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Introduction to Noncommutative QFT

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Abstract

Introduction to the main ideas of noncommutative geometry and noncommutative Quantum Field Theory.

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Contents

1 Commutative Quantum Field Theory

We will present a brief introduction to (commutative) Quantum Field Theory using the path integral formalism. We will discuss the quantisation of scalar fields in Section 1.1, of gauge fields in Section 1.2 and renormalisation in Section 1.3. We will basically follow the presentation provided in the book by Lewis Ryder [1].

Conventions. We will use the conventions of [1]. Hence, the Minkowski metric is given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} . \tag{1.1}$$

1.1 Scalar fields

For a start, we will consider four dimensional Minkowski space-time. The action for a free scalar field is given by

$$\begin{aligned} S_0 &= \int d^4x \, \mathcal{L}_0(\phi, \partial_\mu \phi) \\ &= \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \, . \right) \end{aligned} \tag{1.2}$$

The Euler-Lagrange equation

$$\frac{\partial \mathcal{L}_0}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \phi} = 0$$

is given by the Klein-Gordon equation:

$$(\Box + m^2)\phi = 0 \, . \tag{1.3}$$

The path integral is equal to the vacuum to vacuum amplitude

$$Z_0[J] = \int [D\phi] \, \exp\left(\mathrm{i} \int \mathrm{d^4x} \, \left(\mathcal{L}_0 + \mathrm{J}\phi + \mathrm{i}\epsilon\phi^2 \right) \right) \, , \tag{1.4}$$

where we have coupled the scalar field to an external source J. It is not quantised, and somehow represents the link between the quantum world and macroscopic experiments. The source plays an analogous role to the electromagnetic current. A charged scalar field couples to gauge potential A_μ via this current:

$$\mathcal{L}_{em} = -eJ_\mu A^\mu \,, \tag{1.5}$$

where

$$J_\mu = \mathrm{i}(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^*) \,. \tag{1.6}$$

Let us return to Eqn. (1.4). The integral measure $[D\phi]$ is not well defined,

$$[D\phi] = \prod_{x \in \mathbb{R}^4} d\phi(x) \,. \tag{1.7}$$

There are various techniques to give a well defined meaning to the integration measure. One of them is a lattice regularisation, where space-time is approximated by a four dimensional grid with constant lattice spacing δ, for simplicity. Strictly speaking, one has to perform a Wick rotation from Minkowski space-time to the four dimensional Euclidean space first. Then, the calculations have to be performed there, and the results have to be transformed back to Minkowski space. The formulations in both spaces, Minkowski and Euclidean are equivalent. This transition will be discussed briefly in Section 1.1.1. In the noncommutative realm, this transformation is not at all well understood and still an open question. In the first step of the lattice regularisation, space-time is approximated by a finite lattice with N points, say in each direction and lattice spacing δ. Hence, we have the following identifications

$$[D\phi] \longrightarrow \prod_{i_1=1}^{N} \prod_{i_2=1}^{N} \prod_{i_3=1}^{N} \prod_{i_4=1}^{N} d\phi(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) \,, \tag{1.8}$$

$$\int d^4x \longrightarrow \sum_{n=1}^{N^4} \delta^4 \,, \tag{1.9}$$

$$\partial_1 \phi \Big|_{i,j,k,l} \longrightarrow \frac{1}{\delta} \left(\phi(x_i + \delta, y_j, z_k, t_l) - \phi(x_i, y_j, z_k, t_l) \right) \,, \tag{1.10}$$

$$\dots$$

The results will depend on the volume of the lattice and the lattice spacing. In order to obtain continuum results, one first has to perform the infinite volume limit, $N \to \infty$ and then the continuum limit, $\delta \to 0$. This corresponds to the so-called naive continuum-limit, which might not be the correct limit. For the correct limit, one might have to keep some physical quantities fixed.

The field ϕ in (1.4) is not i.g. a solution of the equation of motion (EOM) (1.3), since all fields are considered in the path integral. We can decompose the scalar field in the following way:

$$\phi \to \phi_0 + \phi\,, \tag{1.11}$$

where ϕ_0 is a solution of the Euler-Lagrange equation. Therefore, we obtain for the action

$$\begin{aligned} S &= -\int d^4x\, \frac{1}{2}\big(\phi(\Box + m^2 - i\epsilon)\phi + \phi_0(\Box + m^2 - i\epsilon)\phi_0 \\ &\quad +\phi(\Box + m^2 - i\epsilon)\phi_0 + \phi_0(\Box + m^2 - \mathrm{i}\epsilon)\phi - 2\phi \mathrm{J} - 2\phi_0 \mathrm{J}\big) \\ &= -\int d^4x\, \frac{1}{2}\big(\phi(\Box + m^2 - i\epsilon)\phi - \phi_0 J\big)\,. \end{aligned} \tag{1.12}$$

In general, a solution to the EOM is given by

$$\phi_0(x) = -\int d^4y\, \Delta_F(x-y)J(y)\,, \tag{1.13}$$

where Δ_F is the Green's function of $L_x = \Box + m^2$, i.e.,

$$L_x\Delta(x-y) = -\delta^{(4)}(x-y)\,. \tag{1.14}$$

Acting with L_x on (1.13), we get

$$\mathrm{LHS} = L_x\phi_0(x) = J(x)\,, \tag{1.15}$$

$$\mathrm{RHS} = -\int d^4y\, L_x\Delta_F(x-y)J(y) = J(x)\,. \tag{1.16}$$

Therefore, the path integral becomes

$$\begin{aligned} Z_0[J] &= \exp\left(-\frac{\mathrm{i}}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y)\right) \\ &\quad \times \int [D\phi] \exp\left(-\frac{\mathrm{i}}{2}\int d^4x\, \phi(\Box + m^2 - \mathrm{i}\epsilon)\phi\right) \\ &= \mathcal{N} \exp\left(-\frac{\mathrm{i}}{2}\int d^4x\, d^4y\, J(x)\Delta_F(x-y)J(y)\right)\,, \end{aligned} \tag{1.17}$$

where $\mathcal{N}$ is a normalization factor.

Propagator. The propagator $\Delta_F(x-y)$ is defined as Green's function of the free field equation:

$$\left(\Box + m^2 - \mathrm{i}\epsilon\right) \Delta_F(x-y) = -\delta^{(4)}(x-y)\,. \tag{1.18}$$

Therefore, we obtain after Fourier transformation

$$\begin{aligned} \Delta_F(x) &= \frac{1}{(2\pi)^4}\int d^4k \frac{e^{-\mathrm{ikx}}}{k^2 - m^2 + \mathrm{i}\epsilon} \\ &= \int \frac{d^3k}{(2\pi)^3}\int \frac{dk_0}{2\pi}\frac{e^{-\mathrm{ik_0t}}}{k_0^2 - \mathbf{k}^2 - m^2 + \mathrm{i}\epsilon} e^{\mathbf{ikx}}\,, \end{aligned} \tag{1.19}$$

where $\mathrm{i}\epsilon$ dictates the path of integration in the complex k_0-plane. The poles of the propagator are

$$k_0 = \pm E_{\mathbf{k}} \mp \mathrm{i}\delta\,, \tag{1.20}$$

where $E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$.

- If $t > 0$: Propagation forward in time; we need to close the contour in the upper complex half-plane,

$$\begin{aligned} \Delta_F(x)^> &= \int \frac{d^3k}{(2\pi)^3}\int \frac{dk_0}{2\pi}\frac{e^{-\mathrm{ik_0t}}}{k_0^2 - \mathbf{k}^2 - m^2 + \mathrm{i}\epsilon} e^{\mathbf{ikx}} \\ &= \int \frac{d^3k}{(2\pi)^3}\frac{\mathrm{i}}{2E_{\mathbf{k}}} e^{\mathbf{ikx} - \mathrm{i}(\mathrm{E}_{\mathbf{k}} - \mathrm{i}\delta)} \\ &\xrightarrow{\delta\to 0} \int \frac{d^3k}{(2\pi)^3}\frac{\mathrm{i}}{2E_{\mathbf{k}}} e^{\mathbf{ikx} - \mathrm{iE}_{\mathbf{k}}}\,, \end{aligned} \tag{1.21}$$

where we have used the residue theorem

$$\oint_\gamma f(z)dz = 2\pi\mathrm{i}\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{Res}(\mathrm{f}, \mathrm{a_k}), \tag{1.22}$$

where γ is a closed curve in the complex z-plane and the function f is holomorphic everywhere except at finitely many points a_k in the interior of γ. In our case, we have

$$f(k) = \frac{1}{(k_0 - E_{\mathbf{k}} + \mathrm{i}\delta)(\mathrm{k}_0 + \mathrm{E}_{\mathbf{k}} - \mathrm{i}\delta)}\,, \tag{1.23}$$

and therefore

$$Res(f, k_0 = E_{\mathbf{k}} - \mathrm{i}\delta) = \frac{1}{2\mathrm{E}_{\mathbf{k}} - 2\mathrm{i}\delta} \xrightarrow{\delta\to 0} \frac{1}{2\mathrm{E}_{\mathbf{k}}}\,. \tag{1.24}$$

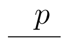
- If $t < 0$: Propagation backwards in time; we need to close the contour in the lower complex half-plane and we obtain

$$\Delta_F(x)^< = \int \frac{d^3k}{(2\pi)^3} \frac{\mathrm{i}}{-2E_{\mathbf{k}}} e^{\mathrm{i}\mathbf{kx}+\mathrm{iE}_{\mathbf{k}}} \,. \tag{1.25}$$

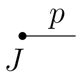
We will use a graphical representation of the free propagator just calculated. The free path integral

$$\begin{aligned} Z_0[J] &= \mathcal{N} e^{-\frac{\mathrm{i}}{2}\int J(x)\Delta_F(x-y)J(y)d^4x d^4y} \\ &= \mathcal{N} e^{-\frac{\mathrm{i}}{2}(2\pi)^4 \int \frac{J(p)J(-p)}{p^2-m^2+\mathrm{i}\epsilon} d^4p} \end{aligned} \tag{1.26}$$

can be expanded in terms of the following basic graphical building blocks:



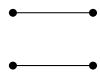
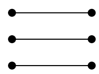
$\frac{\mathrm{i}}{(2\pi)^4} \frac{1}{p^2-m^2+\mathrm{i}\epsilon}$,



$\mathrm{i}(2\pi)^4\mathrm{J(p)}$;

and the partition function (1.26) reads

$$\begin{aligned} Z_0[J]/\mathcal{N} = 1 &- \frac{\mathrm{i}}{2}(2\pi)^4 \int d^4p \frac{J(p)J(-p)}{(p^2-m^2+\mathrm{i}\epsilon)} \\ &- \frac{(2\pi)^8}{8} \int d^4p \frac{J(p)J(-p)}{(p^2-m^2+\mathrm{i}\epsilon)} \int d^4q \frac{J(q)J(-q)}{(q^2-m^2+\mathrm{i}\epsilon)} + \ldots \end{aligned} \tag{1.27}$$

$= 1 + \frac{1}{2}$ $+ \frac{1}{2!}\left(\frac{1}{2}\right)^2$  $+ \frac{1}{3!}\left(\frac{1}{2}\right)^3$  $+ \ldots$ (1.28)

Each line also implies an integration over the corresponding momentum. The functional $Z_0[J]$ is the generating functional of free correlation functions $\tau_n(x_1, \ldots, x_n)$, in the sense that

$$\begin{aligned} \tau_n(x_1,\ldots,x_n) &= \frac{1}{\mathrm{i^n}} \frac{\delta^n Z_0[J]}{\delta J(x_1)\ldots\delta J(x_n)}\Big|_{J=0} \\ &= <0|T\Phi(x_1)\ldots\Phi(x_n)|0> \,. \end{aligned} \tag{1.29}$$

E.g., we have

$$\begin{aligned} \tau_2(x,y) &= \mathrm{i}\Delta_{\mathrm{F}}(\mathrm{x}-\mathrm{y})\,, \\ \tau_3(x,y,z) &= 0\,; \end{aligned} \tag{1.30}$$

in fact, all τ_m with an odd number of arguments - m - vanish identically. Hence, we can write

$$Z_0[J] = \sum_{n=0}^{\infty} \frac{\mathrm{i}^{\mathrm{n}}}{n!} \int d^4x_1 \dots d^4x_n \, J(x_1) \dots J(x_n) \tau_n(x_1, \dots, x_n) \,. \tag{1.31}$$

Generating functional for interacting fields. A scalar field with a quartic selfinteraction can be described by the Lagrangian

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4 \\ &= \mathcal{L}_0 + \mathcal{L}_{int} \end{aligned} \tag{1.32}$$

leading to the following generating functional:

$$Z[J] = \frac{\int [D\phi] \exp\left(\mathrm{iS} + \mathrm{i} \int \mathrm{d^4x \, J}\phi\right)}{\int [D\phi] e^{\mathrm{iS}}} \,. \tag{1.33}$$

The factor in the denominator is independent of the source J and can be written as a normalization factor,

$$Z[J] = \mathcal{N} \int [D\phi] \exp\left(\mathrm{iS} + \mathrm{i} \int \mathrm{d^4x \, J}\phi\right) \,. \tag{1.34}$$

Furthermore, we can write

$$Z[J] = \mathcal{N} \exp\left(\mathrm{i} \int \mathrm{d^4x} \, \mathcal{L}_{\mathrm{int}}(\frac{1}{\mathrm{i}} \frac{\delta}{\delta \mathrm{J}})\right) Z_0[J] \tag{1.35}$$

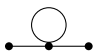
$$= \frac{\exp\left(\mathrm{i} \int \mathrm{d^4x} \, \mathcal{L}_{\mathrm{int}}(\frac{1}{\mathrm{i}} \frac{\delta}{\delta \mathrm{J}})\right) \exp(-\frac{\mathrm{i}}{2} \int d^4x d^4y \, J(x) \Delta_F(x-y) J(y))}{\exp\left(\mathrm{i} \int \mathrm{d^4x} \, \mathcal{L}_{\mathrm{int}}(\frac{1}{\mathrm{i}} \frac{\delta}{\delta \mathrm{J}})\right) \exp(-\frac{\mathrm{i}}{2} \int d^4x d^4y \, J(x) \Delta_F(x-y) J(y))\Big|_{J=0}} \,.$$

The exponential above can be expanded into a Taylor series and examined to each order. In the denominator, the external sources are put to zero. Therefore, these contributions correspond to vacuum bubbles with no external legs, which would be characterized by the sources. The full correlation functions $\tau_n(x_1, \dots, x_n)$ which include the interaction are given by functional differentiating $Z[J]$:

$$\tau_n(x_1, \dots, x_n) = \frac{1}{\mathrm{i}^{\mathrm{n}}} \frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)}\Big|_{J=0} \,. \tag{1.36}$$

As an example, let us consider the 2-point function:

$$\begin{aligned}\tau_2(x_1,x_2) &= \mathrm{i}\Delta_\mathrm{F}(\mathrm{x}_1-\mathrm{x}_2) - \frac{\lambda}{2}\Delta_\mathrm{F}(0)\int \mathrm{d}^4\mathrm{z}\,\Delta_\mathrm{F}(\mathrm{z}-\mathrm{x}_1)\Delta_\mathrm{F}(\mathrm{z}-\mathrm{x}_2) \\ &\quad +\mathcal{O}(\lambda^2)\end{aligned}$$

$$=: \ \mathrm{i}$$ $$-\frac{\lambda}{2}$$  $$+\,\mathcal{O}(\lambda^2) \qquad (1.37)$$

The normalization factor $\mathcal{N}$ removes the vacuum diagrams from the generating functional,

$$\begin{aligned}\mathcal{N} &= \left(\exp\left(\mathrm{i}\int \mathrm{d}^4\mathrm{x}\,\mathcal{L}_\mathrm{int}(\frac{1}{\mathrm{i}}\frac{\delta}{\delta \mathrm{J}})\right) Z_0[J]\Big|_{J=0}\right)^{-1} \\ &= \left(1+\frac{3\mathrm{i}\lambda}{4!}\int d^4x\,\Delta_F(0)^2+\mathcal{O}(\lambda^2)\right)^{-1} \qquad (1.38) \\ &= 1-\frac{3\mathrm{i}\lambda}{4!}\int d^4x\,\Delta_F(0)^2+\mathcal{O}(\lambda^2)\end{aligned}$$

Explicit calculation leads to

$$\begin{aligned}\tau_2(x_1,x_2) &= \frac{\mathrm{i}}{(2\pi)^4}\int d^4p\,\frac{e^{-\mathrm{ip}(\mathrm{x}_1-\mathrm{x}_2)}}{p^2-m^2+\mathrm{i}\epsilon}\left(1+\frac{\frac{\mathrm{i}}{2}\lambda\Delta_F(0)}{p^2-m^2+\mathrm{i}\epsilon}\right) \\ &= \frac{\mathrm{i}}{(2\pi)^4}\int d^4p\,\frac{e^{-\mathrm{ip}(\mathrm{x}_1-\mathrm{x}_2)}}{p^2-m^2+\mathrm{i}\epsilon}\left(1-\frac{\frac{\mathrm{i}}{2}\lambda\Delta_F(0)}{p^2-m^2+\mathrm{i}\epsilon}\right)^{-1} \\ &= \frac{\mathrm{i}}{(2\pi)^4}\int d^4p\,\frac{e^{-\mathrm{ip}(\mathrm{x}_1-\mathrm{x}_2)}}{p^2-m^2+\mathrm{i}\epsilon-\frac{\mathrm{i}\lambda}{2}\Delta_\mathrm{F}(0)}\,, \qquad (1.39)\end{aligned}$$

assuming that $\lambda\Delta_F(0)$ is small. However, we will see that the term $\Delta_F(0)$ introduces divergences into the theory, since

$$\Delta_F(0)=\frac{1}{(2\pi)^4}\int_\Lambda d^4k\,\frac{e^{-\mathrm{ik}(\mathrm{x}-\mathrm{x})}}{k^2-m^2+\mathrm{i}\epsilon}\sim\Lambda^2\,, \qquad (1.40)$$

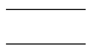
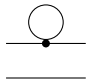
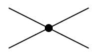
where Λ is a UV cut-off. The expression for $\Delta_F(0)$ shows a quadratic divergence behaviour. Therefore, the one-loop correction effectively modifies the mass, and we can redefine it in the following way:

$$m^2_\text{eff}=m^2+\frac{\mathrm{i}\lambda}{2}\Delta_F(0)\equiv m^2+\delta m^2\,. \qquad (1.41)$$

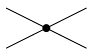
This gives rise to the one-loop renormalisation of the mass, which we will discuss in some detail in Section 1.3.

Similarly, we obtain for the 4-point function

$$\tau_4(x_1, x_2, x_3, x_4) = \frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\Big|_{J=0}$$

$= -3$  $- \frac{\mathrm{i}\lambda}{4!}\Bigg(72$  $+ 24$  $\Bigg) + \mathcal{O}(\lambda^2)$,(1.42)

where

 $\equiv \int d^4z \Delta_F(x_1 - z)\Delta_F(x_2 - z)\Delta_F(x_3 - z)\Delta_F(x_4 - z)\,.$ (1.43)

1.1.1 Transition to Euclidean spaces

1.2 Gauge fields

1.3 Renormalisation

1.3.1 Scalar ϕ^4 theory

1.3.2 Pure Yang-Mills theory

1.4 Compendium

2 Noncommutative Geometry

We will start the discussion of noncommutative geometry with some physical motivations and will then provide the mathematical foundations.

2.1 Motivation

2.1.1 Rotation group

Let us consider the rotation group $SO(3)$ in a three dimensional Euclidean space. The group action is noncommutative. This can be seen be applying two consecutive rotations – (a) 90° around the z-axis, and (b) 90° around the x-axis – on a book, say. Let the book be centered in the coordinate origin.

First applying rotation (a) and then (b) will yield a different result than first applying (b) and then (a). These operations do not commute.

The generators of infinitesimal rotations form a Lie algebra:

$$[T^i, T^j] = \mathrm{i}\epsilon^{\mathrm{ijk}}\mathrm{T}^{\mathrm{k}}\,, \tag{2.1}$$

where $i, j, k = 1, 2, 3$. A finite rotation $g(x)$ is given by

$$g(x) = e^{i\lambda^i(x)T^i}\,.$$

Replacing the generators T^i by coordinate operators $\hat{X}^i$, and we obtain a first example of a noncommutative space:

$$[\hat{X}^i, \hat{X}^j] = \mathrm{i}\epsilon^{\mathrm{ijk}}\hat{\mathrm{X}}^{\mathrm{k}}\,. \tag{2.2}$$

Further assuming the condition that

$$\sum_i \hat{X}^i\hat{X}^i = R^2\,, \tag{2.3}$$

where $R \in \mathbb{R}$ is a central element, the radius; this noncommutative space is called the *fuzzy sphere* [2].

2.1.2 Landau problem

Consider a particle with charge e moving in a homogeneous and constant magnetic field. The quantum mechanical Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e\hat{\mathbf{A}}}{c}\right)^2\,. \tag{2.4}$$

Let us confine the particle to 2D, to the (x, y)-plane with the magnetic field perpendicular to it,

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}\,, \qquad B = \text{const.} \tag{2.5}$$

Therefore, we get for the Hamiltonian

$$\begin{aligned} \hat{H} &= \frac{\hat{p}_x^2}{2m} + \frac{e^2B^2}{2mc^2}\left(\hat{x} - \frac{c}{eB}\hat{p}_y\right)^2 \\ &= \frac{\hat{p}_x^2}{2m} + \frac{1}{2}m\omega_c^2\left(\hat{x} - \frac{\hbar k_y}{m\omega_c}\right)^2\,, \end{aligned} \tag{2.6}$$

with the cyclotron frequency $\omega_c = \frac{eB}{mc}$, and we have replaced $\hat{p}_y$ by its eigenvalues $\hbar k_y$, since $\hat{p}_y$ commutes with $\hat{H}$. The Hamiltonian (2.6) is a shifted harmonic oscillator and has the eigenvalues

$$E_n = \hbar\omega_c(n + \frac{1}{2}) , \qquad n \geq 0 . \tag{2.7}$$

The separation of the eigenvalues is given by

$$\Delta E = \hbar\omega_c = \frac{\hbar\, eB}{mc} .$$

In the limit of strong magnetic fields $B \to \infty$, the energy spacing becomes large, $\Delta E \to \infty$ and only the lowest Landau level is occupied.

Let us turn our attention to the action. It is given by

$$S = \int dt \left(\frac{1}{2} m \dot{x}_i \dot{x}^i - \frac{e}{c} B_{\mu\nu} x^i \dot{x}^i \right) , \tag{2.8}$$

where $B_{\mu\nu}$ is an antisymmetric tensor defining the vector potential A_μ, $B_{\mu\nu} = -B_{\nu\mu}$ and $A_\nu = B_{\mu\nu} x^\mu$. The Poisson bracket between coordinates and momenta is given by

$$\left\{ \pi_i, x^j \right\} = \delta_i^j , \tag{2.9}$$

where

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = m\dot{x}_i + \frac{e}{c} B_{ij} x^j . \tag{2.10}$$

Writing it out explicitly, we get

$$\left\{ \dot{x}_i, x^j \right\} + \frac{eB_{ik}}{c\, m} \left\{ x^k, x^j \right\} = \frac{1}{m} \delta_i^j , \tag{2.11}$$

Let us assume strong magnetic field B and small mass m - i.e., we restrict the particle to the lowest Landau level [3]. In this approximation, eqn. (2.11) simplifies, and we get [4]

$$\left\{ x^i, x^j \right\} = \frac{c\,(B^{-1})^{ij}}{e}. \tag{2.12}$$

The coordinates perpendicular to the magnetic field do not commute, on a classical level. Under quantisation, we replace the Poisson bracket by a commutator,

$$\{ f(x,\pi) \,, g(x,\pi) \} \to \frac{1}{i\hbar} [f(\hat{x},\hat{\pi}) \,, g(\hat{x},\hat{\pi})] \,,$$

and we obtain a noncommutative algebra.

More rigorously, we have to consider the Hamiltonian and analyse the constraints of the system, for details see e.g. [5, 6]. In the limit of strong magnetic field, the canonical momentum given in Eq. (2.10) becomes

$$\pi_i = \frac{e}{c} B_{ij} x^j \,, \tag{2.13}$$

which resembles a so-called *primary constraint*. In order to quantise the Hamiltonian, the Poisson bracket has to be replaced by the *Dirac bracket* $\{\, , \,\}_{DB}$. And the quantisation procedure becomes

$$\{f(x,\pi)\,, g(x,\pi)\}_{DB} \to \frac{1}{i\hbar}[f(\hat{x},\hat{\pi})\,, g(\hat{x},\hat{\pi})]\,.$$

Second class constraints[2] modify the Poisson bracket,

$$\{f,g\}_{DB} = \{f,g\} - \sum_{i,j} \{f,\phi_i\} M^{-1}_{ij} \{\phi_j, g\}\,,$$

where the second class constraints are denoted by ϕ_i, and $M_{ij} = \{\phi_i, \phi_j\}$. Using the definition of a Dirac bracket, one immediately sees that

$$\{x^\sigma, x^\nu\}_{DB} = \frac{c\,(B^{-1})^{\sigma\nu}}{e}. \tag{2.14}$$

2.1.3 Quantizing gravity

When the quantization of gravity was considered thoroughly, it became clear that the usual concepts of space-time are inadequate and that space-time has to be quantized or noncommutative, in some way. This situation has been analyzed in detail by S. Doplicher, K. Fredenhagen and J.E. Roberts in [7]. Measuring the distance l between two particles, energy has to be deposited in that space-time region, proportional to the inverse distance, $E = \frac{hc}{\lambda} \sim \frac{hc}{l}$. As the distance l decreases the Energy E increases. At the Planck length, $l \sim l_{\mathrm{Pl}} = \sqrt{\frac{\hbar G}{c^3}}$, the bailed energy is given by

$$E = hc\sqrt{\frac{c^3}{\hbar G}}\,.$$

[2] These are constraints which have a non-vanishing Poisson bracket with at least one other constraint. If the Poisson bracket of a constraint with any other constraint vanishes, then this constraint is called *first class*.

This corresponds to an inertia

$$M = \frac{E}{c^2} = \frac{h}{c}\sqrt{\frac{c^3}{\hbar G}} \sim \sqrt{\frac{\hbar c}{G}} = M_{Pl} \,,$$

where M_{Pl} denotes the Planck mass. The associated Schwarzschild radius is given by

$$r_S = \frac{2GM}{c^2} = 2\sqrt{\frac{\hbar G}{c^3}} = 2l_{Pl} \,.$$

Therefore, the deposited energy curves space-time to such an extent that a black hole is generated. The limitations arising from the need to avoid the appearance of black holes during a measurement process lead to uncertainty relations between space-time coordinates. This already allows to catch a glimpse of the deep connection between gravity and noncommutative geometry, especially noncommutative gauge theory.

2.1.4 String theory

Another motivation for noncommutative geometry comes from String Theory. In the context of open strings with D-branes in a background magnetic field $B_{\mu\nu}$ (induced by the closed string sector) the endpoints of open strings are confined to the D-branes. Furthermore, their space-time dynamics is described by an effective noncommutative field theory living on the D-branes, in the low energy limit. For details see e.g. [8, 9, 10].

2.2 Basic ideas

Before considering Fig. 1 and the role of algebras in the context of geometry, let us define the notion of a C^* algebra.

Definition 2.1 (Involution). An involution $*$ is a map from a set A to itself,

$$\begin{aligned} * : A &\longrightarrow A \\ a &\longmapsto a^* \,, \end{aligned}$$

satisfying

i) $(a^*)^* = a \,,$

ii) $(ab)^* = b^* a^*$,

iii) $(za + wb)^* = \bar{z}a^* + \bar{w}b^*$,

$\forall a, b \in A$, $\forall z, w \in \mathbb{C}$.

Definition 2.2 (Banach algebra). An algebra A with norm $||\cdot||$ is called a Banach algebra if and only if (iff) the following requirements are satisfied:

i) $(A, +, ||\cdot||)$ is a Banach space, i.e. complete with respect to $||\cdot||$[3].

ii) $(A, +, \cdot)$ is an associative algebra.

iii) $||a \cdot b|| \leq ||a|| ||b||, \qquad \forall a, b \in A$.

Definition 2.3 (C^* algebra). A C^* algebra $\mathcal{C}$ is an involutive Banach algebra with $||a^* a|| = ||a||^2$, $\forall a \in \mathcal{C}$.

The connection between commutative algebras and geometry is depicted in Fig 1. We start with a smooth and compact manifold $\mathcal{M}$. The topology of $\mathcal{M}$ is uniquely determined by the algebra of continuous complex valued functions on $\mathcal{M}$, $C(\mathcal{M})$ with the usual involution (Urysohn's Lemma [11]). The Gel'fand-Naimark theorem [12] relates the function algebra to an Abelian C^*-algebra. The algebra of continuous functions over a compact manifold $\mathcal{M}$ is isomorphic to an Abelian unital C^*-algebra[4]. The algebra of continuous functions vanishing at infinity over a locally compact Hausdorff space, $C^0(\mathcal{M})$, is isomorphic to an Abelian C^*-algebra (not necessarily unital).

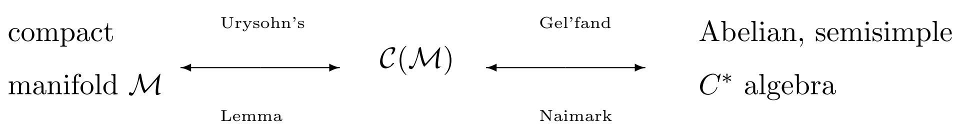


Figure 1: Classical algebraic geometry

points	$\to$	maximal ideals
$x \in \mathcal{M}$	$\mapsto$	ideal $J = \{f \vert f(x) = 0\} \subset C(\mathcal{M})$
coordinates	$\to$	coordinate functions
vector field	$\to$	derivations of the algebra
geometry	$\to$	algebraic geometry

Figure 2: Algebraic geometry

Coordinates on the manifold are replaced by coordinate functions in $C(\mathcal{M})$, vector fields by derivations of the algebra. Points are replaced by maximal ideals, cf. Fig. 2.

The trick in noncommutative geometry is to replace the Abelian C^* algebra by a non-Abelian one and to reformulate as much of the concepts of algebraic geometry as possible in terms of non-Abelian C^* algebras [13, 14].

In the following, this noncommutative algebra $\widehat{\mathcal{A}}$ will be given by the algebra of formal power series generated by the noncommutative coordinate functions $\hat{x}^i$, divided by an ideal $\mathcal{I}$,

$$\widehat{\mathcal{A}} = \frac{\mathbb{C}\langle\langle \hat{x}^1, ...\hat{x}^n \rangle\rangle}{\mathcal{I}}\,, \tag{2.15}$$

where the ideal $\mathcal{I}$ is generated by the commutation relations

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x})\,. \tag{2.16}$$

An element $\hat{f} \in \widehat{\mathcal{A}}$ has the form

$$\hat{f}(\hat{x}) = \sum_{n=0}^{\infty} c_{i_1 \dots i_n} : \hat{x}^{i_1} \dots \hat{x}^{i_n} :, \tag{2.17}$$

where :: indicates an ordering prescription in the noncommutative algebra defining a basis of monomials. This will be discussed in detail in Section 2.3. The validity of the commutation relations (2.16) is incorporated via the ideal $\mathcal{I}$.

[3] A space is complete iff every Cauchy series converges with respect to the norm $||\cdot||$.

[4] A unital algebra contains a multiplicative identity element.

Definition 2.4 (Ideal). Let $(R, +, \cdot)$ be an arbitrary ring. A subset I is called a (two-sided) ideal of R, iff

i) $(I, +)$ is a subgroup of $(R, +)$.

ii) $x \cdot r \in I$ and $r \cdot x \in I$, $\forall x \in I$, $\forall r \in R$.

In our case, the ideal is generated by the commutation relations, and we can define

$$\mathcal{I} := \{c\,\hat{a},\, \hat{r} \cdot \hat{a},\, \hat{a} \cdot \hat{r} \big| \forall c \in \mathbb{C}, \forall \hat{r} \in \widehat{\mathcal{A}}\}\,, \tag{2.18}$$

where $\hat{a} = \hat{x}^i \cdot \hat{x}^j - \hat{x}^j \cdot \hat{x}^i - i\theta^{ij}(\hat{x})$. An element of the quotient algebra $\widehat{\mathcal{A}}$ is then an equivalence class $[\hat{f}] \in \widehat{\mathcal{A}}$. Let us assume that $\hat{g}$ is in the equivalence class $[\hat{f}]$. Then also $\hat{g} + b \in [\hat{f}]$, for all $b \in I$. Let us illustrate this with a simple example.

Example 2.1 (Meaning of ideals). Let us assume $\theta^{ij}(\hat{x}) = \theta^{ij} \in \mathbb{R}$, and let us consider the two functions

$$\hat{f} = \hat{x}^1\hat{x}^2 \qquad \text{and} \qquad \hat{g} = \hat{x}^2\hat{x}^1 + i\theta^{12}\,.$$

We can use relations (2.16) in order to rewrite $\hat{g}$:

$$\hat{g} = \hat{x}^2\hat{x}^1 + i\theta^{12} = \hat{x}^1\hat{x}^2 - (\hat{x}^1\hat{x}^2 - \hat{x}^2\hat{x}^1 - i\theta^{12}) = \hat{x}^1\hat{x}^2 - \hat{a}$$

Therefore, we see that we need not distinguish between $\hat{f}$ and $\hat{g}$, and that they are in the same equivalence class,

$$\hat{g} \in [\hat{f}]\,.$$

In this way, the ideal $\mathcal{I}$ takes into account the commutation relations in the language of algebras. ■

Let us return to the commutation relations for the coordinates (2.16). Since the LHS is a commutator, the RHS has to be antisymmetric in its indices,

$$\theta^{ij}(\hat{x}) = -\theta^{ji}(\hat{x})$$

and has to satisfy the Jacobi identity,

$$[\theta^{ij}(\hat{x}),\, \hat{x}^k] + [\theta^{ki}(\hat{x}),\, \hat{x}^j] + [\theta^{jk}(\hat{x}),\, \hat{x}^i] = 0\,.$$

Most commonly, $\theta^{ij}(\hat{x})$ is chosen to be either constant or linear or quadratic in the generators. In the canonical case, the relations are constant,

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \tag{2.19}$$

where $\theta^{ij} \in \mathbb{R}$ is an antisymmetric matrix, $\theta^{ij} = -\theta^{ji}$. The linear or Lie algebra case

$$[\hat{x}^i, \hat{x}^j] = i\lambda^{ij}_k \hat{x}^k, \tag{2.20}$$

where $\lambda^{ij}_k \in \mathbb{C}$ are the structure constants, basically has been discussed in two different approaches, namely fuzzy spheres [2] and κ-deformation [15, 16, 17]. Last but not least, we have quadratic commutation relations

$$[\hat{x}^i, \hat{x}^j] = (\frac{1}{q}\widehat{R}^{ij}_{kl} - \delta^i_l \delta^j_k)\hat{x}^k \hat{x}^l, \tag{2.21}$$

where $\widehat{R}^{ij}_{kl} \in \mathbb{C}$ is the so-called $\widehat{R}$-matrix corresponding to quantum groups and also related to Statistical Physics. Those structures will be discussed in more detail in the next section. Concerning quantum field theory, we will concentrate on the simplest case in Section 3.

2.2.1 Hopf algebras and quantum groups

Classically, symmetries are described by Lie algebras or Lie groups. A physical space is a representation space of its symmetry algebra. In the deformed case, this is no longer true. Let us consider the canonical commutation relations

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}\,.$$

These relations destroy the invariance of Minkowski space-time under the Poincaré Lie algebra. Nevertheless, there is a generalized notion of symmetry in terms of Hopf algebras and quantum groups, such that deformed spaces are "invariant" under those structures. Therefore, also the commutation relations of the coordinates are left invariant under those transformations.

The interpretation is the following. Space-time is a continuum in the low energy domain, at high energies - Planck energy or maybe below - space-time becomes a "fuzzy", noncommutative space. But the symmetries are not broken, they are deformed to Hopf algebras and quantum groups. There is a well defined commutative limit, e.g. $\theta \to 0$ for canonical deformation, in which the commutative symmetries are recovered.

Definition 2.5 (Hopf algebra). A Hopf algebra A (over $\mathbb{C}$) – see e.g., [18] – consists of an algebra and a co-algebra structure which are compatible with each other. Additionally, there is a map called antipode, which corresponds to the inverse of a group. A is an algebra, i.e., there is a multiplication m and a unit element η,

$$\begin{aligned} m \;&:\; A \otimes A \to A, \\ & \quad\; a \otimes b \mapsto ab, \\ \eta \;&:\; \mathbb{C} \to A, \\ & \quad\; c \mapsto c\,\mathbf{1}_A, \end{aligned}$$

such that the multiplication satisfies the associativity axiom (Fig. 3), where id denotes the identity map, and η the axiom depicted in Fig. 4. Reversing

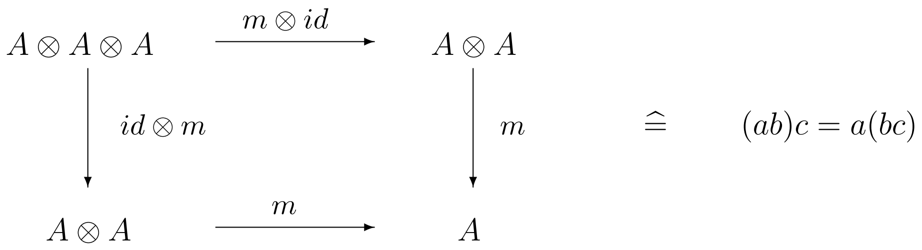


Figure 3: Associativity

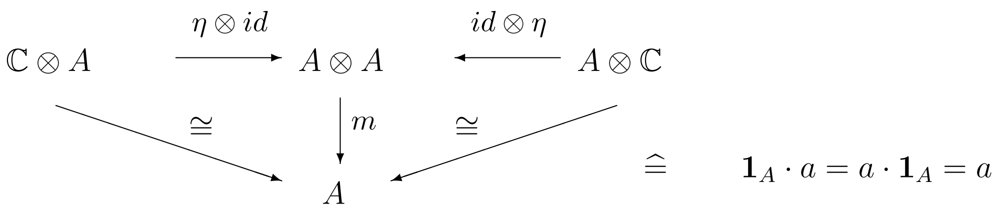


Figure 4: Unity axiom

all the arrows in Figs. 3 and 4 and replacing m by the so-called co-product Δ and η by the co-unit ϵ gives us the axioms for the structure maps of the co-algebra. The co-product and the co-unit,

$$\begin{aligned} \Delta : A &\to A \otimes A, \\ \epsilon : A &\to \mathbb{C} \end{aligned}$$

are dual to m and η, respectively. Compatibility between algebra and co-algebra structure means that the co-product Δ and the co-unit ϵ are algebra homomorphisms, i.e.,

$$\Delta(ab) = \Delta(a)\Delta(b), \tag{2.22}$$
$$\epsilon(ab) = \epsilon(a)\epsilon(b), \tag{2.23}$$

where $a, b \in A$. The antipode $S : A \to A$ satisfies the axiom shown in Fig. 5 below. It is an anti-algebra homomorphism ($S(ab) = S(b)S(a)$).

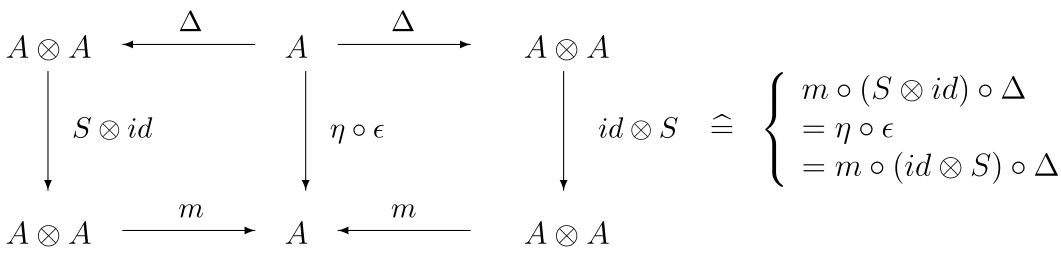


Figure 5: Antipode axiom

Associativity on the algebra side is related to *co-associativity* on the dual co-algebra side, i.e.

$$(ab)c = a(bc) \qquad \Rightarrow id \otimes \Delta(\Delta(a)) = \Delta \otimes id(\Delta(a))\,. \tag{2.24}$$

Let us consider some examples for Hopf algebras.

Example 2.2 (Universal enveloping algebra). Probably the most simple Hopf algebra is the universal enveloping algebra of some Lie algebra. Let us consider a Lie algebra g with generators T^i satisfying

$$[T^i, T^j] = \mathrm{i}\mathrm{f}^{\mathrm{ij}}_{\mathrm{k}}\mathrm{T}^{\mathrm{k}}\,,$$

where f^{ij}_k are the structure constants. The universal enveloping algebra of g, denoted by $U(g)$ is defined as a quotient algebra:

$$U(g) = \frac{T(g)}{\{T^i \otimes T^j - T^j \otimes T^i - [T^i, T^j]\}}\,, \tag{2.25}$$

where $T(g)$ is the so-called tensor algebra, and the ideal in the denominator is generated by the commutation relations of the generators. A generic element of $T(g)$ has the form

$$T(g) \ni t = \sum_{n=0}^{\infty} \alpha_{i_1 \dots i_n} T^{i_1} \otimes \dots \otimes T^{i_n} . \tag{2.26}$$

As we have already seen before, the structure in the algebra is generated by introducing equivalence classes with respect to an appropriate ideal. Then, $U(g)$ is a Hopf algebra with the following structure maps:

$$\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x , \qquad \epsilon(x) = 0 , \qquad S(x) = -x , \tag{2.27}$$

for any $x \in g$. The universal enveloping algebra is co-commutative, which means that

$$\tau \circ \Delta(x) = \Delta(x) , \tag{2.28}$$

with the permutation $\tau(x \otimes y) = y \otimes x$. But it is not commutative, unless g is. ■

Example 2.3 (Function algebra over some (finite) group $\mathcal{G}$). Let $\mathcal{G}$ be an arbitrary, (for simplicity) finite group, then the function algebra $\mathcal{F}(\mathcal{G})$,

$$\mathcal{F}(\mathcal{G}) \ni f : \mathcal{G} \to \mathbb{C} ,$$

is a Hopf algebra due to the following structure maps: The algebra structure of $\mathcal{F}(\mathcal{G})$ is given by

$$\begin{aligned} m &: \mathcal{F}(\mathcal{G}) \otimes \mathcal{F}(\mathcal{G}) \to \mathcal{F}(\mathcal{G}) && (2.29) \\ & \quad m(f_1 \otimes f_2)(g) = f_1(g) f_2(g) , \\ \eta &: \mathbb{C} \to \mathcal{F}(\mathcal{G}) && (2.30) \\ & \quad \eta(k) = k \, \mathbf{1}_{\mathcal{F}(\mathcal{G})} , \end{aligned}$$

where $f_1, f_2 \in \mathcal{F}(\mathcal{G})$ and $g \in \mathcal{G}$. For the co-algebra structure we have

$$\begin{aligned} \Delta &: \mathcal{F}(\mathcal{G}) \to \mathcal{F}(\mathcal{G}) \otimes \mathcal{F}(\mathcal{G}) && (2.31) \\ & \quad \Delta(f)(g_1 \otimes g_2) = f(g_1 g_2) , \\ \epsilon &: \mathcal{F}(\mathcal{G}) \to \mathbb{C} && (2.32) \\ & \quad \epsilon(f) = f(e) , \end{aligned}$$

where $f \in \mathcal{F}(\mathcal{G})$, $g_1, g_2 \in \mathcal{G}$, and e is the unit element of $\mathcal{G}$. Eventually, the antipode is given by

$$S(f)(g) = f(g^{-1}). \tag{2.33}$$

Note that $\mathcal{F}(\mathcal{G})$ is a commutative Hopf algebra, since the algebra of functions is commutative. ■

Example 2.4 (Group Hopf algebra $\mathbb{C}\mathcal{G}$). Let $\mathcal{G}$ be a matrix group of $n \times n$ complex matrices. Then, an element of $\mathbb{C}\mathcal{G}$ is given by

$$\mathbb{C}\mathcal{G} \ni x = \sum_{g \in \mathcal{G}} a_g g \,, \qquad a_g \in \mathbb{C}\,. \tag{2.34}$$

The generators ${t^i}_j$, $i, j = 1, \ldots, n$ form a basis of $\mathbb{C}\mathcal{G}$,

$${t^i}_j = \begin{pmatrix} & \vdots & & \\ \ldots & 1 & \ldots & \ldots \\ & \vdots & & \end{pmatrix}, \tag{2.35}$$

where we have 1 in the i^{th} row and j^{th} column, and 0 everywhere else. Because ${t^i}_j$ generate $\mathbb{C}\mathcal{G}$, it is enough to specify the structure maps for those generators. We have

$$\begin{aligned} \Delta({t^i}_j) &= \sum_k {t^i}_k \otimes {t^k}_j \,, & (2.36) \\ \epsilon({t^i}_j) &= 1\,, & (2.37) \\ S(t^i_j) &= (t^{-1})^i_j \,. & (2.38) \end{aligned}$$

This Hopf algebra is also commutative, since we clearly have

$${t^i}_j {t^m}_n = {t^m}_n {t^i}_j \,. \tag{2.39}$$

■

Definition 2.6 (Quantum group). A quantum group is a Hopf algebra with one additional structure. Let us concentrate on the group Hopf algebra $\mathbb{C}\mathcal{G}$ of some matrix group $\mathcal{G}$. The additional structure is the so-called R-matrix,

$$R : \mathbb{C}\mathcal{G} \otimes \mathbb{C}\mathcal{G} \to \mathbb{C}\,.$$

As we have seen, the Hopf algebra $\mathbb{C}\mathcal{G}$ is commutative. The R-matrix is introduced in order to deform the commutative product in a consistent way. The resulting noncommutative Hopf algebra is called a quantum group. Let us denote this quantum group by $\mathbb{C}\mathcal{G}_q$, since the noncommutativity and R itself are characterised by the parameter q. Let t^i_j be the coordinate functions generating $\mathbb{C}\mathcal{G}_q$. Then, the consistent way to implement a noncommutative multiplication is via the so-called RTT-relations:

$$R^{ij}_{kl}\, t^k_m\, t^l_n = t^j_l\, t^i_k\, R^{kl}_{mn}\,, \tag{2.40}$$

where $R(t^i_k \otimes t^j_l) \equiv R^{ij}_{kl}$. In the commutative limit, $q \to 1$ (which means $R \to \mathbf{1}$, the identity matrix), we recover the commutative multiplication

$$t^k_m\, t^l_n = t^l_n\, t^k_m\,. \tag{2.41}$$

The R-matrix is not an arbitrary matrix, but it needs to satisfy a consistency condition. The R-matrix has to be a solution of the *Quantum-Yang-Baxter-Equation* (QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{2.42}$$

which has to be understood as a matrix equation. Furthermore, we have used the abbreviation $(R_{13})^{ijk}_{lmn} \equiv \delta^j_m R^{ik}_{ln}$; R_{12} and R_{23} are defined accordingly.

Definition 2.7 (Quantum Spaces, $\mathcal{M}_q$). A quantum space $\mathcal{M}_q$ is again defined as a algebra generated by the coordinates $\hat{x}^i$,

$$\mathcal{M}_q \equiv \mathbb{C}\langle\langle \hat{x}^1, \ldots, \hat{x}^n \rangle\rangle / \mathcal{I}, \tag{2.43}$$

where $\mathcal{I}$ is an appropriate ideal. Since $\mathcal{M}_q$ is the symmetry space of a quantum group, the ideal $\mathcal{I}$, which defines the algebraic structure of the quantum space, needs to be invariant under or compatible with the quantum group structure. In order to pursue that goal, let us first introduce

$$\widehat{R} \equiv R \circ \tau\,.$$

In the classical limit, $\widehat{R}$ is just the permutation τ, with $\tau(a \otimes b) = b \otimes a$. The matrix $\widehat{R}$ can be decomposed into projectors,

$$\widehat{R} = \lambda_1 \widehat{P}_S + \lambda_2 \widehat{P}_A, \tag{2.44}$$

where $\widehat{P}_A$ is the q-deformed generalisation of the antisymmetric and $\widehat{P}_S$ of the symmetric projector. Then, the relations

$$\widehat{P}_A{}^{mn}_{\ ij}\hat{x}^i\hat{x}^j = 0 \qquad (2.45)$$

on $\mathcal{M}_q$ do the job. In the commutative limit, (2.45) means that the commutator of two coordinates vanishes. They are also compatible with the structure of the corresponding quantum group $\mathbb{C}\mathcal{G}_q$. This can be seen by applying the so-called co-action ρ

$$\begin{aligned} \rho : \mathcal{M}_q &\rightarrow \mathbb{C}\mathcal{G}_q \otimes \mathcal{M}_q, \\ \rho(\hat{x}^i) &= t^i_j \otimes \hat{x}^j, \end{aligned} \qquad (2.46)$$

on the relations (2.45), since

$$\widehat{P}_A{}^{mn}_{\ ij}(t^i_k \otimes \hat{x}^k)(t^j_l \otimes \hat{x}^l) = t^m_i\, t^n_j \otimes \widehat{P}_A{}^{ij}_{\ kl}\,\hat{x}^k\hat{x}^l = 0. \qquad (2.47)$$

The projector $\widehat{P}_A$ is a polynomial in $\widehat{R}$, and the $\widehat{R}TT$ relations (2.40) can be applied ($R^{ij}_{kl} = \widehat{R}^{ji}_{kl}$), replacing $\hat{R}$ by $\hat{P}_A$.

Differentials, $\hat{\partial}_A$. The partial derivatives $\hat{\partial}_A$ satisfy the same commutation relations as the coordinates [19] (they commute in the limit $q \rightarrow 1$),

$$\widehat{P}_A{}^{ij}_{\ kl}\,\hat{\partial}_i\hat{\partial}_j = 0. \qquad (2.48)$$

This follows from the assumptions on the exterior derivative d. The exterior derivative $d = \xi^A\hat{\partial}_A$ shall have the same properties as in the classical case, namely

$$\begin{aligned} d^2 &= 0, \\ d\hat{x}^A &= \xi^A + \hat{x}^A d, \end{aligned} \qquad (2.49)$$

where the coordinate differentials ξ^A are supposed to anticommute, i.e.,

$$\widehat{P}_S{}^{AB}_{\ CD}\,\xi^C\xi^D = 0. \qquad (2.50)$$

■

Remark: The defining relations for derivatives and coordinate differentials, the modified Leibniz rule, and even the structure of the deformed symmetry group can be reconstructed starting from the commutation relations of the coordinates (e.g. (2.45)). This is done by proposing ansätze and imposing consistency. This means that e.g. if one acts with a partial derivative on the RHS of (2.45), one obtains

$$\hat{\partial}_A(\widehat{P}_A{}^{mn}_{ij}\,\hat{x}^i\hat{x}^j) = (\widehat{P}_A{}^{mn}_{ij}\,\hat{x}^i\hat{x}^j)\hat{\partial}_A\,, \tag{2.51}$$

where we have to use the (ansatz for the) Leibniz-rule in order to permute the partial derivative through the term. Since $\widehat{P}_A{}^{mn}_{ij}\,\hat{x}^i\hat{x}^j = 0$, it has to be zero afterwards as well. This restricts the freedom in the ansatz for the Lebniz rule. In order to find an expression or to restrict the freedom in the ansatz for the commutation relations of partial derivatives, we have to act with two derivatives on (2.45):

$$(\hat{\partial}_B\hat{\partial}_A - \hat{\partial}_A\hat{\partial}_B)(\widehat{P}_A{}^{mn}_{ij}\,\hat{x}^i\hat{x}^j)\,.$$

In general, the solution will not be unique, and we can find more than one differential calculus for a quantum space.

Example 2.5 (Manin plane). $\mathbb{C}\mathcal{G}_q = SL_q(2)$

$$\widehat{R} = \begin{pmatrix} \widehat{R}^{11}_{11} & \widehat{R}^{11}_{12} & \widehat{R}^{11}_{21} & \widehat{R}^{11}_{22} \\ \widehat{R}^{12}_{11} & \cdots & & \\ \widehat{R}^{21}_{11} & & \cdots & \\ \widehat{R}^{22}_{11} & & & \cdots \end{pmatrix} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}\,, \tag{2.52}$$

where $\lambda = q - \frac{1}{q}$. The symmetry algebra is generated by a, b, c, d, with

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,.$$

The RTT-relations (2.40) imply

$$\begin{aligned} ab &= qba & ac &= qca\,, \\ ad &= da + \lambda bc\,, & bc &= cb\,, \\ bd &= qdb\,, & cd &= qdc\,. \end{aligned} \tag{2.53}$$

The $\widehat{R}$-matrix can be decomposed in the following way:

$$\widehat{R} = q\widehat{P}_S - \frac{1}{q}\widehat{P}_A\,,$$

with

$$\widehat{P}_S = \frac{q}{1+q^2}(\widehat{R} + \frac{1}{q}), \qquad , \widehat{P}_A = \frac{-q}{1+q^2}(\widehat{R} - q)\,.$$

From *Definition 2.7* above, we obtain the following relations for the coordinates:

$$\hat{x}^i\hat{x}^j = \frac{1}{q}\widehat{R}^{ij}_{kl}\hat{x}^k\hat{x}^l\,,$$

$i, j = 1, 2$, or explicitly

$$\hat{x}^1\hat{x}^2 = q\hat{x}^2\hat{x}^1\,; \tag{2.54}$$

for the partial derivatives the following choice is consistent:

$$\hat{\partial}_i\hat{\partial}_j = \frac{1}{q}\widehat{R}^{kl}_{ij}\hat{\partial}_k\hat{\partial}_l\,. \tag{2.55}$$

And last but not least, the Leibniz rule is given by

$$\hat{\partial}_i\hat{x}^j = \delta^j_i + \frac{1}{q}\widehat{R}^{-1}{}^{jk}_{il}\hat{x}^l\hat{\partial}_k\,. \tag{2.56}$$

■

Remarks:

- The Leibniz rule for the partial derivatives, e.g. (2.56), and for any other operator follows from their co-product. In general, it has the form

$$\Delta(\hat{\partial}_A) = \hat{\partial}_A \otimes \mathbf{1} + \mathcal{O}_A{}^B \otimes \hat{\partial}_B\,. \tag{2.57}$$

  This leads to the Leibniz-rule

$$\hat{\partial}_A(\hat{f}\hat{g}) = (\hat{\partial}_A\hat{f})\,\hat{g} + (\mathcal{O}_A{}^B\hat{f})\,\hat{\partial}_B\hat{g}\,, \tag{2.58}$$

  where $\mathcal{O}_A{}^B$ is some (differential) operator satisfying

$$\mathcal{O}_A{}^B(\hat{f}\hat{g}) = (\mathcal{O}_A{}^C\hat{f})\,(\mathcal{O}_C{}^B\hat{g})\,,$$

  i.e., it is a homomorphism of the noncommutative function space.

- The co-product describes how an operator acts on a (tensor-) product of states. A nice illustration of the meaning of the co-product can be given in terms of the angular momentum J in quantum mechanics. The co-product describes the addition of angular momentum

$$\Delta J = J \otimes \mathbf{1} + \mathbf{1} \otimes J = J_1 + J_2 \,. \qquad (2.59)$$

  Or you can consider a wavefunction Ψ with spin, it has the form

$$\Psi = \psi(x) \otimes |s> \,,$$

  where $|s> = a|\uparrow> + b|\downarrow>$ for a spin-1/2 particle. Then, the total angular momentum operator acts in the following way on Ψ:

$$J \triangleright \Psi = (\Delta(J)\,\Psi) = (L\psi(x)) \otimes |s> + \psi(x) \otimes (S|s>) \qquad (2.60)$$

  or in other words

$$\Delta J = L + S \,.$$

Example 2.6 (Vertex models). Vertex model are of importance in Statistical Physics. They live on a lattice, consisting of lattice points or vertices and links between them. The links may have different occupations. To the vertex i one attributes a weight ω_i depending on the occupation of the attached links. Not all configurations are allowed and have non-vanishing weight. Let us consider the so-called *six vertex model* as an example. The allowed vertices are depicted in Fig. 6. Not all the weights are independent,

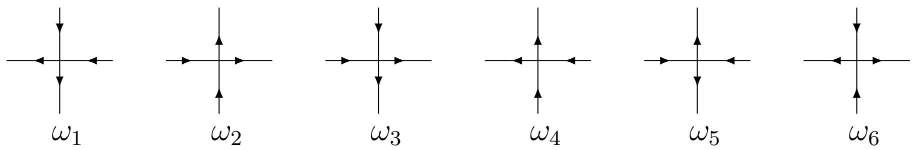


Figure 6: Six vertex model

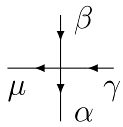
but we have

$$\begin{aligned} \omega_1 = \omega_2 = a \qquad & \omega_3 = \omega_4 = b \,, \\ \omega_5 = \omega_6 = c \,, \qquad & \end{aligned} \qquad (2.61)$$

where a, b, c are some numbers. All physics is encoded in the partition function Z (the equivalent to the Feynman integral):

$$Z = \sum_{\text{all configns}} \prod_{\text{vertices}} \text{weight(vertex)}\,. \tag{2.62}$$

We can parametrize the vertex as a matrix in the following way:



$$= \omega(\mu, \alpha|\beta, \gamma)\,, \tag{2.63}$$

where $\mu, \gamma = 1$ means arrow to the left and $\mu, \gamma = 2$ arrow to the right, resp. $\alpha, \beta = 1$ arrow down and $\alpha, \beta = 2$ arrow up. The remarkable feature of the vertex weights is that they satisfy the Yang-Baxter Equation (2.42), with the identification

$$\omega(\mu\alpha|\beta\gamma) = \hat{R}^{\mu\alpha}_{\beta\gamma}\,. \tag{2.64}$$

The converse is also true: Every solution of the Yang-Baxter Equation yields an exactly solvable vertex model (integrable model). ■

Example 2.7 (Twisted symmetry in the canonical case). As discussed before, Minkowski space with canonical commutation relations does not allow for the usual Poincaré symmetry. Compared to the quantum group case or other more sophisticated examples, calculations can be done more easily and more interesting models can be studied. The commutator of two coordinates is a constant

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}, \tag{2.65}$$

where $\theta^{\mu\nu} = -\theta^{\nu\mu} \in \mathbb{C}$. The derivatives act on coordinates as in the classical case,

$$[\hat{\partial}_\nu, \hat{x}^\mu] = \delta^\mu_\nu. \tag{2.66}$$

However, there are two consistent ways to define commutation relations of derivatives. Simplest choice is

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0. \tag{2.67}$$

Another possibility compatible with the coordinate algebra is obtained by observing that

$$\hat{x}^\mu - \mathrm{i}\theta^{\mu\nu}\hat{\partial}_\nu \tag{2.68}$$

commutes with all coordinates $\hat{x}^\nu$ and all derivatives $\hat{\partial}_\nu$ one may assume that this expression equals some constant, 0 say. Thus, we can define a derivative in terms of the coordinates (for invertible θ),

$$\hat{\partial}_\mu = -\mathrm{i}\theta^{-1}{}_{\mu\nu}\hat{\mathrm{x}}^\nu . \tag{2.69}$$

The commutator of derivatives is given by

$$[\hat{\partial}_\mu, \hat{\partial}_\nu] = \mathrm{i}\theta^{-1}{}_{\mu\nu} . \tag{2.70}$$

We have mentioned that the usual Poincaré symmetry is broken by the existence of the nonnoncommutative structure $\theta^{\mu\nu}$, which is similar to an Ether. Nevertheless, the so-called *twisted Poincaré symmetry* respects the coordinate relations (2.65). It is a Hopf algebra, but not a quantum group. The starting point is the undeformed universal enveloping algebra of the Poincaré algebra, $\mathcal{U}(\mathcal{P})$. This Hopf algebra has the coproduct

$$\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x \,,$$

for any element x. We introduce a twist $\mathcal{F} \in \mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$, which satisfies the cocycle condition:

$$(\mathcal{F} \otimes \mathbf{1})(\Delta \otimes \mathbf{1})\mathcal{F} = (\mathbf{1} \otimes \mathcal{F})(\mathbf{1} \otimes \Delta)\mathcal{F} \,. \tag{2.71}$$

The twist $\mathcal{F}$ is used to define the twisted Poincaré symmetry $\mathcal{U}_\mathcal{F}(\mathcal{P})$: All Hopf algebra structures are unchanged except the co-product:

$$\Delta_\mathcal{F}(x) := \mathcal{F}\Delta(x)\mathcal{F}^{-1} \,. \tag{2.72}$$

In the case of canonical deformation, the twist is rather simple:

$$\mathcal{F} = e^{\frac{\mathrm{i}}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} \,, \tag{2.73}$$

where P_μ are the translation generators (partial derivatives). In order to compute the twisted co-product (2.72) for the generators of the Poincaré algebra – translations P_α and Lorentz generators $M_{\mu\nu}$ – we need to expand the twist and use the algebra relations

$$[P_\mu, P_\nu] = 0 \,, \qquad [P_\alpha, M_{\mu\nu}] = \eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu \,. \tag{2.74}$$

This corresponds to the choice (2.67) for the partial derivatives. Therefore, we obtain

$$\Delta_{\mathcal{F}} P_\mu = P_\mu \otimes \mathbf{1} + \mathbf{1} \otimes P_\mu \,, \tag{2.75}$$

$$\Delta_{\mathcal{F}} M_{\mu\nu} = M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} \tag{2.76}$$
$$-\frac{1}{2}\theta^{\alpha\beta}\left((\eta_{\alpha\mu}P_\nu - \eta_{\alpha\nu}P_\mu) \otimes P_\beta + P_\alpha \otimes (\eta_{\beta\mu}P_\nu - \eta_{\beta\nu}P_\mu)\right) \,.$$

Using the deformed co-product of the generators above and their action on coordinates:

$$\begin{aligned}(P_\mu \hat{x}^\alpha) &= \mathrm{i}\delta^\alpha_\mu \,, \\ (M_{\mu\nu} x^\alpha) &= \mathrm{i}(\hat{\mathrm{x}}_\mu \partial_\nu - \hat{\mathrm{x}}_\nu \partial_\mu)\hat{\mathrm{x}}^\alpha = \mathrm{i}(\hat{\mathrm{x}}_\mu \delta^\alpha_\nu - \hat{\mathrm{x}}_\nu \delta^\alpha_\mu) \,,\end{aligned} \tag{2.77}$$

we can show explicitly that the commutation relations of the coordinates are preserved. Let us start with the translation generators P_μ acting on the LHS of (2.65):

$$\begin{aligned}(P_\mu \left[\hat{x}^\alpha, \hat{x}^\beta\right]) &=: P_\mu \triangleright \left[\hat{x}^\alpha, \hat{x}^\beta\right] \\ = (P_\mu \triangleright \hat{x}^\alpha)\hat{x}^\beta &+ \hat{x}^\alpha(P_\mu \triangleright \hat{x}^\beta) - (P_\mu \triangleright \hat{x}^\beta)\hat{x}^\alpha + \hat{x}^\beta(P_\mu \triangleright \hat{x}^\alpha) = 0 \,,\end{aligned} \tag{2.78}$$

where we have used formula (2.75) for the co-product of P_μ. Acting on the RHS yields the same result:

$$P_\mu \triangleright \mathrm{i}\theta^{\mu\nu} = 0 \,, \tag{2.79}$$

since $\theta^{\mu\nu}$ is constant. Next, consider a Lorentz generator $M_{\mu\nu}$:

$$M_{\mu\nu} \triangleright \left[\hat{x}^\alpha, \hat{x}^\beta\right] =$$
$$= \mathrm{i}\delta^\alpha_\nu \left[\hat{\mathrm{x}}_\mu, \hat{\mathrm{x}}^\beta\right] - \mathrm{i}\delta^\alpha_\mu \left[\hat{\mathrm{x}}_\nu, \hat{\mathrm{x}}^\beta\right] - \mathrm{i}\delta^\beta_\nu \left[\hat{\mathrm{x}}_\mu, \hat{\mathrm{x}}^\alpha\right] + \mathrm{i}\delta^\beta_\mu \left[\hat{\mathrm{x}}_\nu, \hat{\mathrm{x}}^\alpha\right] \tag{2.80}$$
$$-\frac{1}{2}\theta^{\sigma\tau}\left(\eta_{\sigma\mu}P_\nu - \eta_{\sigma\nu}P_\mu\right) \otimes P_\tau \triangleright \left[\hat{x}^\alpha, \hat{x}^\beta\right] \tag{2.81}$$
$$-\frac{1}{2}\theta^{\sigma\tau}P_\sigma \otimes \left(\eta_{\tau\mu}P_\nu - \eta_{\tau\nu}P_\mu\right) \triangleright \left[\hat{x}^\alpha, \hat{x}^\beta\right]$$
$$= \delta^\alpha_\nu \theta^\beta{}_\mu + \delta^\alpha_\mu \theta_\nu{}^\beta + \delta^\beta_\nu \theta_\mu{}^\alpha + \delta^\beta_\mu \theta^\alpha{}_\nu$$
$$-\delta^\alpha_\nu \theta^\beta{}_\mu - \delta^\alpha_\mu \theta_\nu{}^\beta - \delta^\beta_\nu \theta_\mu{}^\alpha - \delta^\beta_\mu \theta^\alpha{}_\nu$$
$$= 0 \,.$$

As before, acting on the RHS of (2.65) yields

$$M_{\mu\nu} \triangleright \mathrm{i}\theta^{\alpha\beta} = 0 \,. \tag{2.82}$$

Therefore, we see that the commutation relations are consistent with the twisted Poincaré Hopf algebra. ■

2.3 Star products

Let us consider the noncommutative algebra of functions $\widehat{\mathcal{A}}$ on a noncommutative space

$$\widehat{\mathcal{A}} = \frac{\mathbb{C}\langle\langle \hat{x}^1, ..., \hat{x}^n \rangle\rangle}{\mathcal{I}}, \tag{2.83}$$

where $\mathcal{I}$ is the ideal generated by the commutation relations of the coordinate functions, and the commutative algebra of functions

$$\mathcal{A} = \frac{\mathbb{C}\langle\langle x^1, ..., x^n \rangle\rangle}{[x^i, x^j]} \equiv \mathbb{C}[[x^1, ..., x^n]], \tag{2.84}$$

i.e., $[x^i, x^j] = 0$. Our aim in this Section is to relate these algebras by an isomorphism. Let us first consider the vector space structure of the algebras, only. In order to construct a vector space isomorphism, we have to choose a basis (ordering) in $\hat{\mathcal{A}}$ - satisfying the Poincaré-Birkhoff-Witt property[5] - e.g., the basis of symmetrically ordered polynomials,

$$1, \quad \hat{x}^i, \quad \frac{1}{2}(\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i), \quad \dots . \tag{2.85}$$

Now we map the basis monomials in $\mathcal{A}$ onto the according symmetrically ordered basis elements of $\widehat{\mathcal{A}}$

$$\begin{aligned} W : \mathcal{A} &\to \widehat{\mathcal{A}}, \\ x^i &\mapsto \hat{x}^i, \\ x^i x^j &\mapsto \frac{1}{2}(\hat{x}^i\hat{x}^j + \hat{x}^j\hat{x}^i) \equiv \,:\hat{x}^i\hat{x}^j:\, . \end{aligned} \tag{2.86}$$

The ordering is indicated by $::$ and may denote any choice of ordering. W is an isomorphism[6] of vector spaces. In order to extend W to an algebra isomorphism, we have to introduce a new noncommutative multiplication $\star$ in $\mathcal{A}$. This $\star$-product is defined by

$$W(f \star g) := W(f) \cdot W(g) = \hat{f} \cdot \hat{g}, \tag{2.87}$$

where $f, g \in \mathcal{A}$, $\hat{f}, \hat{g} \in \widehat{\mathcal{A}}$.

$$(\mathcal{A}, \star) \cong (\widehat{\mathcal{A}}, \cdot), \tag{2.88}$$

i.e., W is an algebra isomorphism. The information on the noncommutativity of $\widehat{\mathcal{A}}$ is encoded in the $\star$-product.

[5] The dimension of the subspace spanned by monomials of a fixed degree is the same as the dimension of the subspace spanned by monomials in the commutative variables of the same degree.

[6] A function f is an isomorphism, iff it is bijective, and f and f^{-1} are homomorphisms.

2.3.1 Construction of a $\star$-product of functions

Let us choose symmetrically ordered monomials as basis in $\widehat{\mathcal{A}}$. The commutation relations of the coordinates are

$$[\hat{x}^i, \hat{x}^j] = \mathrm{i}\theta^{\mathrm{ij}}(\hat{\mathrm{x}}), \tag{2.89}$$

where $\theta^{ij}(\hat{x})$ is an arbitrary expression in the coordinates $\hat{x}$, for now. In just a moment we will discuss the special cases (2.19 - 2.21). The Weyl quantisation procedure [20, 21] is given by the Fourier transformation,

$$\hat{f} = W(f) = \frac{1}{(2\pi)^{n/2}} \int d^n k\, e^{ik_j \hat{x}^j} \tilde{f}(k), \tag{2.90}$$

$$\tilde{f}(k) = \frac{1}{(2\pi)^{n/2}} \int d^n x\, e^{-ik_j x^j} f(x), \tag{2.91}$$

where we have replaced the commutative coordinates by noncommutative ones ($\hat{x}^i$) in the inverse Fourier transformation (2.90). The exponential takes care of the symmetrical ordering, i.e., we use the plane waves as basis:

$$e^{\mathrm{ikx}} \overset{\mathrm{W}}{\mapsto} e^{\mathrm{ik\hat{x}}}.$$

Using eqn. (2.87), we get

$$W(f \star g) = \frac{1}{(2\pi)^n} \int d^n k\, d^n p\, e^{ik_i \hat{x}^i} e^{ip_j \hat{x}^j} \tilde{f}(k) \tilde{g}(p). \tag{2.92}$$

Because of the noncommutativity of the coordinates $\hat{x}^i$, we need the Campbell-Baker-Hausdorff (CBH) formula

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[[A,B],B]-\frac{1}{12}[[A,B],A]+\dots}. \tag{2.93}$$

Clearly, we need to specify $\theta^{ij}(\hat{x})$ in order to evaluate the CBH formula.

Example 2.8 (Weyl-Moyal $\star$-product). Due to the constant commutation relations,

$$[x^\mu \overset{\star}{,} x^\nu] = \mathrm{i}\theta^{\mu\nu},$$

the CBH formula will terminate: Terms with more than one commutator will vanish,

$$\exp(ik_i \hat{x}^i) \exp(ip_j \hat{x}^j) = \exp\left(i(k_i + p_i)\hat{x}^i - \frac{i}{2} k_i \theta^{ij} p_j \right). \tag{2.94}$$

Eqn. (2.92) now reads

$$f \star g\,(x) = \frac{1}{(2\pi)^n} \int d^n k d^n p\, e^{i(k_i+p_i)x^i - \frac{i}{2}k_i\theta^{ij}p_j}\, \tilde{f}(k)\tilde{g}(p)\,, \tag{2.95}$$

and we get for the $\star$-product the Moyal-Weyl product [22]

$$f \star g\,(x) = \exp(\frac{\mathrm{i}}{2}\frac{\partial}{\partial x^i}\,\theta^{ij}\,\frac{\partial}{\partial y^j})\, f(x)g(y)\Big|_{y\to x}. \tag{2.96}$$

The deformed derivatives reduce to the usual partial derivatives:

$$\begin{aligned} [\partial_\mu \overset{\star}{,} \partial_\nu] &= 0\,,\\ \partial_\mu \star x_\nu &= \delta^\mu_\nu + x_\nu \star \partial_\mu\,. \end{aligned}$$

In order to write down an action we need an integration. In this case, we can just use the ordinary 4-dimensional integral:

$$\begin{aligned} \int d^4x\,(f_1 \star f_2 \star \cdots \star f_n)(x) &= \int d^4x\,(f_n \star f_1 \star \ldots f_{n-1})(x) \qquad (2.97)\\ &= \int d^4x\, f_1(x)\,(f_2 \star f_3 \star \cdots \star f_n)(x)\,, \end{aligned}$$

which means that in the special case $n = 2$ the star and the noncommutative effect drops out:

$$\int d^4x\,(f_1 \star f_2)(x) = \int d^4x\, f_1(x) f_2(x)\,. \tag{2.98}$$

This is called the trace-property and is essential for variational calculus. ■.

Example 2.9 (κ-deformation). The coordinates build a Lie algebra

$$[\hat{x}^i, \hat{x}^j] = i\lambda^{ij}_k \hat{x}^k, \tag{2.99}$$

with structure constants λ^{ij}_k. In this case the CBH sum will not terminate and we get

$$\exp(ik_i\hat{x}^i)\exp(ip_j\hat{x}^j) = \exp\left(i(k_i+p_i)\hat{x}^i + \frac{\mathrm{i}}{2}g_i(k,p)\hat{x}^i\right), \tag{2.100}$$

where all the terms containing more than one commutator are collected in $g_i(k,p)$. (2.92) becomes

$$f \star g\,(x) = \frac{1}{(2\pi)^n} \int d^n k d^n p\, e^{i(k_i+p_i)x^i + \frac{\mathrm{i}}{2} g_i(k,p) x^i} \tilde{f}(k)\tilde{g}(p). \qquad (2.101)$$

The symmetrically ordered $\star$-product takes the form

$$f \star g\,(x) = e^{\frac{\mathrm{i}}{2} x^i\, g_i(-\mathrm{i}\frac{\partial}{\partial y}, -\mathrm{i}\frac{\partial}{\partial z})}\, f(y)g(z)\Big|_{\substack{y\to x \\ z\to x}} . \qquad (2.102)$$

In general, it will not be possible to write down a closed expression for the $\star$-product, since the CBH formula can be summed up only for very few examples.

Let us concentrate on the so-called κ-deformed case. We start at the following commutation relations:

$$[\hat{x}^n, \hat{x}^p] = \mathrm{i}a\hat{\mathrm{x}}^{\mathrm{p}} = \frac{\mathrm{i}}{\kappa}\hat{\mathrm{x}}^{\mathrm{p}}, \qquad [\hat{\mathrm{x}}^{\mathrm{q}}, \hat{\mathrm{x}}^{\mathrm{p}}] = 0, \qquad (2.103)$$

where $p, q = 1, \ldots, n-1$. The most general linear quantum space structure compatible with a deformed version of Poincaré symmetry is given by [23]

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\,(\mathrm{a}^\mu \delta^\nu_\sigma - \mathrm{a}^\nu \delta^\mu_\sigma)\,\hat{\mathrm{x}}^\sigma, \qquad (2.104)$$

where a^μ is a constant 4−vector "pointing into the direction of noncommutativity". Its components also play the role of Lie algebra structure constants. Choosing

$$a^\mu = a\delta^{\mu n} \qquad (2.105)$$

we get back to relation (2.103). The symmetrically ordered star product is given by [24]

$$f \star g\,(x) = \int d^4k\, d^4p\, \tilde{f}(k)\tilde{g}(p)\, e^{i(\omega_k+\omega_p)x^n}\, e^{i\vec{x}(\vec{k}e^{a\omega_p} A(\omega_k,\omega_p) + \vec{p}A(\omega_p,\omega_k))}, \qquad (2.106)$$

where $k = (\omega_k, \vec{k})$, and $\vec{x} = (x^2, x^3, x^4)$. We have used the definition

$$A(\omega_k, \omega_p) \equiv \frac{a(\omega_k+\omega_p)}{e^{a(\omega_k+\omega_p)} - 1} \frac{e^{a\omega_k} - 1}{a\omega_k} . \qquad (2.107)$$

$$
\begin{aligned}
\partial_n^\star x^i &= x^i \partial_n^\star, \\
\partial_n^\star x^n &= 1 + x^n \partial_n^\star\,, \\
\partial_i^\star x^j &= \delta_i^j + x^j \partial_i^\star\,, \\
\partial_i^\star x^n &= (x^n + \mathrm{ia})\,\partial_\mathrm{i}^\star\,.
\end{aligned} \tag{2.108}
$$

$$
[\partial_\mu \overset{\star}{,} \partial_\nu] = 0 \tag{2.109}
$$

action on commutative functions

$$
\tilde{\partial}_n \triangleright f(x) = \partial_n f(x), \tag{2.110}
$$

$$
\tilde{\partial}_i \triangleright f(x) = \partial_i \frac{\exp(ia\partial_n) - 1}{ia\partial_n} f(x). \tag{2.111}
$$

The derivatives on the RHS are ordinary derivatives. We can also evaluate the action of the derivatives on a product of two function, i.e., we calculate the modified Leibniz rule. It is given by

$$
\begin{aligned}
\partial_n^\star \triangleright f \star g(x) &= (\partial_n^\star \triangleright f(x)) \star g(x) + f(x) \star (\partial_n^\star \triangleright g(x)), \\
\partial_i^\star \triangleright f \star g(x) &= (\partial_i^\star \triangleright f(x)) \star g(x) + (e^{\mathrm{ia}\partial_\mathrm{n}^\star} \triangleright f(x)) \star (\partial_i^\star \triangleright g(x))\,.
\end{aligned} \tag{2.112}
$$

The boosts and the rotation generators act on functions in the following way,

$$
\begin{aligned}
N^{\star l} \triangleright f(x) &= \Big(x^l \partial_n - x^n \partial_l \\
&\quad + x^l \partial_\mu \partial_\mu \frac{e^{ia\partial_n} - 1}{2\partial_n} + x^\nu \partial_\nu \partial_l \, \frac{a\partial_n + i(e^{ia\partial_n} - 1)}{a\partial_n^2} \Big) f,
\end{aligned} \tag{2.113}
$$

$$
M^{\star rs} \triangleright f(x) = (x^s \partial_r - x^r \partial_s)\, f(x), \tag{2.114}
$$

where $\Delta_{cl} = \sum_{i=1}^{n-1} \partial_i \partial_i$. The action of the rotations $M^{\star rs}$ is easily obtained, since their algebra and co-algebra structure is undeformed.

The definition of a suitable integration is more tricky. We have to introduce a measure function $\mu(x)$ in order to guarantee the trace property,

$$
\begin{aligned}
\int d^4x\, \mu(x)\, (f_1 \star \cdots \star f_n) &= \int d^4x\, \mu(x)\, (f_n \star f_1 \star \cdots \star f_{n-1}) \\
&= \int d^4x\, \mu(x)\, f_n\, (f_1 \star \cdots \star f_n)\ .
\end{aligned} \tag{2.115}
$$

The measure function $\mu(x)$ is given by

$$
\mu(x) = \frac{1}{x^1 x^2 \ldots x^{n-1}}\,. \tag{2.116}
$$

■

Example 2.10 (q-deformed case). The CBH formula cannot be used explicitly, we have to use eqns. (2.86), instead. Let us first write functions as a power series in x^i,

$$f(x) = \sum_J c_J \; (x^1)^{j_1} \cdot \ldots (x^n)^{j_n}, \tag{2.117}$$

where $J = (j_1, \ldots, j_n)$ is a multi-index. In the same way, noncommutative functions are given by power series in ordered monomials

$$\hat{f}(\hat{x}) = \sum_J c_J \;\; : (\hat{x}^1)^{j_1} \cdot \ldots (\hat{x}^n)^{j_n} : . \tag{2.118}$$

In a next step, we have to express the product of two ordered monomials in the noncommutative coordinates again in terms of ordered monomials, i.e., we have to find coefficients a_K such that

$$: (\hat{x}^1)^{i_1} \ldots (\hat{x}^n)^{i_n} : : (\hat{x}^1)^{j_1} \ldots (\hat{x}^n)^{j_n} : = \sum_K a_K : (\hat{x}^1)^{k_1} \ldots (\hat{x}^n)^{k_n} : . \tag{2.119}$$

Knowing the a_K, we know the $\star$-product for monimials. It is simply given by

$$(\hat{x}^1)^{i_1} \ldots (\hat{x}^n)^{i_n} \star (\hat{x}^1)^{j_1} \ldots (\hat{x}^n)^{j_n} = \sum_K a_K (\hat{x}^1)^{k_1} \ldots (\hat{x}^n)^{k_n}, \tag{2.120}$$

using the same coefficients a_K as in (2.119). The whole procedure makes use of the isomorphism W defined in eqns. (2.86) and (2.87). In a last step, we have to generalise the above expression to functions f and g, and express the $\star$-product in terms of ordinary derivatives on the functions f and g, respectively. This merely amounts to replacing q^{i_k} - where i_k refers to the power ot the $\mathrm{k^{th}}$ coordinate in (2.120) - by the differential operator $q^{x^k \partial_k}$, where no summation over k is implied.

For a better illustration, let us consider the Manin plane. The commutation relations are

$$\hat{x}\,\hat{y} = q\,\hat{y}\,\hat{x}.$$

First, we consider normal ordering, i.e., a normal ordered monomial has the form

$$: \hat{y}^3\,\hat{x}^2\,\hat{y} := \hat{x}^2\,\hat{y}^4.$$

Following the above prescription, we end up with the following $\star$-product [25]

$$f \star_N g\,(x,y) = m \circ q^{-y\frac{\partial}{\partial y}\otimes x\frac{\partial}{\partial x}} f(x,y) \otimes g(x,y)\,, \tag{2.121}$$

where m is again the multiplication map, $m(a\otimes b) = ab$. Let us now consider a *symmetric ordering*, where the factor $k!$ in the denominator is replaced by $[k]_{q^{1/2}}!$, e.g.,

$$W[xy] = \frac{\hat{x}\hat{y} + \hat{y}\hat{x}}{[2]_{q^{1/2}}!}, \tag{2.122}$$

where $[a]_{q^b} \equiv \frac{q^{ab}-q^{-ab}}{q^b-q^{-b}}$. The only difference to symmetrically ordered polynomials is the normalisation.

$$f \star_S g\,(x,y) = m \circ q^{-\frac{1}{2}\left(y\frac{\partial}{\partial y}\otimes x\frac{\partial}{\partial x} - x\frac{\partial}{\partial x}\otimes y\frac{\partial}{\partial y}\right)} f(x,y) \otimes g(x,y). \tag{2.123}$$

In the next Section, we will examine the connection between $\star$-products corresponding to different oderings in more detail. ■

Example 2.11 (Quantum mechanics). In Quantum mechanics, cf. [26], we have Heisenberg commutation relations between momenta and position operators,

$$[Q^i, P_j] = i\hbar\delta^i_j, \tag{2.124}$$

$i, j = 1, \ldots, n$. In normal ordering, where all the momenta are on the right and all coordinates on the left, $Q^l P^k$, we get for the $\star$-product

$$f \star_N g\,(Q,P) = m \circ \exp(-i\hbar\partial_{P_i} \otimes \partial_{Q^i}) f(Q,P) \otimes g(Q,P)\,. \tag{2.125}$$

For symmetrical ordering the $\star$-product reads

$$f \star_S g\,(P,Q) = m \circ \exp\left(\frac{\mathrm{i}\hbar}{2}\left(\partial_{Q^i} \otimes \partial_{P_i} - \partial_{P_i} \otimes \partial_{Q^i}\right)\right) f(Q,P) \otimes g(Q,P). \tag{2.126}$$

■

2.3.2 Mathematical approach to $\star$-products

Definition 2.8 (Poisson bracket). Let $\mathcal{M}$ be a smooth manifold, a Poisson bracket is a bi-linear map $\{,\} : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ satisfying

$f, g, h \in \mathcal{C}^\infty(\mathcal{M})$

(i) $\{f, g\} = -\{g, f\}$, antisymmetry

(ii) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$, Jacobi identity

(iii) $\{f, gh\} = \{f, g\}h + g\{f, h\}$, Leibniz rule

Locally, we can always write the Poisson bracket with the help of an antisymmetric tensor

$$\{f, g\} = \alpha^{ij}(x)\, \partial_i f \partial_j g, \tag{2.127}$$

where $\alpha^{ij} = -\alpha^{ji}$. Because of the Jacobi identity θ^{ij} has to satisfy

$$\alpha^{ij}\partial_j \alpha^{kl} + \alpha^{kj}\partial_j \alpha^{li} + \alpha^{lj}\partial_j \alpha^{ik} = 0. \tag{2.128}$$

Definition 2.9 ($\star$-Product). Let $f, g \in \mathcal{C}^\infty(\mathcal{M})$ and $C_i : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$, $i = 1, \ldots, \infty$, be local bi-differential operators. Then we define the product $\star : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})[[h]]$, by

$$f \star g = \sum_{n=0}^{\infty} h^n C_n(f, g), \tag{2.129}$$

such that the following axioms are satisfied:

(i) $\star$ is an associative product.

(ii) $C_0(f, g) = f\, g$, classical limit.

(iii) $\frac{1}{h}\, [f \overset{\star}{,} g] = -\mathrm{i}\, \{\mathrm{f}, \mathrm{g}\}$, in the limit $h \to 0$, semiclassical limit.

The RHS of definition (2.129) is an element of $\mathcal{C}^\infty(\mathcal{M})[[h]]$, the algebra of formal power series in the formal parameter h with coefficients in $\mathcal{C}^\infty(\mathcal{M})$, i.e.,

$$\begin{aligned} \zeta &\in \mathcal{C}^\infty(\mathcal{M})[[h]]\,, \\ \zeta(x) &= \sum_{j=0}^{\infty} h^j \zeta_j(x)\,, \end{aligned} \tag{2.130}$$

where $\zeta_j(x) \in \mathcal{C}^\infty(\mathcal{M})$. Formal power series means that the convergence of the series 2.130 is no issue.

The definition of the $\star$-product can be generalised to a $\mathbb{C}[[h]]$-linear product in $\mathcal{B} = \mathcal{C}^\infty(\mathcal{M})[[h]]$ by

$$\Big(\sum_n f_n h^n\Big) \star \Big(\sum_m g_m h^m\Big) = \sum_{k,l} f_k g_l h^{k+l} + \sum_{k,l\geq 0, m\geq 1} C_m(f_k, g_l) h^{k+l+m} . \quad (2.131)$$

Theorem 1 *(Theorem by M. Kontsevich [27])*
For any Poisson bi-vector field α in some domain of $\mathbb{R}^n$ there exists a $\star$-product. It is given by the following formula:

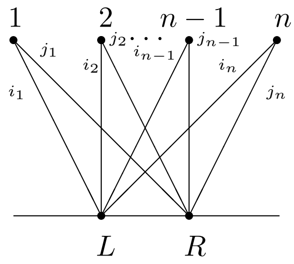
$$f \star_K g = \sum_{n=0}^{\infty} h^n \sum_{\Gamma \in G_n} \omega_\Gamma B_{\Gamma,\alpha}(f, g). \quad (2.132)$$

Let us explain the symbols occuring in (2.132) briefly. To each admissable graph $\Gamma \in G_n$ there is an associated bi-differential operator $B_{\Gamma,\alpha}$ and a weight ω_Γ. The symbol Γ_n, where $n \geq 0$ is the order in the formal deformation parameter, denotes the set of admissable graphs, which is a proper subset of the set of all graphs. It is characterised in the follwing way:
Each graph Γ consists of $n + 2$ vertices labelled by $\{1, 2, \ldots, n, L, R\}$ and $2n$ edges $\{i_k, j_k\}$, with $k = 1, \ldots, n$ starting at the vertex k and pointing to some other vertex. An admissable graph has no loops, i.e., an edge starting and ending at the same vertex. Also, no parallel multiple edges are allowed, edges sharing the same starting and ending vertex.

The vertices $k = 1, \ldots, n$ are distributed over the upper half complex plane, the vertices L and R at the origin and the point $(1, 0)$, respectively. A multi-vectorfield B_Γ is associated to each such graph. The formality theorem assigns a bi-differential operator $B_{\Gamma,\alpha}$ to the multi-vectorfields B_Γ. The edges correspond to derivatives, the vertices to the Poisson structure α. If an edge ends at another vertex in the upper plane, the derivative acts on the Poisson structure associated to that vertex. If it ends on L or R, respectively the derivative acts on the function f or g, respectively. The weight ω_Γ of a graph Γ is given by a complicated complex integration. For details see [27], for the explicit calculations of some weights, see e.g., [28, 29].

In order to illustrate these definitions let us consider some graphs as examples:

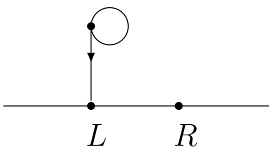
- Graph to n^{th} order for the Weyl-Moyal product:



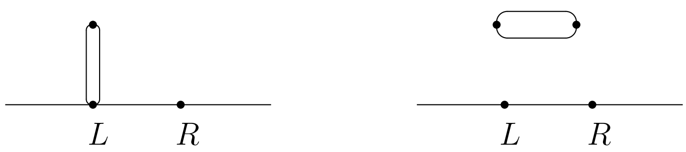
This graph corresponds to the expression

$$\alpha^{i_1 j_1}\alpha^{i_2 j_2}\ldots\alpha^{i_n j_n}\partial_{i_1}\ldots\partial_{i_n}f\,\partial_{j_1}\ldots\partial_{j_n}g\,. \tag{2.133}$$

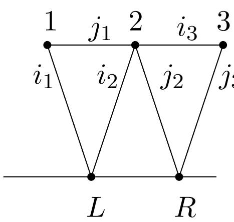
- Example for a not admissable loop graph, $n = 1$:



- Parallel edges are not allowed as well:



- Another graph in third order for non-constant Poisson structure:



The above graph yields in terms of the Poisson structure:

$$\alpha^{i_1 j_1}\left(\partial_{j_1}\partial_{i_3}\,\alpha^{i_2 j_2}\right)\alpha^{i_3 j_3}(\partial_{i_1}\partial_{i_2}f)\,(\partial_{j_2}\partial_{j_3}\,g)\,. \tag{2.134}$$

Changing the ordering in the noncommutative algebra leads to *gauge equivalent* $\star$-products. The $\star$-products are related by a transformation $\mathcal{D}$,

$$\mathcal{D}f \star \mathcal{D}g = \mathcal{D}(f \star' g), \tag{2.135}$$

where

$$\mathcal{D}f = f + \sum_{n\geq 1} (ih)^n \, \mathcal{D}_n(f) \tag{2.136}$$

and $\mathcal{D}_n$ are differential operators of order n.

Let us reconsider the $\star$-products given in Example 2.11 in Section 2.3.1. There, $\star_N$ and $\star_S$ are gauge equivalent $\star$-products. For simplicity, let us consider 1-dimensional Quantum Mechanics. The generalisation to n dimensions is straight forward. In this case, the $\star$-products (2.125) and (2.126) read

$$f \star_N g\,(q,p) = m \circ \exp\left(-i\hbar\partial_p \otimes \partial_q\right)\, f(q,p) \otimes g(q,p),$$
$$f \star_S g\,(q,p) = m \circ \exp\left(\tfrac{\mathrm{i}}{2}(\partial_q \otimes \partial_p - \partial_p \otimes \partial_q)\right) f(q,p) \otimes g(q,p).$$

Using matrices in the exponent, these formulae can be written very succinctly as

$$\begin{aligned} f \star_N g\,(q,p) &= m \circ \exp\left(\frac{\mathrm{i}\hbar}{2}\tilde{\alpha}^{ij}\partial_i \otimes \partial_j\right) f(q,p) \otimes g(q,p), \\ f \star_S g\,(q,p) &= m \circ \exp\left(\frac{\mathrm{i}\hbar}{2}\alpha^{ij}\partial_i \otimes \partial_j\right) f(q,p) \otimes g(q,p), \end{aligned}$$

where

$$\left(\tilde{\alpha}^{ij}\right)_{i,j=q,p} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, \qquad \left(\alpha^{ij}\right)_{i,j=q,p} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.137}$$

The matrix α is the antisymmetric part of $\tilde{\alpha}$. These $\star$-products are connected by the transformation $\mathcal{D}$,

$$\mathcal{D} = \exp\left(\frac{\mathrm{i}\hbar}{4}\theta_S^{ij}\partial_i\partial_j\right) = \exp\left(-\frac{\mathrm{i}\hbar}{2}\partial_q\partial_p\right), \tag{2.138}$$

where θ_S is the symmetric part of $\tilde{\alpha}$, i.e., $\tilde{\alpha}^{ij} = \alpha^{ij} + \theta_S^{ij}$. We have

$$f \star_S g = \mathcal{D}^{-1}\left(\mathcal{D}f \star_N \mathcal{D}g\right). \tag{2.139}$$

The same is true in case of the Manin plane. There we have

$$(\tilde{\alpha}^{ij})_{i,j=q,p} = \begin{pmatrix} 0 & 0 \\ -2y\otimes x & 0 \end{pmatrix}, \quad (\alpha^{ij})_{i,j=q,p} = \begin{pmatrix} 0 & x\otimes y \\ -y\otimes x & 0 \end{pmatrix},$$

$$\left(\theta_S^{ij}\right)_{i,j=q,p} = \begin{pmatrix} 0 & -x\otimes y \\ -y\otimes x & 0 \end{pmatrix}. \tag{2.140}$$

The transformation $\mathcal{D} = \exp\left(\frac{\mathrm{i}\hbar}{4}\theta_S^{ij}\partial_i\partial_j\right)$ connects the normal ordered $\star$-product (2.121) and the symmetrically ordered $\star$-product (2.123) (with q-numbers as normalisation factors). Again, we have $f \star_S g = \mathcal{D}^{-1}\left(\mathcal{D}f \star_N \mathcal{D}g\right)$.

In Example **2.11**, we have considered the covariance of a quantum space under the twisted Poincaré symmetry. Therefore, let us briefly discuss this property in more general terms, see e.g. [30] and references therein. Covariance of a space $\mathcal{M}$ under the symmetry algebra $\mathcal{U}_\hbar(\mathbf{g})$ means that $\mathcal{M}$ is a $\mathcal{U}_\hbar(\mathbf{g})-$module algebra:

$$g \triangleright m_\star(x\otimes y) = m_\star(\Delta_\hbar(g)\triangleright(x\otimes y)) = m_\star(g_{(1)}\triangleright x\otimes g_{(2)}\triangleright y), \tag{2.141}$$

where $g\in\mathcal{U}_\hbar(\mathbf{g})$, $x, y\in\mathcal{M}$, and $m_\star$ denotes the $\star$-multiplication on $\mathcal{M}$. Drinfel'd's theorem [31] establishes an isomorphism $\mathcal{U}_\hbar(\mathbf{g}) \cong (\mathcal{U}(\mathbf{g})[[\hbar]], \Delta_\hbar, \epsilon_\hbar, S_\hbar)$, where

$$\Delta_\hbar(g) = \mathcal{F}\Delta(g)\mathcal{F}^{-1}.$$

Δ is the classical co-product of $\mathcal{U}(\mathbf{g})$, $\mathcal{F}$ the so-called Drinfel'd twist. Covariant $\star$-products are defined in [30] using some appropriate Drinfel'd twist by

$$x\star y = m_\star(x\otimes y) = m(\mathcal{F}^{-1}\triangleright(x\otimes y))\,, \tag{2.142}$$

where m denotes the commutative, pointwise multiplication. The twist is defined uniquely by the requirement of covariance up to a central 2-coboundary.

3 Noncommutative Quantum Field Theory

In this section, we will consider Euclidean 4 dimensional space endowed with a Weyl-Moyal star-product.

3.1 Scalar models

For describing real scalar fields on a space with Weyl-Moyal star-product, we will here employ a straight forward approach. We simply replace the pointwise product in the commutative action by the non-local star-product. In a second step, we use the identities (2.97) and (2.98) in order to simply the expressions. The action then reads

$$\begin{aligned} S &= \int d^4x \left(\frac{1}{2}\partial_\mu\phi \star \partial_\mu\phi - \frac{m^2}{2}\phi\star\phi - \frac{\lambda^4}{4!}\phi\star\phi\star\phi\star\phi \right) \\ &= \int d^4x \left(\frac{1}{2}\partial_\mu\phi\, \partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda^4}{4!}\phi\star\phi\star\phi\star\phi \right) \end{aligned} \tag{3.1}$$

The first one to consider this action was T. Filk [32] who derived the corresponding Feynman rules, noticing that the propagator is exactly the same as in commutative space, i.e.

$$G(k) = \frac{1}{(2\pi)^4}\frac{1}{k^2+m^2}\,, \tag{3.2}$$

while the vertex gains phase factors in the momenta. In order to see this, let us examine the interaction part of (3.1):

$$\begin{aligned} \Gamma_{\text{int}} &= \int d^4x \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi(x) \\ &= \int d^4x \frac{1}{(2\pi)^8}\int (\Pi_{i=1}^4 d^4k_i)\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4) \\ &\qquad \times e^{\mathrm{ik_1x}}\star(e^{\mathrm{ik_2x}}\star(e^{\mathrm{ik_3x}}\star e^{\mathrm{ik_4x}}))) \\ &= \int d^4x \frac{1}{(2\pi)^8}\int (\Pi_{i=1}^4 d^4k_i)\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4) \\ &\qquad \times e^{\mathrm{i(k_1+k_2+k_3+k_4)x}} e^{-\frac{\mathrm{i}}{2}\sum_{i<j}k_i\theta k_j} \\ &= \frac{1}{(2\pi)^4}\int (\Pi_{i=1}^4 d^4k_i)\phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4) \\ &\qquad \times \delta^{(4)}(k_1+k_2+k_3+k_4) e^{-\frac{\mathrm{i}}{2}\sum_{i<j}k_i\theta k_j}\,, \end{aligned} \tag{3.3}$$

with $k_i\theta k_j = k_i^\alpha \theta_{\alpha\beta} k_j^\beta$. The vertex-function is defined by

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = \frac{\delta^4 S}{\delta\phi(p_1)\delta\phi(p_2)\delta\phi(p_3)\delta\phi(p_4)}\,,$$

which yields

$$\Gamma^{(4)}(p_1,p_2,p_3,p_4) = \frac{\lambda}{3}(2\pi)^4\delta^{(4)}(p_1+p_2+p_3+p_4) \\ \times\Bigg(\cos(\frac{1}{2}k_1\tilde{k}_2)\cos(\frac{1}{2}k_3\tilde{k}_4)+\cos(\frac{1}{2}k_1\tilde{k}_3)\cos(\frac{1}{2}k_2\tilde{k}_4) \\ +\cos(\frac{1}{2}k_1\tilde{k}_4)\cos(\frac{1}{2}k_2\tilde{k}_3)\Bigg) \tag{3.4}$$

As a consequence, new types of Feynman graphs appear: In addition to the ones known from commutative space, where no phases depending on internal loop momenta appear (*planar graphs*) and which exhibit the usual UV divergences, so-called *non-planar graphs* come into the game which are regularized by phases depending on internal momenta. Explicit one-loop calculations have been performed in [33, 34, 35, 36, 37] and the infamous UV/IR mixing problem has been discovered: Due to the phases in the non-planar graphs, their UV sector is regularized on the one hand, but on the other hand this regularization implies divergences for small external momenta instead. We will consider the example of the 2-point tadpole and discuss the UV/IR mixing property in detail in the next subsection.

3.1.1 UV/IR mixing

For example the two point tadpole graph is approximately given by the integral

$$\Pi(\Lambda,p) = \frac{\lambda}{3}\int d^4k\frac{2+\cos(k\tilde{p})}{k^2+m^2} \equiv \Pi^{\text{pl}}(\Lambda)+\Pi^{\text{n-pl}}(p)\,. \tag{3.5}$$

The planar contribution is as usual quadratically divergent in the UV cutoff Λ, i.e.

$$\Pi^{\text{pl}} \sim \Lambda^2\,.$$

The difference to the commutative case is the factor 2/3 and the non-planar contribution. This contribution is regularized by the cosine, and we get

$$\Pi^{\text{n-pl}} \sim \frac{1}{\tilde{p}^2}\,, \tag{3.6}$$

which shows that the original UV divergence is not present any more, but reappears when $\tilde{p} \to 0$ (where the phase is 1) representing a new kind of infrared divergence. Since both divergences are related to one another, one speaks of *UV/IR mixing*. At one-loop level, this is no problem though. It corresponds to a counter term

$$\int d^4p\, \tilde{\phi}(p) \frac{1}{\tilde{p}^2}\, \tilde{\phi}(-p)\,, \tag{3.7}$$

which is well behaved even in the limit $\tilde{p} \to 0$. But higher loop insertions then lead to a term of the form

$$\int d^4p\, \tilde{\phi}(p) \frac{1}{(\tilde{p}^2)^n} \tilde{\phi}(-p)\,, \tag{3.8}$$

where n is the number of insertions. Clearly, this term exhibits a serious IR singularity. It is this mixing which renders the action (3.1) non-renormalizable. Two different strategies to cure UV/IR mixing are known. Both modify the propagator by adding an additional term quadratic in the fields: An oscillator term and a $1/\tilde{p}^2$-term, respectively. In what follows, we will briefly review those approaches.

3.1.2 Ways out

Grosse-Wulkenhaar model. Adding an oscillator potential and after some awkward rewritting, the action (3.1) becomes [38, 39]

$$\begin{aligned} S \;=\; & \int d^4x \left(\frac{1}{2} \phi \star [\tilde{x}_\nu \overset{\star}{,} [\tilde{x}^\nu \overset{\star}{,} \phi]] + \frac{\Omega^2}{2} \phi \star \{\tilde{x}_\nu \overset{\star}{,} [\tilde{x}^\nu \overset{\star}{,} \phi]\} \right. \\ & \left. + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right)\,, \end{aligned} \tag{3.9}$$

where $\tilde{x}_\nu = \theta^{-1}_{\nu\alpha} x^\alpha$, and we have used $\mathrm{i}\,\partial_\mu \mathrm{f} = [\tilde{\mathrm{x}}_\mu \overset{\star}{,} \mathrm{f}]$. This action is covariant, i.e.

$$S[\phi; \mu, \lambda, \Omega] \mapsto \Omega^2 S[\phi; \frac{\mu}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}]\,, \tag{3.10}$$

under the so-called Langmann-Szabo duality transformation [40] between position and momenta:

$$\hat{\phi}(p) \longleftrightarrow \pi^2 \sqrt{|det\Theta|}\, \phi(x)\,, \qquad p_\mu \longleftrightarrow 2\tilde{x}_\mu\,, \tag{3.11}$$

where $\hat{\phi}(p_a) = \int d^4x_a \mathrm{e}^{(-1)^{\mathrm{a}}\mathrm{i}\mathrm{p}_{\mathrm{a},\mu}\mathrm{x}_{\mathrm{a},\mu}}\,\phi(\mathrm{x_a})$. The index a is labelling the legs of vertex and propagator, resp. and defines the direction of the according momentum. This becomes a symmetry at $\Omega = 1$.Due to oscillator term, the propagator is modified and an IR damping is implemented. The propagator is given by the Mehler kernel:

$$K_M(p,q) = \frac{\omega^3}{8\pi^2}\int_0^\infty \frac{d\alpha}{\sinh^2\alpha}\, \mathrm{e}^{-\frac{\omega}{4}(p-q)^2\coth\frac{\alpha}{2} - \frac{\omega}{4}(p+q)^2\tanh\frac{\alpha}{2}}\,, \tag{3.12}$$

where $\omega = \Theta/\Omega$. The IR damping is also responsible for a proper handling of the UV/IR mixing problem. The model is renormalisable to all orders in perturbation theory. The propagator depends on two momenta, an incoming and outgoing momentum, since the explicit x-dependence of the action breaks translation invariance. Therefore, also momentum conservation is broken. Remarkably, the oscillator term can be interpreted as coupling of the scalar field to the curvature of some specific noncommutative background [41].

$1/p^2$-model. In the second approach, a non-local term is added to the action (3.1). In momentum space, it reads [42]

$$S_{nl} = \int d^4p\, \frac{a}{2}\tilde{\phi}(p)\frac{1}{\tilde{p}^2}\,\tilde{\phi}(-p)\,. \tag{3.13}$$

This is exactly the counter term (3.7) we have discussed before. The resulting action is translation invariant, and thus momentum conservation holds. The term (3.13) implements IR damping for the propagator, i.e. $G(p) \to 0$, for $p \to 0$. The modified propagator has the form

$$G(p) = \frac{1}{p^2 + m^2 + \frac{a^2}{p^2}}\,. \tag{3.14}$$

The damping effect of the propagator becomes obvious when one considers higher loop orders. An n-fold insertion of the divergent one-loop result (3.6) into a single large loop can be written as

$$\Pi^{n\mathrm{np-ins.}}(p) \approx \lambda^2 \int d^4k\, \frac{\mathrm{e}^{\mathrm{i}k\tilde{\mathrm{p}}}}{\left(\tilde{k}^2\right)^n \left[k^2 + m^2 + \frac{a'^2}{k^2}\right]^{n+1}}\,, \tag{3.15}$$

neglecting any effects due to recursive renormalization and approximating the insertions of irregular single loops by the most divergent (quadratic) IR divergence. For the model (3.1), i.e. $a = 0$, the integrand is proportional to $(k^2)^{-n}$, for $k^2 \to 0$, as we have already mentioned. But $a \neq 0$ implies that the integrand behaves like

$$\frac{1}{\left(\tilde{k}^2\right)^n \left[\frac{a'^2}{k^2}\right]^{n+1}} = \frac{\tilde{k}^2}{(a'^2)^{n+1}}\,, \tag{3.16}$$

which is independent of the loop order n. Using multiscale analysis, the perturbative renormalisability of this model to all orders could be shown [42].

3.1.3 Minkowski space-time

In this section, we want to briefly discuss or rather mention the difficulties that arise when we consider noncommutative Minkowski space-time. They occur when we consider $\theta^{0\mu} \neq 0$ and is due to the fact that the interaction part of the Lagrangian now depends on infinitely many time derivatives acting on the fields. This corresponds to a non-locality in time. In the commutative case, field theories on Minkowski space-time and Euclidean space are related by Wick rotation. However, the generalisation of the Wick rotation to the noncommutative realm is not understood up to now.

A way to deal with fields on noncommutative Minkowski space has been proposed by S. Doplicher *et al.* [7] and further developed for noncommutative scalar ϕ^4 theory by several authors [43, 44, 45]. It is termed "interaction point time ordered perturbation theory" (IPTOPT) and is based on the following idea: Consider the Gell-Mann–Low formula applied to the field operators $\hat{\phi}$ of a scalar $\hat{\phi}^4$ theory

$$\begin{aligned}\left\langle 0|T\{\hat{\phi}_H(x_1)\ldots\hat{\phi}_H(x_n)\}|0\right\rangle &= \sum_{m=0}^{\infty}\frac{(-\mathrm{i})^{\mathrm{m}}}{m!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \ldots \int_{-\infty}^{\infty} dt_m \\ &\times \left\langle 0|T\{\hat{\phi}_I(x_1)\ldots\hat{\phi}_I(x_n)\hat{V}(t_1)\ldots\hat{V}(t_m)\}|0\right\rangle .\end{aligned} \tag{3.17}$$

The subscripts H and I denote the Heisenberg picture and the interaction picture, respectively; V is the interaction part of the Hamiltonian

$$\hat{V}(z^0) = \int d^3z \frac{\kappa}{4!}\hat{\phi}(z)\star\hat{\phi}(z)\star\hat{\phi}(z)\star\hat{\phi}(z)\,. \tag{3.18}$$

The idea is that the time-ordering operator T acts on the time components of the x_i and on the so-called *time stamps* $t_1, \ldots, t_m$. For example, considering the interaction (3.18) with an alternative representation for the star products

$$V(z^0) = \frac{\kappa}{4!} \prod_{i=1}^{3} \int \frac{d^4 s_i d^4 l_i}{(2\pi)^4} e^{\mathrm{i s_i l_i}} \\ \times \phi(z - \frac{1}{2}\tilde{l}_1)\phi(z + s_1 - \frac{1}{2}\tilde{l}_2)\phi(z + s_1 + s_2 - \frac{1}{2}\tilde{l}_3)\phi(z + s_1 + s_2 + s_3)\,,$$

the time ordering only affects z^0 and no other time components (like e.g. l_i^0 etc.). This leads to modified Feynman rules. For example, the propagator of the commutative ϕ^4 theory,

$$G(x, x') = \int \frac{d^4 k}{(2\pi)^4} \frac{e^{\mathrm{ik(x-x')}}}{k^2 + m^2 - \mathrm{ie}}\,, \tag{3.19}$$

is generalised to the so-called *contractor*

$$G_C(x, t; x', t') = \int \frac{d^4 k}{(2\pi)^4} \frac{\exp\left[\mathrm{ik(x - x') + ik^0(x^0 - t - (x'^0 - t'))}\right]}{k^2 + m^2 - \mathrm{ie}} \tag{3.20}$$
$$\times \left[\cos\big(\omega_k(x^0 - t - (x'^0 - t'))\big) - \frac{\mathrm{ik}^0}{\omega_k} \sin\big(\omega_k(x^0 - t - (x'^0 - t'))\big)\right].$$

For $x^0 = t$ and $x'^0 = t'$ (which is the case if $\theta^{0\mu} = 0$), the contractor reduces to (3.19). This approach seems promising in some respects, meaning that one may extend the formalism to noncommutative gauge fields, although (among many others) the question of unitarity is still unclear [46].

Finally, one should also remark that similar work, i.e. considerations concerning proper time ordering when dealing with noncommutative time, have been done by D. Bahns *et al.* [47, 48].

3.2 Gauge models

As one generally assumes the commutator $\theta^{\mu\nu}$ to be very small (as mentioned in the introduction perhaps even of the order of the Planck length squared), it certainly makes sense to also consider an expansion of a noncommutative theory in terms of that parameter. In the expanded approach, noncommutative gauge theory is based on essentially three principles,

- Covariant coordinates,
- Locality and classical limit,
- Gauge equivalence conditions.

Let ψ be a noncommutative field with infinitesimal gauge transformation

$$\widehat{\delta}\psi(x) = \mathrm{i}\alpha \star \psi(\mathrm{x})\,, \tag{3.21}$$

where α denotes the gauge parameter. The $\star$-product of a field and a coordinate does not transform covariantly,

$$\widehat{\delta}(x \star \psi(x)) = \mathrm{i}\,\mathrm{x} \star \alpha(\mathrm{x}) \star \psi(\mathrm{x}) \neq \mathrm{i}\,\alpha(\mathrm{x}) \star \mathrm{x} \star \psi(\mathrm{x})\,. \tag{3.22}$$

Therefore, one has to introduce covariant coordinates [49]

$$X^\mu \equiv x^\mu + g\theta^{\mu\alpha} A_\alpha\,, \tag{3.23}$$

such that

$$\widehat{\delta}(X^\mu \star \psi) = \mathrm{i}\alpha \star (\mathrm{X}^\mu \star \psi)\,. \tag{3.24}$$

Hence, covariant coordinates and the gauge potential transform under a noncommutative gauge transformation in the following way

$$\widehat{\delta}X^\mu = \mathrm{i}\,[\alpha \stackrel{\star}{,} \mathrm{X}^\mu]\,,\ \mathrm{g}\,\widehat{\delta}\mathrm{A}^\mu = \mathrm{i}\theta^{-1}_{\mu\alpha}\,[\alpha \stackrel{\star}{,} \mathrm{x}^\alpha] + \mathrm{ig}\,[\alpha \stackrel{\star}{,} \mathrm{A}^\mu]\ , \tag{3.25}$$

where we have assumed that θ is non-degenerate. Other covariant objects can be constructed from covariant coordinates, such as the field strength,

$$\mathrm{ig}\theta^{\mu\alpha}\theta^{\nu\beta}\mathrm{F}_{\alpha\beta} = [\mathrm{X}^\mu \stackrel{\star}{,} \mathrm{X}^\nu] - \mathrm{i}\theta^{\mu\nu}, \qquad \widehat{\delta}\mathrm{F}^{\mu\nu} = \mathrm{i}\,[\alpha \stackrel{\star}{,} \mathrm{F}^{\mu\nu}]\ . \tag{3.26}$$

3.2.1 Seiberg-Witten maps

For simplicity, we will set the coupling constant $g = 1$ in this section. The star product can be written as an expansion in a formal parameter θ,

$$f \star g = f \cdot g + \sum_{n=1}^{\infty} \theta^n C_n(f,g)\,.$$

In the commutative limit $\theta \to 0$, the star product reduces to the pointwise product of functions. One may ask, if there is a similar commutative limit for the fields. The solution to this question was given for Abelian gauge groups by [9],

$$\begin{aligned}
\widehat{A}_\mu[A] &= A_\mu + \frac{\theta}{2}\theta^{\sigma\tau}\left(A_\tau\partial_\sigma A_\mu + F_{\sigma\mu}A_\tau\right) + \mathcal{O}(\theta^2)\,, \\
\widehat{\psi}[\psi, A] &= \psi + \frac{\theta}{2}\theta^{\mu\nu}A_\nu\partial_\mu\psi + \mathcal{O}(\theta^2)\,, \\
\widehat{\alpha} &= \alpha + \frac{\theta}{2}\theta^{\mu\nu}A_\nu\partial_\mu\alpha + \mathcal{O}(\theta^2)\,.
\end{aligned} \tag{3.27}$$

The origin of this map lies in string theory. There, the resulting gauge theory depends on the applied regularization scheme [9]. Pauli-Villars regularization provides us with classical gauge invariance

$$\delta A_i = \partial_i\lambda, \tag{3.28}$$

whence point-splitting regularization comes up with noncommutative gauge invariance

$$\widehat{\delta}_\lambda\widehat{A}_i = \partial_i\widehat{\Lambda} + \mathrm{i}\left[\widehat{\Lambda} \stackrel{\star}{,} \widehat{\mathrm{A}}_\mathrm{i}\right]\,. \tag{3.29}$$

N. Seiberg and E. Witten argued that consequently there must be a local map from ordinary gauge theory to noncommutative gauge theory

$$\widehat{A}[A],\ \widehat{\Lambda}[\lambda, A]\,, \tag{3.30}$$

satisfying

$$\widehat{A}[A + \delta_\lambda A] = \widehat{A}[A] + \widehat{\delta}_\lambda\widehat{A}[A]\,, \tag{3.31}$$

where δ_α denotes an ordinary gauge transformation and $\widehat{\delta}_\alpha$ a noncommutative one. The Seiberg-Witten (SW) maps are solutions of the so-called "gauge-equivalence relation" (3.31).

By locality we mean that in each order in the noncommutativity parameter θ there is only a finite number of derivatives. Let us remember that we consider arbitrary gauge groups. The noncommutative gauge fields $\widehat{A}$ and gauge parameters $\widehat{\Lambda}$ are enveloping algebra valued. Let us choose a symmetric basis in the enveloping algebra, T^a, $\frac{1}{2}(T^aT^b + T^bT^a)$, ..., such that

$$\begin{aligned}
\widehat{\Lambda}(x) &= \widehat{\Lambda}_a(x)T^a + \widehat{\Lambda}^1_{ab}(x)\ :T^aT^b: + \ldots\,, \\
\widehat{A}_\mu(x) &= \widehat{A}_{\mu a}(x)T^a + \widehat{A}_{\mu ab}(x)\ :T^aT^b: + \ldots\,.
\end{aligned} \tag{3.32}$$

Eqn. (3.31) defines the SW maps for the gauge field and the gauge parameter. However, it is more practical to find equations for the gauge parameter and the gauge field alone [50]. First, we will concentrate on the gauge parameters $\widehat{\Lambda}$. We already encountered the consistency condition

$$\widehat{\delta}_\alpha \widehat{\delta}_\beta - \widehat{\delta}_\beta \widehat{\delta}_\alpha = \widehat{\delta}_{-\mathrm{i}[\alpha \stackrel{\star}{,} \beta]} \,,$$

which more explicitly reads

$$\mathrm{i}\widehat{\delta}_\alpha \widehat{\beta}[\mathrm{A}] - \mathrm{i}\widehat{\delta}_\beta \widehat{\alpha}[\mathrm{A}] + \Big[\widehat{\alpha}[\mathrm{A}] \stackrel{\star}{,} \widehat{\beta}[\mathrm{A}]\Big] = (\widehat{[\alpha,\beta]})[\mathrm{A}] \,. \tag{3.33}$$

We can expand $\widehat{\alpha}$ in terms of θ,

$$\widehat{\alpha}[A] = \alpha + \alpha^1[A] + \alpha^2[A] + \mathcal{O}(\theta^3) \,, \tag{3.34}$$

where α^n is $\mathcal{O}(\theta^n)$. The consistency relation (3.33) can be solved order by order in θ:

$$\begin{aligned} 0^{\text{th}} \text{ order}: \; \alpha^0 &= \alpha \,, \\ 1^{\text{st}} \text{ order}: \; \alpha^1 &= \frac{\theta}{4}\theta^{\mu\nu} \left\{\partial_\mu \alpha, A_\nu\right\} \\ &= \frac{\theta}{2}\theta^{\mu\nu} \partial_\mu \alpha_a A_{\mu b} : T^a T^b : \,. \end{aligned} \tag{3.35}$$

For fields $\widehat{\psi}$ the condition

$$\delta_\alpha \widehat{\psi}[A] = \widehat{\delta}_\alpha \widehat{\psi}[A] = \mathrm{i}\widehat{\alpha}[\mathrm{A}] \star \widehat{\psi}[\mathrm{A}] \tag{3.36}$$

has to be satisfied. In other words, the ordinary gauge transformation induces a noncommutative gauge transformation. We expand the fields in terms of the non-commutativity

$$\widehat{\psi} = \psi^0 + \psi^1[A] + \psi^2[A] + \dots \,, \tag{3.37}$$

and solve Eqn. (3.36) order by order in θ. In first order, we have to find a solution to

$$\delta_\alpha \psi^1[A] = \mathrm{i}\alpha\psi^1 + \mathrm{i}\alpha^1\psi - \frac{\theta}{2}\theta^{\mu\nu}\partial_\mu \alpha \partial_\nu \psi \,. \tag{3.38}$$

It is given by

$$\begin{aligned}
0^{\text{th}} \text{ order}: \ \psi^0 &= \psi\,, \\
1^{\text{st}} \text{ order}: \ \psi^1 &= -\frac{\theta}{2}\theta^{\mu\nu}A_\mu\partial_\nu\psi + \frac{\mathrm{i}\theta}{4}\theta^{\mu\nu}A_\mu A_\nu\psi\,.
\end{aligned} \tag{3.39}$$

The gauge fields $\widehat{A}_\mu$ have to satisfy

$$\delta_\alpha \widehat{A}_\mu[A] = \partial_\mu\widehat{\alpha}[A] + \mathrm{i}\left[\widehat{\alpha}[\mathrm{A}] \stackrel{\star}{,} \widehat{\mathrm{A}}_\mu[\mathrm{A}]\right]\,. \tag{3.40}$$

Using the expansion

$$\widehat{A}_\mu[A] = A^0_\mu + A^1_\mu[A] + A^2_\mu[A] + \dots\,, \tag{3.41}$$

and solving (3.40) order by order, we end up with

$$\begin{aligned}
0^{\text{th}} \text{ order}: \ A^0_\mu &= A_\mu\,, \\
1^{\text{st}} \text{ order}: \ A^1_\mu &= -\frac{\theta}{4}\theta^{\tau\nu}\left\{A_\tau, \partial_\nu A_\mu + F_{\nu\mu}\right\}\,,
\end{aligned} \tag{3.42}$$

where $F_{\nu\mu} = \partial_\nu A_\mu - \partial_\mu A_\nu - \mathrm{i}\,[\mathrm{A}_\nu, \mathrm{A}_\mu]$. Similarly, we have for the field strength $\widehat{F}_{\mu\nu}$

$$\begin{aligned}
\delta_\alpha \widehat{F}_{\mu\nu} &= \mathrm{i}\left[\widehat{\alpha}, \widehat{\mathrm{F}}_{\mu\nu}\right] \\
\text{and } \widehat{F}_{\mu\nu} &= F_{\mu\nu} + \frac{\theta}{2}\theta^{\sigma\tau}\left\{F_{\mu\sigma}, F_{\nu\tau}\right\} - \frac{\theta}{4}\theta^{\sigma\tau}\left\{A_\sigma, (\partial_\tau + \mathcal{D}_\tau)F_{\mu\nu}\right\}\,,
\end{aligned} \tag{3.43}$$

where $\mathcal{D}_\mu F_{\tau\nu} = \partial_\mu F_{\tau\nu} - \mathrm{i}\,[\mathrm{A}_\mu, \mathrm{F}_{\tau\nu}]$.

3.2.2 Oscillator models

As a first step, a BRST invariant action including an oscillator term has been proposed in [51]:

$$S = \int d^4x \left(\frac{1}{4}F_{\mu\nu} \star F^{\mu\nu} + s(\bar{c} \star \partial_\mu A_\mu) - \frac{1}{2}B^2 + \frac{\Omega^2}{8}s(\tilde{c}_\mu \star \mathcal{C}_\mu)\right)\,, \tag{3.44}$$

where $\mathcal{C}_\mu$ contains the crucial new terms:

$$\mathcal{C}_\mu = \{\{\tilde{x}_\mu \stackrel{\star}{,} A_\nu\} \stackrel{\star}{,} A_\nu\} + [\{\tilde{x}_\mu \stackrel{\star}{,} \bar{c}\} \stackrel{\star}{,} c] + [\bar{c} \stackrel{\star}{,} \{\tilde{x}_\mu \stackrel{\star}{,} c\}]\,, \tag{3.45}$$

and $\tilde{c}_\mu$ is a new parameter which also transforms under BRST. The noncommutative field strength is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}\,[\mathrm{A}_\mu \stackrel{\star}{,} \mathrm{A}_\nu]$. Summing up, the action (3.44) is invariant under the following BRST transformation:

$$\begin{aligned} sA_\mu &= D_\mu c, \quad s\bar{c} = B, \qquad sc = igc \star c, \\ sB &= 0, \qquad s\tilde{c}_\mu = \tilde{x}_\mu \,. \end{aligned} \tag{3.46}$$

The above set of transformations is nilpotent. The propagator of the gauge field is given by Mehler kernel (3.12). One-loop calculations have been preformed in [52]. A power counting formula has been obtained and the corrections to the vertex functions have been computed. Remarkably, the one-point tadpole is UV-divergent. Therefore, the action (3.44) is not stable under one-loop corrections, and a linear counter terms needed.

It seems natural to look for a more general action. The so-called induced gauge action [53, 54] contains the terms of (3.44) and more. It is invariant under noncommutative $U(1)$ transformations. The starting point is the scalar ϕ^4 model with oscillator potential (3.9). The scalar field is then coupled to an external gauge field. The dynamics of the gauge field is given by the divergent contributions of the one-loop effective action generalising the method of heat kernel expansion to the noncommutative realm. The induced action is given by

$$\begin{aligned} S \;=\; \int d^4x \Bigg\{ &\frac{3}{\theta}(1-\rho^2)(\tilde{\mu}^2-\rho^2)(\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \\ &+\frac{3}{2}(1-\rho^2)^2\big((\tilde{X}_\mu \star \tilde{X}_\mu)^{\star 2} - (\tilde{x}^2)^2\big) - \frac{\rho^4}{4} F_{\mu\nu}F_{\mu\nu} \Bigg\}, \end{aligned} \tag{3.47}$$

where $\rho = \frac{1-\Omega^2}{1+\Omega^2}$, $\tilde{\mu}^2 = \frac{m^2\theta}{1+\Omega^2}$. Furthermore, the field strength is given by

$$F_{\mu\nu} = -i[\tilde{x}_\mu, A_\nu]_\star + i[\tilde{x}_\nu, A_\mu]_\star - i[A_\mu, A_\nu]_\star \,,$$

and $\tilde{X}_\mu$ denote the covariant coordinates, $\tilde{X}_\mu = \tilde{x}_\mu + A_\mu$. In the limit $\Omega \to 0$ (i.e., $\rho \to 1$), we recover the usual noncommutative Yang-Mills action. An interesting limit is $\Omega \to 1$ (i.e., $\rho \to 0$), where we obtain a pure matrix model. It has a non-trivial vacuum, which makes the quantization more difficult. The computation of propagator and Feynman rules and also one-loop calculations are work in progress.

An alternative model has been proposed in [55]. The gauge model is constructed on a specific curved noncommutative background space, the so-called truncated Heisenberg space. In two dimensions the action reads

$$S = \int d^2x \Big((1-\alpha^2) F_{12}^{\star 2} - 2(1-\alpha^2)\mu\, F_{12} \star \phi + (5-\alpha^2)\mu^2\phi^2 \qquad (3.48)$$

$$+4\mathrm{i}\alpha\, \mathrm{F}_{12} \star \phi^{\star 2} + (\mathrm{D_i}\phi)^2 - \alpha^2 \left\{\mathrm{p_i} + \mathrm{A_i} \stackrel{\star}{,} \phi\right\}^2 \Big) , \qquad (3.49)$$

where α is some parameter and μ has dimension of a mass.

3.2.3 $1/p^2$ model

The same strategy as in the scalar case is applied here, the IR divergence is added as a counter term. Considering the action

$$S = \int d^4x\, F_{\mu\nu} \star F_{\mu\nu} \qquad (3.50)$$

for noncommutative $U(1)$ theory, the vacuum polarization shows the following IR divergent contribution:

$$\Pi_{\mu\nu} \propto \frac{\tilde{p}_\mu \tilde{p}_\nu}{(\tilde{p}^2)^2} \,. \qquad (3.51)$$

A gauge invariant implementation of the above is given by the term [56]

$$\int d^4x\, F_{\mu\nu} \frac{1}{\tilde{D}^2 D^2} F_{\mu\nu} \,. \qquad (3.52)$$

The inverse covariant derivatives in the above expression need to be expanded in terms the gauge field. Hence, vertices with arbitrary number of photon legs occur. This situation might still be treatable, but it is simpler to use a localised version of (3.52). Basically, there are two different ways to implement the localization:

- By introducing an antisymmetric field $B_{\mu\nu}$ [57]:

  $$\int d^4x F_{\mu\nu} \frac{a^2}{\tilde{D}^2 D^2} F_{\mu\nu} \to \int d^4x \left(a B_{\mu\nu} F_{\mu\nu} - B_{\mu\nu} \star \tilde{D}^2 D^2 B_{\mu\nu} \right) \,. \qquad (3.53)$$

  But this field is physical and introduces additional degrees of freedom. Therefore, the model is not pure noncommutative $U(1)$ gauge theory any more but describes different physics.

- Secondly, BRST doublet structures are employed in [58]. The additional fields needed for the localization of (3.52) build BRST doublets. This avoids the introduction of new physical degrees of freedom. Unfortunately, the model presented in [58] is not renormalizable.

The virtue of the latter approach is the implementation of the IR damping as a so-called "soft breaking". This is in analogy to the Gribov-Zwanziger approach to undeformed QCD [59, 60], where an IR modification of the propagator is suggested to cure the Gribov ambiguities. The UV renormalizability is not altered. In [61], the "soft breaking" approach has been developed further. As a result the following action is proposed:

$$S = S_{\text{inv}} + S_{\text{gf}} + S_{\text{aux}} + S_{\text{soft}} + S_{\text{ext}}\,, \tag{3.54}$$

$$S_{\text{inv}} = \int d^4x\, \frac{1}{4} F_{\mu\nu} F_{\mu\nu}\,, \tag{3.55}$$

$$S_{\text{gf}} = \int d^4x\, s(\bar{c}\partial_\mu A_\mu)\,, \tag{3.56}$$

$$S_{\text{aux}} = \int d^4x\, s\left(\bar{\psi}_{\mu\nu} B_{\mu\nu}\right)\,, \tag{3.57}$$

$$S_{\text{soft}} = \int d^4x\, s\left((\bar{Q}_{\mu\nu\alpha\beta} B_{\mu\nu} + Q_{\mu\nu\alpha\beta} \bar{B}_{\mu\nu}) \frac{1}{\tilde{\square}} (f_{\alpha\beta} + \sigma \frac{\theta_{\alpha\beta}}{2} \tilde{f}) \right)\,, \tag{3.58}$$

$$S_{\text{ext}} = \int d^4x\, (\Omega^A_\mu s A_\mu + \Omega^c s c)\,, \tag{3.59}$$

where $f_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the commutative $U(1)$ field strength, $\Theta_{\alpha\beta} = \epsilon\,\theta_{\alpha\beta}$ and $\tilde{f} = \theta_{\alpha\beta} f_{\alpha\beta}$, $\tilde{\square} = \tilde{\partial}_\mu \tilde{\partial}_\mu = \theta_{\mu\alpha}\theta_{\mu\beta}\partial_\alpha\partial_\beta$. For convenience, ϵ has mass dimension -2, whereas $\theta_{\mu\nu}$ is rendered dimensionless. The additional sources $\bar{Q}, Q, \bar{J}, J$ ensure BRST invariance of (3.54). In the IR, they take their physical values:

$$\begin{aligned} \bar{Q}_{\mu\nu\alpha\beta}|_{phys} &= 0, \qquad \bar{J}_{\mu\nu\alpha\beta}|_{phys} = \frac{\gamma^2}{4}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}), \\ Q_{\mu\nu\alpha\beta}|_{phys} &= 0, \qquad J_{\mu\nu\alpha\beta}|_{phys} = \frac{\gamma^2}{4}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha})\,. \end{aligned} \tag{3.60}$$

Inserting the physical values and integrating out the field $B_{\mu\nu}$ the following action is obtained:

$$S_{\text{phys}} = \int d^4x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \gamma^4 \left[\partial_\mu A_\nu \frac{1}{2\tilde{\square}^2} f_{\mu\nu} + \left(\sigma + \frac{\theta^2}{4}\sigma^2 \right) (\tilde{\partial} A) \frac{1}{\tilde{\square}^2} (\tilde{\partial} A) \right] + s\,(\bar{c}\partial_\mu A_\mu) \right)\,. \tag{3.61}$$

The term proportional to γ^4 breaks gauge invariance. It is called "soft breaking" since the parameter γ has dimension of mass. We have used the commutative field strength in this expression although it is not covariant under noncommutative gauge transformations. But it only appears in the breaking term and cannot make it worse, since gauge invariance is already violated. The advantage is that only the propagation but not the interaction is modified due to the "soft breaking".

The full action (3.54) is invariant under the following set of BRST transformations:

$$\begin{gathered} sA_\mu = D_\mu c\,, \quad sc = igcc\,, \quad s\bar{c} = b\,, \quad sb = 0\,, \\ s\bar{\psi}_{\mu\nu} = \bar{B}_{\mu\nu}\,, \quad s\bar{B}_{\mu\nu} = 0\,, \quad sB_{\mu\nu} = \psi_{\mu\nu}\,, \quad s\psi_{\mu\nu} = 0\,, \\ s\bar{Q} = \bar{J}\,, \quad s\bar{J} = 0\,, \quad sQ = J\,, \quad sJ = 0\,. \end{gathered} \tag{3.62}$$

The fields ψ and B, resp. $\bar{\psi}$ and $\bar{B}$ and the sources Q and J, resp. $\bar{Q}$ and $\bar{J}$ are BRST doublets. Let us discuss the Feynman rules for (3.54). The vertex functions are the same as in the usual noncommutative $U(1)$ theory defined by the action (3.50). The propagator is more complicated, it reads

$$G^A_{\mu\nu}(k) = \left(k^2 + \frac{\gamma^4}{\tilde{k}^2}\right)^{-1} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - \frac{\bar{\sigma}^4}{(k^2 + (\bar{\sigma}^4 + \gamma^4)\frac{1}{\tilde{k}^2})} \frac{\tilde{k}_\mu \tilde{k}_\nu}{(\tilde{k}^2)^2}\right)\,, \tag{3.63}$$

where

$$\bar{\sigma} = 2\gamma^4 \left(\sigma + \frac{\theta^2\sigma^2}{4}\right)\,.$$

But for 1-loop calculation, it can be approximated by

$$G^A_{\mu\nu} \sim \frac{1}{k^2}(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}), \quad k^2 >> 1\,, \tag{3.64}$$

since both UV and IR divergences result from high momentum range in the loop. This ignores the IR damping, but as we have seen the damping has no effect at one-loop. Considering higher loop insertions of a single tadpole (cf. (3.15)) the damping of the propagators between the single loops is essential and renders the result independent of the number of inserted loops - at least in the scalar case, for the gauge model discussed here this still needs to be shown.

A power counting formula,

$$d_G = 4 - E_A - E_{c\bar{c}}\,, \tag{3.65}$$

where E_ϕ denotes the number of external ϕ-legs, and one-loop results have been obtained in [61]. The correction to the vacuum polarization is given by

$$\Pi_{\mu\nu} = \frac{2g^2}{\epsilon^2\pi^2}\frac{\tilde{p}_\mu\tilde{p}_\nu}{(\tilde{p}^2)^2} + \frac{13g^2}{3(4\pi)^2}(p^2\delta_{\mu\nu} - p_\mu p_\nu)\ln\Lambda\,, \tag{3.66}$$

where Λ denotes a momentum cut-off. Remarkably, the one-loop correction is transversal. Furthermore, we obtained the following results for the vertices:

$$\begin{aligned}
\Gamma^{3A,IR}_{\mu\nu\rho} &= -\frac{2ig^3}{\pi^2}\cos\frac{\epsilon p_1\tilde{p}_2}{2}\sum_{j=1,2,3}\frac{\tilde{p}_{j,\mu}\tilde{p}_{j,\nu}\tilde{p}_{j,\rho}}{\epsilon(\tilde{p}_j^2)^2}\,, && (3.67)\\
\Gamma^{3A,UV}_{\mu\nu\rho} &= -\frac{17g^2}{6(4\pi)^2}\ln\Lambda\,\tilde{V}^{3A,\text{tree}}_{\mu\nu\rho}(p_1,p_2,p_3)\,, && (3.68)\\
\Gamma^{4A,UV}_{\mu\nu\rho\sigma} &= -\frac{5}{8\pi^2}\ln\Lambda\,\tilde{V}^{4A,\text{tree}}_{\mu\nu\rho\sigma}\,, && (3.69)
\end{aligned}$$

where $V^{3A,\text{tree}}_{\mu\nu\rho}$ and $V^{4A,\text{tree}}_{\mu\nu\rho\sigma}$ denote the tree level vertex functions. Regarding the three-point function, the IR divergent result (3.67) corresponds to a counter term

$$S^{3A,\text{corr}} = \int d^4x\, g^3\,\{A_\mu \stackrel{\star}{,} A_\nu\}\,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\epsilon\,\tilde{\Box}^2}A_\rho\,. \tag{3.70}$$

Such a term can readily be introduced into the "soft breaking" part of the action S_{soft} in (3.54). But in order to do so, we have to restore BRST invariance in the UV regime. Again, this can be achieved by introducing sources Q' and J', which form a BRST doublet,

$$sQ' = J'\,, \qquad sJ' = 0\,. \tag{3.71}$$

Consequently, we insert the following terms into S_{soft}:

$$\int d^4x \left(J'\,\{A_\mu \stackrel{\star}{,} A_\nu\}\,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\tilde{\Box}^2}A_\rho - Q's\left(\{A_\mu \stackrel{\star}{,} A_\nu\}\,\frac{\tilde{\partial}_\mu\tilde{\partial}_\nu\tilde{\partial}_\rho}{\tilde{\Box}^2}A_\rho\right)\right)\,. \tag{3.72}$$

This term is BRST invariant by itself. In the IR, the sources take on their physical values

$$J' = g\gamma'^2,\ Q' = 0\,, \tag{3.73}$$

and the counter term in (3.70) leads to a renormalization of γ', which is another parameter of mass-dimension 1.

The above one-loop result leads to a negative β-function:

$$\beta = -\frac{7g^3}{12\pi^2}\,.$$

List of abbreviations

EOM	equations of motion
LHS	left hand side
QYBE	Quantum-Yang-Baxter-Equation
RHS	right hand side

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