[\hbar]])\widetilde{\mathbf{H}}^{(0)}_{\mathrm{BRST}}(\mathcal{A}[[\hbar]]) \xrightarrow{\cong} \mathscr{C}^\infty(C)^{\mathrm{G}}[[\hbar]] = \pi^*\mathscr{C}^\infty(M_{\mathrm{red}})[[\hbar]]$$

- $\mathscr{C}^\infty(C)^{\mathrm{G}}[[\hbar]]$-valued inner product on $\mathscr{C}^\infty(C)[[\hbar]]$

$$\langle \phi, \psi \rangle(c) = \int_{\mathrm{G}} \left(\boldsymbol{\iota}^* \left(\overline{\mathrm{prol}(\phi)} \star \mathrm{prol}(\psi) \right) \right) (g^{-1} \triangleright c) \, \mathrm{d}^{\mathrm{left}} g$$

- Representation $f \bullet \phi = \boldsymbol{\iota}^*(f \star \mathrm{prol}(\phi))$ for $f \in \mathscr{C}^\infty(M)[[\hbar]]$ with

$$\langle \phi, f \bullet \psi \rangle = \langle \overline{f} \bullet \phi, \psi \rangle \Longrightarrow \int_{\mathrm{G}} \boldsymbol{\iota}^* f(g^{-1} \triangleright c) \, \mathrm{d}^{\mathrm{left}} g = \int_{\mathrm{G}} \overline{\boldsymbol{\iota}^* \overline{f}}(g^{-1} \triangleright c) \, \mathrm{d}^{\mathrm{left}} g$$