Diplomarbeit

Thermal corrections to the sunset diagram

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To Max and Bello
who introduced me to the beauty of mathematics

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Chapter 1

Introduction

The marriage of two of the most successful theories in physics, quantum mechanics and statistical mechanics, has led to the explanation of many interesting physical phenomena in the past, such as black body radiation or phase transitions. In the 1950s the concept was extended to non-relativistic field theory at finite temperature and density by Matsubara [46] in the context of condensed matter physics and generalized to relativistic field theories by Fradkin [2] in the mid 1960s. After having lain dormant for some years, finite temperature field theories (as they are now called) have been rediscovered and systematically developed in the last decades.

Today quantum field theories at finite temperature and density provide a theoretical framework to describe the physics of extreme environments: the deconfined phase of quarks and gluons called "quark-gluon-plasma" at very high temperatures and color super-conductivity in the CFL (color-flavor-locked) phase at very high densities. The first phenomenon is commonly believed to have occurred in the early universe and there is evidence that a quark-gluon plasma has been produced in ultrarelativistic heavy ion collisions at CERN [47]. The existence of CFL matter, on the other hand, might have interesting astrophysical consequences [24].

However, gauge theories at finite temperature are quite complicated to handle and give rise to many technical difficulties that are only partly resolved, especially when dealing with higher order corrections in perturbation theory. To explore the technical aspects of these higher order corrections we have decided to study the simple toy model of the massive neutral scalar field at finite temperature where calculations are much simpler due to the lack of gauge dependence, but nevertheless non-trivial. The present work is organized as follows: in this chapter we set up the framework of perturbation theory at finite temperature for the scalar field which is meant as an introductory chapter for people not acquainted with the field and will also be used to fix our conventions and notations. In chapter 2 we specialize the theory to the $\lambda\phi^4$ model and – after fixing the Feynman rules and the definition of the self-energy at finite temperature – calculate the self-energy to first order. In chapter 3 we go on to calculate the second order of the self-energy that involves mainly the "sunset" diagram which has given this work its title. We keep the full momentum dependence of the diagram and do as many integrals as possible analytically (the most technical calculations have been relegated to appendix C). In chapter 4 we proceed to renormalize the results and find the full momentum dependent self-energy to second order. As this result will still contain some contributions that we were not able to evaluate analytically for general momenta we present some numerical evaluation of these constituents in chapter 5. In chapter 6 we go on to derive analytic results for a specific external momentum and finally in the last chapter we give a short introduction to the resummation of perturbation theory and calculate the thermal quasiparticle mass to second order in the coupling constant. Furthermore, since we have found it convenient when reading monographs by other authors we have appended a summary of the contents at the end of each chapter.

As a further motivation, it should be pointed out that the result for the self-energy to two-loop order will allow us to calculate the entropy of the system to three-loop order. The reason for this is provided by the proof in [45] that if the contribution to the entropy in the Φ-derivable approximation coming from the interactions between the quasiparticles of a hot plasma was vanishing to two-loop order the result for the entropy to this order would only involve the self-energy to first loop order. Contrary to more complicated theories, in $\lambda\phi^4$ theory this proof is easily extended to three and two loops respectively, so that the ensuing exact 3-loop result for the entropy may be used to benchmark the quality of different approximations that one ultimately has to resort to in more realistic theories such as QCD.

1.1 Transition amplitude for bosons

We will start by calculating the transition amplitude for a quantum mechanical bosonic system, described by a Lagrangian density $\mathcal{L}\left(\phi, \partial_\mu \phi\right)$, where $\phi(x)$ is a real scalar field[1]. We assume that there exists a field operator $\hat{\phi}\left(\mathbf{x}\right)$ (in the Schrödinger picture) and its conjugate momentum operator $\hat{\pi}(x)$. At a time $t = t_i$ any quantum state of the system may be formally expanded in the basis of eigenstates of the field operator $\hat{\phi}_{t=t_i}\left(\mathbf{x}\right)$. Such an eigenstate will be labelled $|\phi>$ and the action of the field operator at the time $t = t_i$ is then given by

$$\hat{\phi}_{t=t_i}\left(\mathbf{x}\right)|\phi>=\phi\left(\mathbf{x}\right)|\phi>,$$

where $\phi(\mathbf{x})$ is the eigenvalue of the state $|\phi>$ with respect to $\hat{\phi}_{t=t_i}\left(\mathbf{x}\right)$ (being actually a function of $\mathbf{x}$). We require $|\phi>$ to satisfy the usual completeness and

[1] The approach we follow is taken from [1].

orthogonality conditions

$$\begin{aligned}\int d\phi(\mathbf{x})|\phi >< \phi| &= 1 \\ < \phi_a|\phi_b > &= \delta\left[\phi_a(\mathbf{x}) - \phi_b(\mathbf{x})\right] \end{aligned} \tag{1.1}$$

at any given time t. Similarly, the eigenstates of the conjugate momentum operator $\hat{\pi}\left(\mathbf{x},t\right)$ are labelled by $|\pi >$ and their eigenvalues with respect to the momentum operator are given by

$$\hat{\pi}_{t=t_i}\left(\mathbf{x}\right)|\pi >= \pi\left(\mathbf{x}\right)|\pi > .$$

The completeness and orthogonality conditions are

$$\begin{aligned}\frac{1}{2\pi}\int d\pi\ |\pi >< \pi| &= 1 \\ < \pi_a|\pi_b > &= \delta\left[\pi_a\left(\mathbf{x}\right) - \pi_b\left(\mathbf{x}\right)\right] . \end{aligned} \tag{1.2}$$

Notice that in field theory the overlap $< \phi|\pi >$ is given by[2]

$$< \phi|\pi >= \exp\left[i\int d^3x\ \pi\left(\mathbf{x}\right)\phi\left(\mathbf{x}\right)\right]. \tag{1.3}$$

Finally, the Hamiltonian of the system is defined as usual by

$$\hat{H} = \int d^3x\mathcal{H} = \int d^3x\left(\hat{\pi}(x)\partial_0\hat{\phi}(x) - \mathcal{L}\right),$$

where we will assume in the following that $\hat{H}$ is time-independent.

Suppose that the system is in a state $|\phi_i >$ at the time $t = t_i$. Since we have assumed our Hamiltonian to be time-independent the time evolution of this state can be expressed using the time evolution operator $e^{-i\hat{H}t}$ acting on the initial state $|\phi_i >$,

$$|\phi_{i,t} >= e^{-i\hat{H}(t-t_i)}|\phi_i > .$$

The transition amplitude A is now defined as the matrix element of the initial state at a time $t = t_f$ given by $|\phi_{i,t_f} >$ and some final state $|\phi_f >$

$$A =< \phi_f|\phi_{i,t=t_f} > .$$

[2] In quantum mechanics one would have $< x|p >= \exp\left[ipx\right]$. Generalizing for a finite number of degrees of freedom one would obtain $\sum_{i=1}^{N} p_i x_i$ in the exponential; finally going into the continuum limit with $N \to \infty$ one obtains the result (1.3).

Using the time evolution operator this is nothing other than the matrix element of the time evolution operator in the basis $|\phi>$:

$$A =< \phi_f | e^{-i\hat{H}(t_f - t_i)} | \phi_i > . \tag{1.4}$$

In order to obtain a practical means of evaluating this amplitude we divide the time interval $[t_i; t_f]$ into N equal steps $\Delta t = (t_f - t_i)/N$. Then we insert a complete set of states, alternating between $\frac{1}{2\pi} \int d\pi_k |\pi_k >< \pi_k|$ and $\int d\phi_k |\phi_k >< \phi_k|$ at each time interval Δt_k. We obtain for the transition amplitude A:

$$\begin{aligned} A &= \int \left(\prod_{k=1}^{N} \frac{d\pi_k d\phi_k}{2\pi} \right) < \phi_f | \pi_N >< \pi_N | e^{-i\hat{H}\Delta t} | \phi_N >< \phi_N | \pi_{N-1} > \times \\ & \quad \times < \pi_{N-1} | e^{-i\hat{H}\Delta t} | \phi_{N-1} >< \phi_{N-1} | \pi_{N-2} > \times ... \\ & \quad ... \times < \phi_2 | \pi_1 >< \pi_1 | e^{-i\hat{H}\Delta t} | \phi_1 >< \phi_1 | \phi_i > . \end{aligned}$$

If we increase the number of steps in such a way that $N \gg 1$, the time intervals become very small $(\Delta t \ll 1)$ and we may expand

$$e^{-i\hat{H}\Delta t} \simeq 1 - i\hat{H}\Delta t$$

so that

$$\begin{aligned} < \pi_k | e^{-i\hat{H}\Delta t} | \phi_k > &\simeq < \pi_k | \phi_k > - iH_k \Delta t < \pi_k | \phi_k > + O\left((\Delta t)^2 \right) \\ &\simeq < \pi_k | \phi_k > \left(e^{-iH_k \Delta t} \right) + O\left((\Delta t)^2 \right) \end{aligned} \tag{1.5}$$

with $H_k = \int d^3x \, \mathcal{H}\left(\pi_k(\mathbf{x}), \phi_k(\mathbf{x}) \right)$. Furthermore we know that $< \phi_1 | \phi_i >= \delta(\phi_1 - \phi_i)$ and $< \phi_{k+1} | \pi_k >= \exp\left[i \int d^3x \; \pi_k(\mathbf{x}) \, \phi_{k+1}(\mathbf{x}) \right]$. Putting together the last steps we obtain for the transition amplitude A

$$A = \lim_{N\to\infty} \int \left(\prod_{k=1}^{N} \frac{d\pi_k d\phi_k}{2\pi} \right) \delta\left(\phi_1 - \phi_i \right) \exp iS \tag{1.6}$$

with

$$S = \int d^3x \left[\sum_{j=1}^{N} \Delta t \left(\pi_j \left(\frac{\phi_{j+1} - \phi_j}{\Delta t} \right) - \mathcal{H}(\pi_j, \phi_j) \right) \right] .$$

where $\phi_{N+1} = \phi_f$. In the continuum limit $N \to \infty$ we associate

$$\frac{\phi_{j+1} - \phi_j}{\Delta t} \to \frac{\partial \phi}{\partial t}, \quad \sum_{j=1}^{N} \Delta t \to \int_{t_i}^{t_f} dt,$$

$$\pi_j(\mathbf{x}) \to \pi(\mathbf{x}, t), \quad \phi_j(\mathbf{x}) \to \phi(\mathbf{x}, t),$$

$$\lim_{N\to\infty}\int\left(\prod_{k=1}^{N}\frac{d\pi_k d\phi_k}{2\pi}\right)\to\int\mathcal{D}\pi\int\mathcal{D}\phi \tag{1.7}$$

Here we defined the functions $\phi(\mathbf{x},t)$ and $\pi(\mathbf{x},t)$ in such a way that at $t=t_i$ they correspond to the eigenfunctions $\phi_1(\mathbf{x})$ and $\pi_1(\mathbf{x})$, respectively, and at a later time are defined as

$$\phi(\mathbf{x},t_i+t)=\phi_{1+t/\Delta t}(\mathbf{x}) \tag{1.8}$$

(similar for π). Note that at $t=t_f$ this definition corresponds to the final state eigenfunction $\phi_f(\mathbf{x})$. With these definitions the transition amplitude becomes

$$A=\int\mathcal{D}\pi\int_{\phi(\mathbf{x},t_i)=\phi_i}^{\phi(\mathbf{x},t_f)=\phi_f}\mathcal{D}\phi\ \exp iS[\pi,\phi] \tag{1.9}$$

with S now the classical action for the scalar field

$$S=\int_{t_i}^{t_f}dt\int d^3x\left(\pi(\mathbf{x},t)\frac{\partial\phi(\mathbf{x},t)}{\partial t}-\mathcal{H}\left(\phi(\mathbf{x},t),\pi(\mathbf{x},t)\right)\right).$$

The symbols $\mathcal{D}\pi$ and $\mathcal{D}\phi$ denote functional integrations and should be thought of as an infinite product as defined in (1.6). Note that we have omitted the delta function $\delta(\phi_1-\phi_i)$ in this continuum limit. It actually represents the fact that the transition amplitude still depends on the initial field configuration ϕ_i which is not integrated over. Keeping this in mind we may safely drop the term in the continuum limit but have to reintroduce it whenever we return to the discrete version of the transition amplitude.

The transition amplitude A is now expressed as a path integral, that is, an integral over all possible paths connecting the initial field configuration $\phi_i(\mathbf{x})$ to the final field configuration $\phi_f(\mathbf{x})$, which are fixed functions. The advantage of this representation over the operator representation of the transition amplitude is that the path integral is described only in terms of classical variables – such as the the classical field action S – where manipulations become trivial. Moreover, the path integral representation can easily be generalized to incorporate sources and eventually be used to derive various Green functions of the theory in a simple and straightforward manner.

Interestingly, in the next section we will find out that there is a way of writing the most important function in thermodynamics – the partition function – *completely analogously* to (1.9), thereby opening up the possibility of using the main machinery of standard quantum field theory to describe systems in equilibrium at a given temperature.

1.2 Partition function for bosons

Let us first review some basic properties of statistical quantum mechanics: We want to describe a quantum mechanical system in contact with a heat bath (with a given temperature $T = 1/\beta$) in respect to which the system is at rest. The system is then characterized by the partition function Z given by

$$Z = \mathrm{Tr}\hat{\rho}$$

where $\hat{\rho}$ is called the density operator. The system may be thought of to be composed of two weakly interacting subsystems S_1 and S_2 with their proper density operators ρ_1 and ρ_2. As a consequence of statistical independence (c.f. [5]) it follows that $\rho = \rho_1\rho_2$. Furthermore, because of Liouville's theorem the density operator is conserved along a trajectory in phase space which implies that ρ itself consists only of conserved quantities. Therefore $\ln\rho$ has to be a linear combination of the conserved (and mutually commuting) quantities of the system, which – since we have placed ourselves in the rest frame of the heat bath[3] – can be written as

$$\ln\rho = -\beta\hat{H} + \alpha_i\hat{Q}_i.$$

Here $\hat{H}$ is the Hamilton operator of the system, $\hat{Q}_i$ the conserved charge operators and α_i their respective Lagrange multipliers. The density operator takes its simplest form if there are no conserved charges at all so that the α_i are zero; this corresponds to the "canonical" ensemble in quantum statistical mechanics. If the only conserved charge operator is given by the number operator N (with which we associate the Lagrange multiplier $\alpha = \beta\mu$ and μ the chemical potential), we refer to the ensemble as "grand canonical" ensemble. Besides heat, the system is also allowed to exchange particles with a reservoir, since N may e.g. be the fermionic number operator and the system may go on creating particle-antiparticle pairs as long as the total number of fermions is conserved. Since this is a central process in quantum field theories it is usually the grand canonical ensemble that offers the most natural description of systems in thermal equilibrium with a heat bath and we shall make use of it in the following.

Collecting the above identities we find that the partition function of a quantum grand canonical ensemble is given by

$$Z = \mathrm{Tr}e^{-\beta(\hat{H}-\mu\hat{N})}.$$

Once the partition function of a system has been calculated, all the other standard thermodynamic properties such as pressure, particle number, entropy and energy may be determined from $Z(\beta)$ by[4]

$$P = T\frac{\partial\ln Z}{\partial V}, \quad N = T\frac{\partial\ln Z}{\partial\mu}$$

$$S = \frac{\partial\,(T\ln Z)}{\partial T}, \quad E = -PV + TS + \mu N,$$

[3] For an approach to formulate a covariant theory see [22].

[4] In the infinite volume limit.

making $Z(\beta)$ the single most important function in thermodynamics. Moreover, to calculate expectation values of some operator B for a given statistical ensemble one has to take the statistical ensemble-average $< B >_\beta$ which is defined as

$$< B >_\beta = \frac{1}{Z} Tr\left[\rho B\right].$$

In the following, we shall be interested in the theory describing a real scalar field which possesses no conserved charges; therefore (and because it simplifies calculations) we will set $\mu = 0$ from now on. The partition function is then only a function of β and will be denoted as $Z(\beta)$.

The KMS relation Before actually tackling the partition function let us first derive an interesting property that will inspire us to some intellectual leap that is at the foundation of thermal quantum field theory.

Let $B_1(t)$ and $B_2(t')$ be two operators in the Heisenberg picture, that is

$$\begin{aligned} B_1(t) &= e^{iHt} B_1 e^{-iHt} \\ B_2(t') &= e^{iHt'} B_2 e^{-iHt'} \end{aligned} \tag{1.10}$$

at times t and t', respectively. Then the statistical ensemble average of the product of these two operators is given by

$$< B_1(t) B_2(t') >_\beta = \frac{1}{Z(\beta)} Tr\left[e^{-\beta H} B_1(t) B_2(t')\right]. \tag{1.11}$$

Because of the cyclicity of the trace we may pull through $B_2(t')$ to the leftmost side in the trace and insert identity in the form $e^{\beta H} e^{-\beta H}$ on the rightmost side,

$$< B_1(t) B_2(t') >_\beta = \frac{1}{Z(\beta)} Tr\left[B_2(t') e^{-\beta H} B_1(t) e^{\beta H} e^{-\beta H}\right].$$

Pulling through the $e^{-\beta H}$ to the left again we obtain

$$< B_1(t) B_2(t') >_\beta = \frac{1}{Z(\beta)} Tr\left[e^{-\beta H} B_2(t') e^{-\beta H} B_1(t) e^{\beta H}\right].$$

Looking at the definition of the operators (1.10) we are tempted to rewrite the expression

$$e^{-\beta H} B_1(t) e^{\beta H} = e^{-\beta H + iHt} B_1 e^{\beta H - iHt} = B_1(t + i\beta),$$

finding the interesting relation for the statistical ensemble average

$$< B_1(t) B_2(t') >_\beta = < B_2(t') B_1(t + i\beta) >_\beta, \tag{1.12}$$

which is known as the KMS (Kubo-Martin-Schwinger) relation. It is a sort of periodicity relation in *imaginary* time, the latter having been introduced by formally interpreting $e^{-\beta H}$ as evolution operator.

Imaginary time It should be stressed that the concept of imaginary time is only formal. We do not expect physical time to be a complex variable and indeed, when calculating physical quantities have to make time real again[5]. However, when turning our attention back to the partition function it should become clear in a moment why nevertheless it is appealing to venture into the realm of imaginary time.

To evaluate the trace of the partition function we insert a complete set of eigenvectors of the field operator (1.1):

$$Z(\beta) = \int d\phi_i < \phi_i | e^{-\beta H} | \phi_i > .$$

Taking a close look at the definition of the transition amplitude (1.9) and allowing for imaginary time arguments we are led to interpret the partition function as integral over the transition amplitude with $\phi_f = \phi_i$ and $t_f = t_i - i\beta$. Using our path-integral formulation for the transition amplitude (1.9) we obtain for the partition function

$$Z(\beta) = \int d\phi_i \int \mathcal{D}\pi \int_{\phi(\mathbf{x},t_i)=\phi_i}^{\phi(\mathbf{x},t_i-i\beta)=\phi_i} \mathcal{D}\phi \quad \exp iS[\pi,\phi]$$

with

$$S[\pi\, \phi] = \int_{t_i}^{t_i - i\beta} dt \int d^3x \left(\pi(\mathbf{x},t) \frac{\partial \phi(\mathbf{x},t)}{\partial t} - \mathcal{H}\left(\phi(\mathbf{x},t), \pi(\mathbf{x},t)\right) \right) .$$

Using our definition of the integration measure (1.7) and remembering that we have dropped the delta function $\delta(\phi_1 - \phi_i)$ that represents the dependence of the transition amplitude on the initial field configuration we now see that the extra integration $d\phi_i$ provided by the trace exactly cancels the delta function. For the partition function – contrary to the transition amplitude – the initial field configuration is integrated over. The path-integral formulation of the partition function for bosons is then given by

$$Z(\beta) = \int \mathcal{D}\pi \int_{periodic} \mathcal{D}\phi \;\; \exp iS[\pi,\phi], \tag{1.13}$$

where the term "periodic" means that the integration over the field is restricted so that $\phi(\mathbf{x}, t_i) = \phi(\mathbf{x}, t_f)$.

1.3 Propagators

As can be seen by comparing (1.9) and (1.13) the partition function takes the role of the transition amplitude if we allow time to take complex values, thereby

[5]The next section will be dealing with this problem.

introducing temperature into the theory. By formally treating the partition function as transition amplitude we may construct a "thermal" quantum field theory (TQFT), that is, a theory that incorporates the temperature of the system in a natural way. To construct this TQFT we will use analogies to $T = 0$ quantum field theory wherever possible. However, there will be numerous cases where we have to resort to new methods and techniques in order to give a proper meaning to the quantities we are to calculate.

First of all we want to define Green functions in TQFT. In standard ($T = 0$) quantum field theory they are defined as time ordered product of field operators or – alternatively in the path integral approach – by functional derivatives of the generating functional $Z(J)$; the latter is defined [3] by a generalization of the transition amplitude (1.9) where one modifies the action by introducing a source J that couples to the field,

$$S \rightarrow S + \int_{t_i}^{t_f} dt \int d^3x \phi(x) J(x).$$

The Green functions in $T = 0$ theory are then defined by functional differentiation of $Z(J)$ with respect to J. As outlined above, using the analogy between transition amplitude and the partition function of a grand canonical ensemble we define[2] a generating functional $Z(\beta, J)$ in TQFT by introducing a source into the definition of the partition function $Z(\beta)$ similar to the above substitution,

$$Z(\beta, J) := \int \mathcal{D}\pi \int_{periodic} \mathcal{D}\phi \exp\left[i \int_{t_i}^{t_i - i\beta} dt \int d^3x \left(\pi \frac{\partial \phi}{\partial t} - \mathcal{H} + J\phi \right) \right].$$

Again using the analogy to standard quantum field theory we define the thermal propagator $G(x, y)$ as the second functional derivative of $Z(\beta, J)$ with respect to J and normalizing by $Z(\beta)$[6]

$$G(x, y) := \frac{1}{Z(\beta)} \left. \frac{\delta^2 Z(\beta, J)}{\delta J(x) \delta J(y)} \right|_{J=0}.$$

Using the definition of functional differentiation (A.1) we get for $G(x, y)$:

$$G(x, y) = \frac{1}{Z(\beta)} \int \mathcal{D}\pi \int_{periodic} \mathcal{D}\phi \; \phi(x)\phi(y) \exp\left[i \int_{t_i}^{t_i - i\beta} dt \int d^3x \left(\pi \frac{\partial \phi}{\partial t} - \mathcal{H} + J\phi \right) \right]. \tag{1.14}$$

We now will reverse the steps leading from (1.4) to (1.9): first we discretize the integration measures and therefore the time variable into N steps

$$\int \mathcal{D}\pi \int_{periodic} \mathcal{D}\phi \rightarrow \lim_{N \to \infty} \prod_{i=1}^{N} \frac{d\pi_i d\phi_i}{2\pi}, \quad \phi_1 = \phi_{N+1}.$$

[6]In $T = 0$ theory this defines the causal propagator (the vacuum expectation value of the product of two time-ordered field operators).

Recalling the definition for $\phi(x) = \phi(\mathbf{x}, t_a)$ given in (1.8) we have to replace the time variable by a discrete index. Since we want to know which index has the lower value we rewrite the product $\phi(x)\phi(y)$ as

$$\phi(x)\phi(y) = \phi_a(\mathbf{x})\phi_b(\mathbf{y})\left(\Theta(a-b) + \Theta(b-a)\right)$$

$(y^0 = t_b)$ and when changing $a \leftrightarrow b$ in the last term obtain

$$\phi(x)\phi(y) = \phi_a(\mathbf{x})\phi_b(\mathbf{y}) + \phi_a(\mathbf{y})\phi_b(\mathbf{x}), \quad a > b.$$

The exponential, in its discretized version, is separated into three parts[7]

$$\exp\left[i\Delta t\sum_{j=1}^{N} S_j\right] = \exp\left[i\Delta t\sum_{j=1}^{b-1} S_j\right]\exp\left[i\Delta t\sum_{j=b}^{a-1} S_j\right]\exp\left[i\Delta t\sum_{j=a}^{N} S_j\right]$$

and

$$S_j = \int d^3x\pi_j\frac{\phi_{j+1}-\phi_j}{\Delta\tau} - \mathcal{H}\left(\pi_j, \phi_j\right).$$

Reversing now (1.5) we get

$$\exp\left[i\Delta t S_j\right] = <\phi_{j+1}|\pi_j><\pi_j|e^{-i\Delta t\hat{H}}|\phi_j> + O\left((\Delta t)^2\right)$$

where $\hat{H}$ is once again an operator. We now put all these into the definition of $G(x,y)$ and get

$$\begin{aligned}
Z(\beta)G(x,y) \quad = \quad & \lim_{N\to\infty}\prod_{i=1}^{N}\int\frac{d\pi_i d\phi_i}{2\pi} <\phi_1|\pi_N><\pi_N|e^{-i\Delta t\hat{H}}|\phi_N> \times ... \\
& ...\times <\phi_{a+1}|\pi_a><\pi_a|e^{-i\Delta t\hat{H}}|\phi_a>\phi_a\left(\mathbf{x}\right)<\phi_a|\pi_{a-1}> \times ... \\
& ...\times <\phi_{b+1}|\pi_b><\pi_b|e^{-i\Delta t\hat{H}}|\phi_b>\phi_b\left(\mathbf{y}\right)<\phi_b|\pi_{b-1}> \times ... \\
& ...\times <\phi_2|\pi_1><\pi_1|e^{-i\Delta t\hat{H}}|\phi_1> + \\
& + \left(\mathbf{x}\leftrightarrow\mathbf{y}\right)
\end{aligned}$$

There are several things to note in this formula: first of all the term $<\phi_1|$ at the very beginning is the implementation of the condition $\phi_1 = \phi_{N+1}$ coming from the periodicity. Secondly, the eigenfunctions ϕ_a, ϕ_b can be thought of as the action of the field operator $\hat{\phi}$ on the states $|\phi_a>$ and $|\phi_b>$, respectively. We can write

$$\begin{aligned}
\phi_a\left(\mathbf{x}\right)|\phi_a> \quad &= \quad \hat{\phi}\left(\mathbf{x}\right)|\phi_a> \\
\phi_b\left(\mathbf{y}\right)|\phi_b> \quad &= \quad \hat{\phi}\left(\mathbf{y}\right)|\phi_b>.
\end{aligned}$$

[7]Here we assume $N > a > b > 0$. We shall see later that this assumption is justified because $G(x,y)$ will turn out periodic in imaginary time.

Having traded the dependence on the eigenfunctions ϕ_a and ϕ_b for quantum mechanical operators we can now do all of the $d\pi$ and all but one of the $d\phi$ integrations using the completeness and orthogonality conditions (1.1) and (1.2). Together with the definition of Δt we get

$$Z(\beta)G(x,y) = \int d\phi_1 < \phi_1|e^{-i(t_i-i\beta-t_a)\hat{H}}\hat{\phi}\,(\mathbf{x})\,e^{-i(t_a-t_b)\hat{H}}\hat{\phi}(\mathbf{y})e^{-i(t_b-t_i)\hat{H}}|\phi_1> + (\mathbf{x}\leftrightarrow\mathbf{y})\,, \tag{1.15}$$

where t_a and t_b may in general be complex. However, if we require $G(x,y)$ to be analytical we will find a condition restricting t_a and t_b. Note that the r.h.s of (1.15) is nothing but a trace of some complicated operator. Without loss of generality, we may change the basis in which the trace is evaluated into the eigenstates of the Hamiltonian $\hat{H}$. These states will be labelled by $|\lambda>$ and their eigenvalues with respect to the action of $\hat{H}$ will be denoted as E_λ. We obtain

$$\begin{aligned}Z(\beta)G(x,y) = \sum_{\lambda,\nu} < \lambda|e^{-i(t_i-i\beta-t_a)\hat{H}}\hat{\phi}\,(\mathbf{x})\,e^{-it_a\hat{H}}|\nu>< \nu|e^{it_b\hat{H}}\hat{\phi}(\mathbf{y})e^{-i(t_b-t_i)\hat{H}}|\lambda> \\ + (\mathbf{x}\leftrightarrow\mathbf{y})\,,\end{aligned}$$

where we have inserted one more complete set of eigenstates of the Hamiltonian $|\nu>$. By letting $\hat{H}$ act on the states $|\lambda>$ or $|\nu>$ we obtain the (positive) energy eigenvalues E_λ and E_ν. Therefore we find for $Z(\beta)G(x,y)$

$$Z(\beta)G(x,y) = \sum_{\lambda,\nu} e^{-\beta E_\lambda + i(E_\lambda-E_\nu)(t_a-t_b)} < \lambda|\hat{\phi}(\mathbf{x})|\nu>< \nu|\hat{\phi}(\mathbf{y})|\lambda> + (\mathbf{x}\leftrightarrow\mathbf{y})\,. \tag{1.16}$$

Because of the term $(\mathbf{x}\leftrightarrow\mathbf{y})$ the amplitudes are purely real; assuming that the exponentials dominate the convergence of the sum, analyticity is achieved if the term in the exponential is negative definite. Because the eigenvalues of the Hamiltonian are defined to be positive we obtain as requirement of analyticity

$$0 \geq \mathrm{Im}(t_a - t_b) \geq -\beta.$$

To simplify the argument we will from now on assume Re t_a = Re $t_b = 0 = t_i$ and elaborate on general t at the end of this section. With this simplification we obtain (returning to the definition (1.15) of $G(x,y)$)

$$Z(\beta)G(x,y) = \int d\phi_1 < \phi_1|e^{-\beta\hat{H}+it_a\hat{H}}\hat{\phi}\,(\mathbf{x})\,e^{-i(t_a-t_b)\hat{H}}\hat{\phi}\,(\mathbf{y})\,e^{-it_b\hat{H}}|\phi_1> + (\mathbf{x}\leftrightarrow\mathbf{y})\,,$$

where we have once again used the completeness condition for $|\phi_a>$. By virtue of (1.10) we can write $\hat{\phi}(\mathbf{x})$ and $\hat{\phi}(\mathbf{y})$ as operators in the Heisenberg picture at times t_a and t_b respectively,

$$\begin{aligned} e^{+it_a\hat{H}}\hat{\phi}(\mathbf{x})\,e^{-it_a\hat{H}} &= \hat{\phi}_H(\mathbf{x},t_a) \\ e^{+it_b\hat{H}}\hat{\phi}(\mathbf{y})\,e^{-it_b\hat{H}} &= \hat{\phi}_H(\mathbf{y},t_b)\,. \end{aligned}$$

Comparing with (1.11) we finally find

$$\begin{aligned} G(x,y) &= <\hat{\phi}_H(\mathbf{x},t_a)\,\hat{\phi}_H(\mathbf{y},t_b)>_\beta \Theta\left(it_a-it_b\right)+(\mathbf{x}\leftrightarrow\mathbf{y}) \\ &= <T_c\hat{\phi}_H(\mathbf{x},t_a)\,\hat{\phi}_H(\mathbf{y},t_b)>_\beta \end{aligned} \tag{1.17}$$

where we have introduced a contour-ordering operator T_c (similar to a time-ordering operator in standard quantum field theory) that is defined as

$$\begin{aligned} T_c\hat{\phi}_H(\mathbf{x},t_a)\,\hat{\phi}_H(\mathbf{y},t_b) &= \hat{\phi}_H(\mathbf{x},t_a)\,\hat{\phi}_H(\mathbf{y},t_b)\,\Theta\left(it_a-it_b\right) \\ &\quad +\hat{\phi}_H(\mathbf{y},t_b)\,\hat{\phi}_H(\mathbf{x},t_a)\,\Theta\left(it_b-it_a\right), \end{aligned}$$

and acts by ordering the operators from left to right with monotonously increasing imaginary part of the time argument.

But what is meant by this operator if – going back to general t – t_a or t_b are complex? Interestingly, the above phrase still holds true but the mathematical definition has to be changed somewhat. For general, complex t we have to define how we get from the starting point t_i in the complex t-plane to the end point $t_i-i\beta$. This is done by defining a contour with monotonously decreasing imaginary part (a necessary condition to insure analyticity [23]) in the complex t-plane that – starting from t_i – hits t_b and t_a and finally arriving at $t_i-i\beta$ (this is why T_c is usually referred to as contour-ordering operator). The choice of the contour is of course not unambiguous and we will see in the next section that it gives rise to different formalisms, of which we will present the two most famous examples, the so-called Matsubara and Keldysh formalisms.

Note that n-point thermal Green functions can be defined in complete analogy to the thermal propagator and standard quantum field theory; they turn out to have the same analytic structure as in $T=0$ field theory with vacuum expectation values replaced by thermal averages (c.f.[25], [20]).

1.4 ITF versus RTF

As could be anticipated already in the previous section the form of the thermal Green functions is dependent on the choice of the contour in the complex time plane. However, note that the physical quantities are actually *real time* Green functions which – as they should – are independent of the formalism.

Since the main difference between the two formalisms consists of the treatment of the time variable whereas the spatial variables do not play any important role

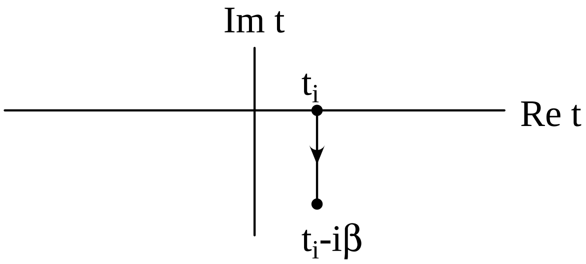


Figure 1.1: The contour T_1 giving rise to the Matsubara formalism

we will from now on drop any dependence on $\mathbf{x}$,$\mathbf{y}$. Using furthermore the identity (1.16) we find that the thermal propagator solely depends on the difference $(t_a - t_b)$

$$G(t_a, t_b) = G(t_a - t_b).$$

Let us now investigate the properties of the thermal propagator for two different contours, T_1 and T_2, that connect an "initial" time t_i in the complex t-plane with the "final" time $t_i - i\beta$ and show how to obtain real-time retarded and advanced propagators that are used in linear response theory [1], [2], [8].

1.4.1 Matsubara formalism (ITF)

The first and simplest contour T_1 consists of a straight line going from t_i to $t_i - i\beta$. It gives rise to the Matsubara or imaginary time formalism (ITF) which we will describe in the following.

Since T_1 follows a straight line parallel to the imaginary time axis, the real part of the time stays the same on the contour; without loss of generality we may therefore assume it to be equal to zero. Then t is purely imaginary and it makes sense to trade it for the imaginary time variable $\tau = it$, with τ purely real and $0 < \tau < \beta$ to insure analyticity. The thermal propagator $G(t_a - t_b)$ then turns into the imaginary time propagator $G^i(\tau_a - \tau_b) = G(t_a - t_b)$. As we have seen (1.17) the thermal propagator corresponds to the thermal average of two contour-ordered field operators,

$$G^i(\tau) =< T_c \hat{\phi}_H(-i\tau)\, \hat{\phi}_H(0) >_\beta .$$

But since the ordering of the operators is already dictated by T_c we can apply the KMS relation (1.12) to find the interesting property

$$G^i(\tau) = G^i(\tau - \beta).$$

The imaginary time propagator $G^i(\tau)$ is periodic in imaginary time with the period of inverse temperature! This periodicity in turn allows us to write $G^i(\tau)$

as a Fourier series

$$G^i(\tau) = \frac{1}{\beta}\sum_n e^{-i\omega_n\tau}\tilde{G}^i(i\omega_n) \tag{1.18}$$

where the summation goes from $n = -\infty$ to $n = +\infty$. The frequencies ω_n take discrete values

$$\omega_n = \frac{2\pi n}{\beta}$$

and are called "Matsubara frequencies". $\tilde{G}^i$ is referred to as "Matsubara propagator" and it is given by the inverse transformation of (1.18):

$$\tilde{G}^i(i\omega_n) = \int_0^\beta d\tau e^{i\tau\omega_n} G^i(\tau). \tag{1.19}$$

Spectral function Note that the Matsubara propagator is only defined for imaginary and discrete frequencies $i\omega_n$, which in the following we will have to continue to real and continuous ones. To do so, however, it is useful to introduce the standard notation for the Matsubara propagator which we will derive in this paragraph. Since we have $0 < \tau < \beta$, T_c orders the operators in $G^i(\tau)$ as

$$G^i(\tau) =< T_c\hat{\phi}(-i\tau)\hat{\phi}(0) >_\beta =< \hat{\phi}(-i\tau)\hat{\phi}(0) >_\beta = G^>(-i\tau) \tag{1.20}$$

where we have introduced another propagator $G^>(t)$ which (together with its counterpart $G^<(t)$) is defined by

$$\begin{aligned} G^>(t) &= < \hat{\phi}(t)\hat{\phi}(0) >_\beta \\ G^<(t) &= < \hat{\phi}(0)\hat{\phi}(t) >_\beta . \end{aligned} \tag{1.21}$$

Note that because of (1.12) these two propagators are related as

$$G^<(t) = G^>(t + i\beta) \tag{1.22}$$

and also fulfill the relation

$$G^>(t) - G^<(t) =< [\hat{\phi}(t), \hat{\phi}(0)] >_\beta \tag{1.23}$$

where $[A, B]$ denotes the commutator between the two fields A and B. Using the Fourier transform of $G^>(t)$ and (1.20) we can write (1.19) as

$$\begin{aligned} \tilde{G}^i(i\omega_n) &= \int_0^\beta d\tau e^{i\tau\omega_n} \int \frac{dk_0}{2\pi} e^{-k_0\tau}\tilde{G}^>(k_0) \\ &= \int \frac{dk_0}{2\pi}\frac{\tilde{G}^>(k_0)}{i\omega_n - k_0}\left(e^{-k_0\beta} - 1\right). \end{aligned} \tag{1.24}$$

In order to write (1.24) in a more compact form it is useful to introduce the spectral function $\rho(k_0)$ at this point. It is defined as

$$\rho(k_0) = \tilde{G}^{>}(k_0) - \tilde{G}^{<}(k_0).$$

Because of (1.22) $\tilde{G}^{<}(k_0)$ and $\tilde{G}^{>}(k_0)$ are related as

$$\tilde{G}^{<}(k_0) = e^{-\beta k_0}\tilde{G}^{>}(k_0),$$

and therefore $\rho(k_0)$ becomes

$$\rho(k_0) = \tilde{G}^{>}(k_0)\left(1 - e^{-\beta k_0}\right).$$

$\tilde{G}^{i}(i\omega_n)$ then can be written in the compact form

$$\tilde{G}^{i}(i\omega_n) = -\int \frac{dk_0}{2\pi}\frac{\rho(k_0)}{i\omega_n - k_0}. \tag{1.25}$$

Analytic continuation Now that we are in possession of a compact notation for $\tilde{G}^{i}(i\omega_n)$ we may approach the problem of finding its analytic continuation $i\omega_n \to z$ from discrete imaginary values $i\omega_n$ to arbitrary values of z. It was shown in [36] that the necessary and sufficient conditions for the analytic continuation to be unique are given by

- $|\tilde{G}^{i}(z)| \to 0$ for $|z| \to \infty$
- $\tilde{G}^{i}(z)$ analytic in the complex plane outside the real axis.

The result of this analytic continuation,

$$\tilde{G}^{i}(z) = -\int \frac{dk_0}{2\pi}\frac{\rho(k_0)}{z - k_0}, \quad \mathrm{Im} z \neq 0 \tag{1.26}$$

is sometimes referred to as analytic propagator.

We finally want to show that this analytic continuation is special in the sense that for $z = k_0 \pm i\varepsilon$, $\varepsilon \to +0$ we obtain the retarded and advanced real time propagators from the analytic propagator. For $z = k_0 \pm i\varepsilon$ (1.26) becomes

$$\tilde{G}^{i}(k_0 \pm i\varepsilon) = -\int \frac{dk_0'}{2\pi}\frac{\rho(k_0')}{k_0 - k_0' \pm i\varepsilon},$$

which – using

$$\rho(k_0') = \int \frac{dt}{2\pi} e^{ik_0' t}\left(G^{>}(t) - G^{<}(t)\right)$$

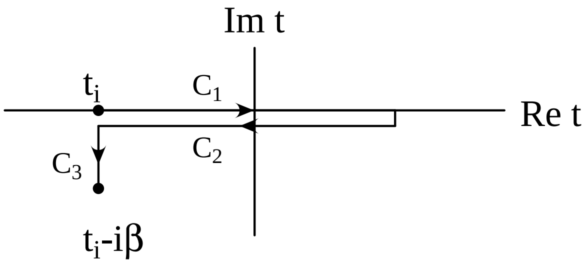


Figure 1.2: The contour T_2 giving rise to the Keldysh formalism

and the integral representation of the step function given in the appendix – can be written as

$$\tilde{G}^i(k_0 \pm i\varepsilon) = i \int dt e^{ik_0 t} \left(\Theta(\pm t)\right) \left(G^>(t) - G^<(t)\right).$$

Together with (1.23) we find that $G^i(k_0 \pm i\varepsilon)$ is the Fourier transform of

$$\begin{aligned} iG^r(t) &= i\Theta(t) < [\hat{\phi}(t), \hat{\phi}(0)] >_\beta \\ iG^a(t) &= i\Theta(-t) < [\hat{\phi}(0), \hat{\phi}(t)] >_\beta, \end{aligned}$$

which are recognized to be the retarded and advanced thermal propagators, respectively.

It should be pointed out that for general n-point Green functions with $n > 2$ the analytic continuation becomes more involved. However, it has been discussed in [37] how to continue the imaginary time Green functions to obtain retarded and advanced real time ones.

1.4.2 Real time formalism (RTF)

Let us now turn our attention to the contour T_2 that first connects a real time t_i with a real time t_f (C_1) then back from t_f to t_i with infinitesimal imaginary part (C_2) and finally straight down to $t_i - i\beta$ (C_3). Contrary to the imaginary time formalism, where Green functions can only be calculated for time arguments $[t_i, t_i - i\beta]$ and after that have to be continued to real times, in the RTF or Keldysh formalism those real-time Green functions can be calculated directly. For $t_i \to -\infty$ it can be argued that only C_1 and C_2 contribute to the Green functions [2], [20]. If we distinguish between operators that have been inserted on C_1 from those inserted on C_2 (dubbed type 1 and type 2 fields respectively) the thermal propagator takes the form of a 2×2 matrix

$$G(t_a, t_b) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$

where $G_{ij} =< T_c\hat{\phi}_i(t_a)\hat{\phi}_j(t_b) >$ and $\hat{\phi}_{1,2}$ are operators inserted on C_1 and C_2 respectively. For G_{12} and G_{21} the action of T_c then becomes trivial since type 1 fields are always "earlier" on the contour than type 2 fields, whereas for G_{11} and G_{22} T_c turns into a time and anti-time ordering operator, respectively. It can be shown [20] that the retarded and advanced real time propagators are then given by

$$\begin{aligned} iG^r &= G_{11} - G_{12} \\ iG^a &= G_{11} - G_{21}. \end{aligned}$$

The main advantage of the Keldysh formalism is that one does not have to do an analytic continuation in order to obtain real time results in contrast to the ITF formalism. However, one has to pay the price of a doubling of the fields that render higher order calculations more involved.

1.5 Summary

In this chapter we derived the path integral formulations of the transition amplitude and the partition function for bosons and showed that we could identify the two results by formally introducing imaginary time. We then made use of the standard field theoretical machinery to obtain the thermal propagator that turned out to be given by the thermal average of two contour-ordered field operators. In the last section we specified two different contours and showed how to obtain the physical real time Green functions for the Matsubara formalism that we are going to use in the following chapters.

Chapter 2

The neutral scalar field at finite temperature

In this chapter we shall be interested in setting up perturbation theory at finite temperature and eventually calculate the self energy of the neutral scalar field. Using the Matsubara formalism we will set up perturbation theory in momentum space as well as give the Feynman rules to obtain the self-energy to any given order, closely following the approach taken in [1]; the time coordinate t can therefore be taken to be purely imaginary ($t = -i\tau$, Im $\tau = 0$) and the results have to be continued analytically at the end of the calculation. We have found it convenient to work in the path integral formalism and only rarely will introduce operator analogues for reasons of compact notation.

The most general renormalizable Lagrangian for a neutral scalar field in 4 space-time dimensions [1] is given by

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu\phi\right)\left(\partial_\mu\phi\right) - \frac{1}{2}m^2\phi^2 - U(\phi) \tag{2.1}$$

where $\phi(x)$ is a real field and the interaction $U(\phi)$ is

$$U(\phi) = \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4 \tag{2.2}$$

with $\lambda \geq 0$ in order to insure stability of the vacuum. The conjugate momentum to ϕ is

$$\pi = \frac{\partial\mathcal{L}}{\partial\left(\partial_0\phi\right)} = \frac{\partial\phi}{\partial t} \tag{2.3}$$

and the Hamiltonian density is defined as usual by

$$\mathcal{H} = \pi\frac{\partial\phi}{\partial t} - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{1}{2}m^2\phi^2 + U(\phi).$$

Since $\phi(x)$ is real there is no conserved charge and – accordingly – no chemical potential μ (which has been used already in the introduction to simplify calculations). From (1.13) we easily find the partition function in imaginary time

$$Z(\beta) = \int \mathcal{D}\pi \int_{periodic} \mathcal{D}\phi \exp\left[\int_0^\beta d\tau \int d^3x \left(i\pi\frac{\partial\phi}{\partial\tau} - \mathcal{H}\right)\right].$$

To evaluate this expression we note that the exponential can be written as

$$\begin{aligned} i\pi\frac{\partial\phi}{\partial\tau} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - U(\phi) &= -\frac{1}{2}\left(\pi - i\frac{\partial\phi}{\partial\tau}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 \\ &\quad -\frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - U(\phi). \end{aligned}$$

Since $\phi(\tau, \mathbf{x})$ is a real field $\pi(\tau, \mathbf{x})$ is also real by construction (2.3) and therefore equal to its complex conjugate $\pi^*(\tau, \mathbf{x})$. Upon shifting the $d\pi$ integration the π dependence consists only of a gaussian integral and can be integrated out to give a numerical factor. However, doing the calculation carefully [35], one finds that this factor is in fact temperature dependent so that the partition function reads

$$Z(\beta) = N'(\beta) \int_{periodic} \mathcal{D}\phi e^{-S_E} \tag{2.4}$$

with[1]

$$S_E = \frac{1}{2}\int_0^\beta d\tau \int d^3x \left(\left(\frac{\partial\phi}{\partial\tau}\right)^2 + (\nabla\phi)^2 + m^2\phi^2 + 2U(\phi)\right). \tag{2.5}$$

It should be pointed out that $N'(\beta)$ is actually an infinite quantity stemming from the fact that $\mathcal{D}\pi$ itself was not a well-defined measure in the first place. In standard quantum field theory this would not pose a problem since any constant factor can simply be absorbed into $\mathcal{D}\phi$ without changing the dynamics of the system. In fact, it is easy to see that also at finite temperature the thermodynamics of the system are not changed by a *constant* factor (even if it is infinite) since all the thermodynamical quantities such as pressure, entropy and energy density depend only on the derivative of $\ln Z(\beta)$. Unfortunately the multiplicative factor $N'(\beta)$ in (2.4) is dependent on the temperature and therefore has to be taken care of.

2.1 Thermal propagator

We now want to specify the thermal propagator that was already defined in the first chapter for the case of the neutral scalar field. Using (1.14) and once again

[1]Note that S_E (where E stands for Euclidean) and S from the previous section are related as $S_E = -iS$.

integrating out the momentum dependence we obtain

$$G^i(x,y) = \frac{1}{Z(\beta)} \int_{periodic} \mathcal{D}\phi\ \phi(x)\phi(y)e^{-S_E}. \tag{2.6}$$

Note that because of translational invariance $G^i(x_1, x_2)$ depends only on $(x_1 - x_2)$; the thermal propagator $G^i(x_1, x_2)$ will therefore be denoted as $G^i(x) = G^i(x_1 - x_2)$. To set up perturbation theory in momentum space we do a Fourier transform of the fields

$$\phi(\tau, \mathbf{x}) = \sum\!\!\!\!\!\!\!\int_P e^{-i\omega_n\tau - i\mathbf{p}\mathbf{x}}\tilde{\phi}_n(\mathbf{p}), \tag{2.7}$$

where we have introduced the short notation

$$\sum\!\!\!\!\!\!\!\int_P = T\sum_{n=-\infty}^{n=+\infty}\int \frac{d^3p}{(2\pi)^3}.$$

Upon inserting the above transformation into (2.6) there are several things to note: Firstly, we immediately notice that the periodicity condition is by construction fulfilled by the fields in Fourier space, $\tilde{\phi}_n$. Furthermore, when calculating the free propagator, we will find that there are no surface contributions in the free action as there would have been in position space and the evaluation of the "free" partition function is greatly simplified. However, as a consequence of the principle of conservation of misery, there is a price to pay for all the niceties that frequency space offers us: we also have to transform the measure $\mathcal{D}\phi$ into frequency space. Naively, one might think that it is sufficient to use the Jacobian determinant to transform one into the other. Unfortunately, according to [9] this does not give the correct expression and one has to define a new approximation of the path instead of discretizing time. Doing this we find that the "new" measure $\mathcal{D}\tilde{\phi}$ is given by

$$\mathcal{D}\tilde{\phi} = N'' \lim_{j\to\infty} \prod_{k=-j}^{k=+j} d\tilde{\phi}_k$$

where N'' is a temperature independent factor according to the literature [35], [1].

Fortunately, since we will limit ourselves to calculate only the propagator (2.6) and corrections thereof, we *neither* have to worry about the exact transformation coefficient *nor* about the temperature dependent term $N'(\beta)$ in (2.4) since any such terms in the numerator are cancelled exactly by their analogues in $Z(\beta)$ in the denominator. If we were to calculate the partition function to lowest order, however, we would have to take $N'(\beta)$ into account and in fact would need its contribution to cancel another infinite temperature dependent term resulting from the calculation [35].

Using the transformation (2.7) into (2.6) we obtain for the Matsubara propagator[2]

$$\begin{aligned}
\tilde{G}^i(i\omega_n, \mathbf{k}) &= \int_0^\beta d\tau \int d^3x e^{i\mathbf{kx}+i\omega_n\tau} G(x) \\
&= \frac{1}{Z(\beta)} \int \mathcal{D}\tilde{\phi} \sum\!\!\!\!\!\!\int_{P_1} \sum\!\!\!\!\!\!\int_{P_2} \tilde{\phi}_{n_1}(\mathbf{p_1})\tilde{\phi}_{n_2}(\mathbf{p_2}) e^{-S_E} \times \\
&\quad \times \int_0^\beta d\tau \int d^3x \exp\left[i(\omega_n - \omega_{n_1})\tau + i(\mathbf{k} - \mathbf{p_1})\mathbf{x}\right], \qquad (2.8)
\end{aligned}$$

and denote $\tilde{G}^i(i\omega_n, \mathbf{k}) = D(K)$ in the following to make things more readable ($K = (\mathbf{k}, \omega_n)$ may be thought of as Euclidean four-vector with $K^2 = \omega_n^2 + k^2$).

Since we integrate over all field configurations $\tilde{\phi}_n$ one can convince oneself that the result of the integration will only be nonzero if $\tilde{\phi}_{n_2}(\mathbf{p_2}) = \tilde{\phi}^*_{n_1}(\mathbf{p_1})$. Otherwise there will be a phase left in the integrand which – upon integration – will return zero. Going back to the definition of $\tilde{\phi}_n$ in (2.7) and using that the field is real we find the relation

$$\tilde{\phi}^*_n(\mathbf{p}) = \tilde{\phi}_{-n}(-\mathbf{p}) \qquad (2.9)$$

so that $n_2 = -n_1$ and $\mathbf{p_2} = -\mathbf{p_1}$ in order for $D(K)$ not to vanish. The condition is implemented using a Kronecker delta $\delta_{n_1,-n_2}$ and – since the momentum is taken to be continuous – a Dirac delta function $(2\pi)^3\delta^3(\mathbf{p_2} + \mathbf{p_1})$. Integration over $d\tau$ and d^3x is done using the relations

$$\begin{aligned}
\int d^3x e^{i\mathbf{x}(\mathbf{p}+\mathbf{p}')} &= (2\pi)^3\delta^3(\mathbf{p} + \mathbf{p}') \\
\int_0^\beta d\tau e^{i\tau(\omega_n + \omega_{n'})} &= \beta\delta_{-n,n'} \qquad (2.10)
\end{aligned}$$

to finally obtain

$$D(K) = \frac{1}{\beta Z(\beta)} \int \mathcal{D}\tilde{\phi}\; \tilde{\phi}_n(\mathbf{k})\tilde{\phi}_{-n}(-\mathbf{k}) e^{-S_E} \qquad (2.11)$$

for the full Matsubara propagator of the neutral scalar field.

2.1.1 The free neutral scalar field

In order to set up perturbation theory we first have to specialize (2.11) for the simplest model imaginable, that of a free neutral scalar field. The Lagrangian of

[2]Remember that $G^i(x) = < T_c\hat{\phi}(x)\hat{\phi}(0) >_\beta$ and therefore there is no dependence on n_2 or $\mathbf{p_2}$ in the second exponential in (2.8).

the model is given by (2.1) with the interaction term $U(\phi)$ set to zero, $U(\phi) = 0$. Inserting (2.7) into (2.5) with vanishing $U(\phi)$ and using (2.10) we obtain[3]

$$S_{E,0} = \frac{1}{2}\sum\!\!\!\!\!\!\int_P \tilde{\phi}_n(\mathbf{p}) \left(\omega_n^2 + p^2 + m^2\right) \tilde{\phi}_{-n}(-\mathbf{p}). \tag{2.12}$$

Because the field is real (2.9), the free action only depends on the absolute values of the Fourier amplitudes and not their phases which can be integrated out. With S_E replaced by $S_{E,0}$ (2.11) becomes the free Matsubara propagator

$$D_0(K) = \frac{\prod_i \int d\tilde{\phi}_i \left|\tilde{\phi}_n(\mathbf{k})\right|^2 e^{-S_{E,0}}}{\beta \prod_i \int d\tilde{\phi}_i e^{S_0}}$$

where we have written out the integration measure as a product with i from $-\infty$ to $+\infty$. Writing $e^{-S_{E,0}}$ formally as

$$e^{-S_{E,0}} = \prod_i \prod_{\mathbf{p}'} \exp\left[-\frac{1}{2\beta}\left|\tilde{\phi}_i(\mathbf{p}')\right|^2 (\omega_i^2 + p'^2 + m^2)\right]$$

and realizing that all the integrations save the one for $i = n$, $\mathbf{p}' = \mathbf{k}$ appear in the numerator and denominator we only have to evaluate

$$\frac{\int_{-\infty}^{+\infty} dx\ x^2 e^{-1/2ax^2}}{\int_{-\infty}^{+\infty} dx\ e^{-1/2ax^2}} = \frac{1}{a}$$

so that the free Matsubara propagator is simply given as

$$D_0(K) = \frac{1}{\omega_n^2 + k^2 + m^2}. \tag{2.13}$$

2.2 Interactions and Feynman diagrams for $\lambda\phi^4$ theory

Contrary to the previous section it is not possible to carry out the functional integration in closed form for non-vanishing interaction term $U(\phi)$ of the form (2.2). We therefore have to resort to approximation techniques, e.g. expansion of e^{-S_E} in powers of the interaction term (which are legitimate if the interaction term is weak). The action S_E can be decomposed as

$$S_E = S_{E,0} + S_{E,I}$$

[3]In order to distinguish terms of the free field from other models we reserve the subscript 0 for quantities calculated in a model where the interactions are turned off. The action $S_{E,0}$ will accordingly be called "free" action.

with $S_{E,0}$ the action of the free neutral scalar field (2.5) and $S_{E,I}$ containing the interaction

$$S_{E,I} = \frac{\lambda}{4!}\int_0^\beta d\tau \int d^3x\phi^4, \tag{2.14}$$

where we have taken $\frac{\lambda}{4!}\phi^4$ for the interaction term $U(\phi)$. Expanding in powers of S_I the exponential of the action becomes

$$e^{-S_E} = e^{-S_{E,0}}\sum_{l=0}^{\infty}\frac{(-S_{E,I})^l}{l!}. \tag{2.15}$$

Setting up perturbation theory is done by using (2.15) in the definition of the partition function to obtain

$$\ln Z(\beta) = \ln\left[\int \mathcal{D}\phi e^{-S_{E,0}}\right] + \ln\left[1+\sum_{l=1}^{\infty}\frac{\int \mathcal{D}\phi e^{-S_{E,0}}(-S_{E,I})^l}{l!\int \mathcal{D}\phi e^{-S_{E,0}}}\right]$$

and $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - + \ldots$ to expand the second logarithm[4]. The logarithm of the partition function is then given by

$$\ln Z(\beta) = \ln Z_0 + \sum_{l=1}^{\infty}\ln Z_l \tag{2.16}$$

where the first term is

$$\ln Z_0 = \ln\left[\int \mathcal{D}\phi e^{S_0}\right]$$

and the other terms constitute a power series in λ.

2.2.1 First order corrections

The lowest order correction to $\ln Z_0$, the partition function of the free scalar theory, is given by

$$\ln Z_1 = \frac{\int \mathcal{D}\phi e^{-S_{E,0}}S_{E,I}}{\int \mathcal{D}\phi e^{-S_{E,0}}}. \tag{2.17}$$

[4]Assuming that

$$|x| = \left|\sum_{l=1}^{\infty}\frac{1}{l!}\frac{\int \mathcal{D}\phi e^{S_0}S_I^l}{\int \mathcal{D}\phi e^{S_0}}\right| < 1.$$

Transforming ϕ into Fourier space (2.14) becomes

$$S_{E,I} = \frac{\lambda}{4!}\int_0^\beta d\tau \int d^3x \sum\!\!\!\!\!\!\int_{P_1} \ldots \sum\!\!\!\!\!\!\int_{P_4} \tilde{\phi}(\mathbf{p_1})_{n_1} \ldots \tilde{\phi}(\mathbf{p_4})_{n_4} e^{-i\tau(\omega_{n_1}+\ldots+\omega_{n_4})-i\mathbf{x}(\mathbf{p_1}+\ldots+\mathbf{p_4})}$$

so that after doing the d^3x and $d\tau$ integrations which give a Kronecker and a Dirac delta, respectively, we obtain

$$\begin{aligned}\ln Z_1 &= -\frac{\lambda}{4!}\sum\!\!\!\!\!\!\int_{P_1} \ldots \sum\!\!\!\!\!\!\int_{P_4} \beta\delta_{n_1+\ldots+n_4}(2\pi)^3\delta^3\left(\mathbf{p_1}+\ldots+\mathbf{p_4}\right)\times \\ &\quad \times \frac{\int \mathcal{D}\tilde{\phi} e^{-S_{E,0}}\tilde{\phi}_{n_1}\ldots\tilde{\phi}_{n_4}}{\int \mathcal{D}\tilde{\phi} e^{-S_{E,0}}}.\end{aligned}$$

In the previous section we have seen that the integrals vanish if there are phases left in the integrand. We therefore have either $n_3 = -n_1$, $n_4 = -n_2$ or the other two permutations. The integrals then factorize and – using the same techniques as in the previous section – we quickly arrive at the result

$$\ln Z_1 = -\frac{\lambda}{4!}3\left(\sum\!\!\!\!\!\!\int_P D_0(P)\right)^2 \beta\delta_{0,0}(2\pi)^3\delta^3(0).$$

The factor $\beta\delta_{0,0}(2\pi)^3\delta^3(0)$ is due to energy conservation and – for finite volume – would in fact be important for $\ln Z_1$ to be a proper thermodynamical quantity. Since we are only interested in the corrections to the propagator however, the factor does not have any relevance for further calculations and will be dropped consequently.

2.2.2 Self-energy

Corrections to the propagator are usually expressed via the introduction of the self-energy which we define similar to standard quantum field theory [6] as

$$\Pi(K) = D_0^{-1}(K) - D^{-1}(K).$$

Expanding $\Pi(K)$ formally as a power series in the coupling constant

$$\Pi = \sum_{l=1}^{\infty} \Pi_l$$

with $\Pi_l \sim \lambda^l$, one obtains an expansion for the full Matsubara propagator

$$D(K) = \frac{1}{D_0^{-1}(K) - \Pi(K)} = D_0(K) + D_0(K)\Pi_1(K)D_0(K) + \ldots. \tag{2.18}$$

To obtain an expression of Π_l for any given $l > 0$ we make use of a nice identity that we shall derive presently. First note that – after transforming ϕ into Fourier space – the free action (2.12) can be written as $S_{E,0} = \frac{1}{2\beta}\sum_n \int \frac{d^3p}{(2\pi)^3} D_0^{-1}(P)\left|\tilde{\phi}_n(\mathbf{p})\right|^2$. Armed with this insight the relation

$$\frac{\delta}{\delta D_0^{-1}(K)} \int \mathcal{D}\tilde{\phi} e^{-S_{E,0}} \sum_{l=0}^{\infty} \frac{(-S_{E,I})^l}{l!} = \int \mathcal{D}\tilde{\phi} e^{-S_{E,0}} \left(-\frac{1}{2\beta}\left|\tilde{\phi}_n(\mathbf{k})\right|^2\right) \sum_{l=0}^{\infty} \frac{(-S_{E,I})^l}{l!}$$

is easily derived; by finally using (2.4) and (2.11) we find the powerful identity

$$D(K) = -2\frac{\delta}{\delta D_0^{-1}(K)} \ln Z\,(\beta). \tag{2.19}$$

Since now both $D(K)$ and $\ln Z$ are given as a power series in λ we can compare (2.18) and (2.16) order by order to obtain $\Pi_l(K)$, e.g. for $l = 1$

$$\begin{aligned} D_0(K)\Pi_1 D_0(K) &= -2\frac{\delta \ln Z_1}{\delta D_0^{-1}(K)} = +2D_0^2(K)\frac{\delta \ln Z_1}{\delta D_0(K)} \\ \Pi_1 &= -\frac{1}{2}\lambda \sum\!\!\!\!\!\!\int_P D_0(P), \end{aligned}$$

where we have dropped the reference to K as argument of the self-energy since Π_1 does not depend on any external momentum.

2.2.3 Interpretation as Feynman diagrams

Calculation of higher order corrections to the self-energy (and therefore the propagator) is a tedious process if we have to keep track of all the factors of β, signs and permutation factors. Fortunately a powerful notation exists that allows us to directly write down the contributions to the self-energy of various order: Feynman diagrams. Similar to standard $\lambda\phi^4$ quantum field theory the building blocks of the Feynman diagrams are straight lines (free propagators) and vertices (4-legged crosses) which we use to construct diagrams that contribute to the self-energy in the usual way. By going through the same steps as in the previous section (calculating $\ln Z_l$, inserting it in (2.19) and comparing to (2.18)) we find the following Feynman rules for the self-energy to l^{th} order [1]

- Draw all connected diagrams that contribute to the given order
- Throw away all diagrams that are not 1PI (1 particle irreducible, diagrams that do not separate into two parts if one internal line is cut).
- Determine the combinatoric factor S for each diagram
- For each vertex, assign a factor $-\lambda\beta\delta_{\omega_{in},\omega_{out}}(2\pi)^3\delta^3\left(\mathbf{p_{in}} - \mathbf{p_{out}}\right)$. Because of energy conservation there will be a factor $\beta\delta_{0,0}(2\pi)^3\delta^3(0)$ left over which we will drop.

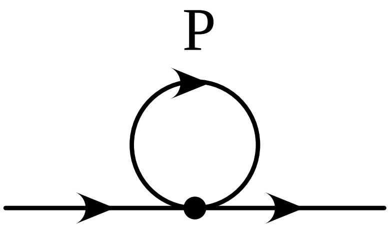


Figure 2.1: The tadpole diagram

- For each internal line, assign a factor $\sum\!\!\!\!\!\!\int_P D_0(P)$.
- Truncate the external lines.

The fact that we just have to consider connected diagrams goes back to the introduction of the logarithm of the partition function; all non-connected diagrams are cancelled because of the expansion of the logarithm.

The rules for determining the combinatoric factor S of a diagram can be summarized as follows: first count the number of possibilities to attach the first external line to the diagram and denote it e_1. Do the same for the second external line to obtain e_2. Having attached the external lines to the diagram, determine the number of possibilities i to draw the internal lines of the diagram. The symmetry factor of the diagram with v vertices is then given as the product of e_1, e_2 and i divided by a factor of 4! for each vertex and divided by $v!$ (coming from the development of the exponential)

$$S = \frac{e_1 e_2 i}{v!(4!)^v}.$$

2.3 Self-energy at finite temperature to first order

When applying the above Feynman rules to obtain the self-energy to first order we create the so-called "tadpole" diagram given in figure 2.1. Its algebraic form we already found in the previous section to be

$$\Pi_1 = -\frac{\lambda}{2} T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + p^2 + m^2},$$

where we have written out the sum-integral and the free propagator. The principal problem in evaluating this expression lies in the calculation of the infinite sum. Fortunately, there exists a technique of rewriting the sum in terms of contour integrals which can readily be evaluated. Since the technique is rather lengthy we do not include it in the main text but refer the interested reader to the appendix. Evaluating the sum we obtain

$$\Pi_1 = -\frac{\lambda}{2} \int \frac{d^3p dp_4}{(2\pi)^4} \frac{1}{p_4^2 + p^2 + m^2} - \frac{\lambda}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{p^2 + m^2}} \frac{1}{e^{\beta\sqrt{p^2+m^2}} - 1}.$$

It is interesting to see that the first term is actually the self-energy to first order in standard "Euclidean" quantum field theory (compare e.g. [4]). The second term contains a Bose-Einstein factor

$$n(\sqrt{p^2+m^2}) = \frac{1}{e^{\beta\sqrt{p^2+m^2}}-1}$$

and therefore vanishes at $T=0$. We may interpret it as contribution of thermal excitations of bosonic nature that are not present at $T=0$. Therefore standard quantum field theory results are often referred to as "vacuum" results from the viewpoint of TQFT and the self-energy to first order in TQFT may be written as

$$\Pi_1 = \Pi_1^{vac} + \Pi_1^{\beta}$$

with Π_1^{β} the thermal correction to the vacuum (standard quantum field) theory.

Note that in contrast to the vacuum contribution the thermal correction does not contain any ultraviolet divergences since the Bose-Einstein factor acts as a cutoff function. Furthermore, because we calculate a massive theory, both results are infrared finite. In order to evaluate the vacuum contribution we have to extract the ultraviolet divergence which we will do by dimensional regularization [4]. The number of space-time dimensions is changed from $4 \to 4-2\varepsilon$ with $\varepsilon \to 0$ at the end of the calculation. We define the new – dimensionless – coupling constant

$$\lambda = \lambda' \mu^{2\varepsilon}$$

with μ an arbitrary constant of mass dimension. The vacuum contribution is then given as[5]

$$\Pi_1^{vac} = -\frac{\lambda'}{2}\mu^{2\varepsilon}\int \frac{d^{4-2\varepsilon}P}{(2\pi)^{4-2\varepsilon}}\frac{1}{P^2+m^2}.$$

This integral is readily evaluated using (A.12) to be

$$\Pi_1^{vac} = -\frac{\lambda'}{2}\frac{\mu^{2\varepsilon}}{(4\pi)^{2-\varepsilon}}\Gamma(\varepsilon-1)\left(m^2\right)^{1-\varepsilon}$$

where $\Gamma(x)$ is the Gamma function defined in the appendix. The ultraviolet pole can now be extracted using (A.4) and its rearrangement $\Gamma(\varepsilon) \simeq \frac{1}{\varepsilon}\left(e^{-\gamma}\right)^{\varepsilon} + O(\varepsilon)$. Expanding the other terms to first order in ε we obtain for the vacuum contribution

$$\Pi_1^{vac} = \frac{\lambda''}{2}m^2\left[\frac{1}{\varepsilon}+1-\ln\hat{m}^2\right], \tag{2.20}$$

[5] As before, P is defined as Euclidean four-vector with $P^2 = p_4^2 + p^2$.

with $\lambda'' = \frac{\lambda'}{16\pi^2}$ and $\hat{m}^2 = \frac{m^2 e^\gamma}{4\pi\mu^2}$.

The thermal correction – as pointed out before – is manifestly finite and does not have to be regularized. The angular integrations are trivial and the result for the thermal contribution is therefore

$$\Pi_1^\beta = -\frac{\lambda}{2}\frac{1}{2\pi^2}T_1(m)$$

with

$$T_1(m) = \int_0^\infty dp \frac{p^2}{\sqrt{p^2+m^2}} n\left(\sqrt{p^2+m^2}\right).$$

Collecting the results for the vacuum and the thermal contribution we find the self-energy to first order to be given by

$$\Pi_1 = \frac{\lambda''}{2} m^2 \left[\frac{1}{\varepsilon} + 1 - \ln \hat{m}^2\right] - 4\lambda''\mu^{2\varepsilon}T_1(m) \qquad (2.21)$$

Taking the limit $m \to 0$ to a massless theory in the above formula we find that the divergent part vanishes and the only contribution comes from the thermal integral T_1:

$$\lim_{m\to 0} \Pi_1 = -4\lambda''\mu^{2\varepsilon}T_1(0).$$

2.4 Summary

In this chapter we specified the results of the previous chapter to a TQFT for the neutral scalar field with a $\lambda\phi^4$ interaction term. We calculated the free propagator and set up perturbation theory for the partition function. Moreover, we defined the self-energy of the neutral scalar field at finite temperature and gave the Feynman diagrammatic rules to obtain it to any given order in the coupling constant. Finally we calculated the self energy to first order which we found to be given by a temperature independent "vacuum" part that corresponds to standard quantum field theory and a thermal part that vanishes for $T \to 0$. In a last step we extracted the UV-divergences of the vacuum part using dimensional regularization.

Chapter 3

Self-energy in $\lambda\phi^4$ to second order

In the previous chapter we calculated the self-energy to first order in the coupling constant in the $\lambda\phi^4$ TQFT. In this chapter we want to calculate the correction to *second* order in λ which will prove to be much more involved.

Using the Feynman rules of the previous chapter it turns out that the self-energy correction to λ^2 involves the two diagrams shown in figure 3.1 and 3.2. The principal difficulty will lie in the calculation of the second diagram, usually referred to as the "sunset" graph. The reason for this is that the sunset diagram – unlike the other contributing graph – does depend on the external momentum rendering calculations rather tedious already in $T = 0$ quantum field theory. Fortunately, the first diagram, which we will call "double-scoop" will be relatively easy to evaluate.

3.1 The double-scoop diagram

Using the Feynman rules defined above the algebraic form of double-scoop diagram is given by

$$\Pi_2^A = S_A\lambda^2 \sum\!\!\!\!\!\!\!\int_P \sum\!\!\!\!\!\!\!\int_Q \sum\!\!\!\!\!\!\!\int_R D_0(P)D_0(Q)D_0(R)(2\pi)^3\delta^3(\mathbf{p}+\mathbf{q})\beta\delta_{\omega_p+\omega_q,0}$$

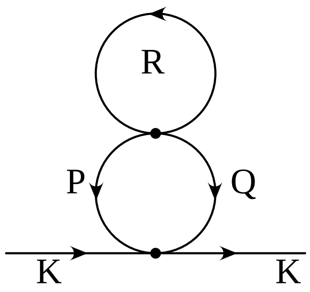


Figure 3.1: The double-scoop diagram.

with the symmetry factor $S_A = \frac{(8.3).4.3}{(4!)^2.2} = \frac{1}{4}$ and we have written ω_p as short-hand notation for ω_{n_p}. Using the delta functions we realize that the expression factorizes into

$$\Pi_A^{(2)} = \frac{\lambda^2}{4}\left(\sum\!\!\!\!\!\!\!\int_R D_0(R)\right)\left(\sum\!\!\!\!\!\!\!\int_P D_0^2(P)\right).$$

Fortunately, we already know the expression for the first factor since we have solved it already in order to calculate the tadpole diagram of the previous section and the result can be found in the appendix (B.10). The second factor can be obtained by the same result by using

$$D_0^2(P) = -\frac{\partial}{\partial m^2}D_0(P).$$

The result for the double-scoop diagram is therefore given by

$$\Pi_2^A = -\frac{1}{8}\frac{\partial}{\partial m^2}\left(\lambda\sum\!\!\!\!\!\!\!\int_P D_0(P)\right)^2$$

which – when writing out all the terms – becomes

$$\begin{aligned}\Pi_2^A \;=\; & -\frac{\lambda''^2}{4}\left[\frac{m^2}{\varepsilon^2} - \frac{1}{\varepsilon}\left(m^2(2\ln\hat{m}^2 - 1) + 8\mu^{2\varepsilon}T_1(m) + 8m^2\mu^{2\varepsilon}\frac{\partial}{\partial m^2}T_1(m)\right)\right. \\ & + m^2\left(2\ln^2\hat{m}^2 - 2\ln\hat{m}^2 + \frac{\pi^2}{6} + 1\right) + 8\mu^{2\varepsilon}T_1(m)\ln\hat{m}^2 \\ & \left. + 8\mu^{2\varepsilon}m^2(1-\ln\hat{m}^2)\partial_{m^2}T_1(m) + 64\mu^{4\varepsilon}T_1(m)\partial_{m^2}T_1(m)\right]. \end{aligned} \tag{3.1}$$

Note that Π_2^A contains temperature-dependent divergences that are due to products of divergent terms from the vacuum loops and finite thermal loops. In order for the self-energy to be finite, these thermal divergences will have to cancel *automatically* when we renormalize the vacuum theory in the next chapter.

3.2 The sunset diagram

Now we will focus on calculating the main part of our work, the sunset diagram shown in figure 3.2. Advancing step by step we will first calculate the thermal sums, then the angular integrals and finally extract the divergent terms to obtain the real part of the self-energy[1] to second order. Using our Feynman rules, its algebraic form is given by

$$\Pi_2^B(\mathbf{k},\omega_k) = S_B\lambda^2\sum\!\!\!\!\!\!\!\int_P\sum\!\!\!\!\!\!\!\int_Q\sum\!\!\!\!\!\!\!\int_R D_0(P)D_0(Q)D_0(R)(2\pi)^3\delta^3(\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{k})\,\beta\delta_{\omega_p+\omega_q+\omega_r+\omega_k,0}$$

[1]The imaginary part was already calculated in [34].

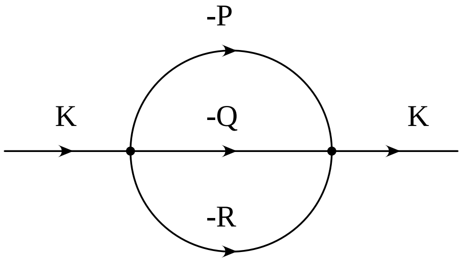


Figure 3.2: The sunset diagram.

where the symmetry factor is determined to be $S_B = \frac{8.4.6}{(4!)^2.2} = \frac{1}{6}$. Because of the Kronecker delta the thermal sums cannot be done separately, rendering the calculation rather awkward. This problem (and the resulting pages of painstaking work) can be circumvented, however, by using the identity (c.f.[1])

$$\beta\delta_{\omega_p+\omega_q+\omega_r+\omega_k,0} = \int_0^\beta d\theta\ e^{i(\omega_p+\omega_q+\omega_r+\omega_k)\theta} = \frac{e^{i(\omega_p+\omega_q+\omega_r+\omega_k)\beta}-1}{i(\omega_p+\omega_q+\omega_r+\omega_k)}, \tag{3.2}$$

which allows to factorize the thermal sums. Moreover, all the sums are now of the form $\sum_n e^{i\omega_n\theta}D_0(P)$, calculated in the appendix to be

$$T\sum_n \frac{e^{i\omega_n\theta}}{\omega_n^2+p^2+m^2} = \frac{1}{2\varepsilon_p}\frac{\cosh\left[(\theta-\beta/2)\varepsilon_p\right]}{\sinh\left[\beta\varepsilon_p/2\right]}. \tag{3.3}$$

Inserting this result, the contribution of the sunset diagram to the self-energy is then given by

$$\Pi_2^B(\mathbf{k},\omega_k) = \frac{\lambda^2}{6}\int \frac{d^3pd^3qd^3r}{(2\pi)^6}\delta^3\left(\mathbf{p}+\mathbf{q}+\mathbf{r}+\mathbf{k}\right)W(p,q,r,\omega_k) \tag{3.4}$$

with

$$W(p,q,r,\omega_k) = \int_0^\beta d\theta \prod_{i=1}^3 \frac{1}{2\varepsilon_i}\frac{\cosh\left[(\theta-\beta/2)\varepsilon_i\right]}{\sinh\left[\beta\varepsilon_i/2\right]}e^{i\omega_k\theta}$$

where $\varepsilon_i = (\varepsilon_p, \varepsilon_q, \varepsilon_r)$. The last remains of the thermal sums is now the integration over $d\theta$ in W which we will do presently. Using the identities

$$\begin{aligned}
e^{i\omega_k\theta} &= e^{i\beta\omega_k/2}\left(\cosh\left[i(\theta-\beta/2)\omega_k\right]+\sinh\left[i(\theta-\beta/2)\omega_k\right]\right)\\
\cosh\alpha\cosh\beta &= 1/2\left(\cosh\left(\alpha+\beta\right)+\cosh\left(\alpha-\beta\right)\right)\\
\cosh\alpha\sinh\beta &= 1/2\left(\cosh\left(\alpha+\beta\right)-\sinh\left(\alpha-\beta\right)\right)
\end{aligned}$$

we can split the integrand consisting of the exponential and the product over the cosini hyperbolici into a cosh and a sinh part. Substituting now $\theta-\beta/2=\bar{\theta}$, the

$d\bar{\theta}$ integration limits are symmetric so that the sinh part (which is antisymmetric) does not contribute to the integral. W becomes

$$W = e^{i\omega_k\beta/2} \prod_{i=1}^{3} \frac{1}{2\varepsilon_i 2\sinh\left[\beta\varepsilon_i/2\right]} \int_{-\beta/2}^{+\beta/2} d\bar{\theta} f\left(\bar{\theta}\right)$$

with

$$f\left(\bar{\theta}\right) = \sum_{\sigma,\tau,\kappa} \cosh\left[\bar{\theta}(\varepsilon_p + \sigma\varepsilon_q + \tau\varepsilon_r + i\kappa\omega_k)\right]$$

and the sum is over $\sigma, \tau, \kappa = \pm$. The integration is done easily and – using $e^{\pm i\omega_k\beta} = e^{\pm i2\pi n} = 1$ – the result can be written as

$$\int_{-\beta/2}^{+\beta/2} d\theta f(\bar{\theta}) = \frac{1}{8} e^{-i\omega_k\beta/2} \sum_{\sigma,\tau,\kappa} \frac{2\sinh\left[\beta/2 E_{\sigma\tau}\right]}{E_{\sigma\tau} + i\kappa\omega_k}$$

with

$$E_{\sigma\tau} = \varepsilon_p + \sigma\varepsilon_q + \tau\varepsilon_r.$$

Rewriting the sini hyperbolici as Bose-Einstein distribution functions by using

$$\frac{1}{2\sinh\left[\beta\varepsilon_i/2\right]} = e^{\beta\varepsilon_i/2} n(\varepsilon_i),$$

we obtain for W the rather unwieldy result

$$\begin{aligned} W \;=\; & \frac{n(\varepsilon_p)n(\varepsilon_q)n(\varepsilon_r)}{2\varepsilon_p 2\varepsilon_q 2\varepsilon_r} \sum_{\kappa} P \left\{ \frac{e^{\beta(\varepsilon_p+\varepsilon_q+\varepsilon_r)} - 1}{\varepsilon_p + \varepsilon_q + \varepsilon_r + i\kappa\omega_k} + \frac{e^{\beta(\varepsilon_p+\varepsilon_p)} - e^{\beta\varepsilon_r}}{\varepsilon_p + \varepsilon_q - \varepsilon_r + i\kappa\omega_k} + \right. \\ & \left. + \frac{e^{\beta(\varepsilon_p+\varepsilon_r)} - e^{\beta\varepsilon_q}}{\varepsilon_p - \varepsilon_q + \varepsilon_r + i\kappa\omega_k} + \frac{e^{\beta(\varepsilon_q+\varepsilon_r)} - e^{\beta\varepsilon_p}}{-\varepsilon_p + \varepsilon_q + \varepsilon_r + i\kappa\omega_k} \right\}. \end{aligned}$$

Fortunately, this expression can be greatly simplified by using the symmetries $p \leftrightarrow q$, $p \leftrightarrow r$ and $q \leftrightarrow r$ of the integrand as well as the identity (B.1). Applying those we find W as a sum of terms with zero, one and two thermal factors $n(\varepsilon_i)$, respectively:

$$W = W_0 + 3\frac{n(\varepsilon_p)}{2\varepsilon_p} W_1 + 3\frac{n(\varepsilon_p)n(\varepsilon_q)}{2\varepsilon_p 2\varepsilon_q} W_2, \tag{3.5}$$

with W_0, W_1 and W_2, given by

$$\begin{aligned} W_0 &= \frac{1}{2\varepsilon_p 2\varepsilon_q 2\varepsilon_r} \sum_{\kappa} \frac{1}{\varepsilon_p + \varepsilon_q + \varepsilon_r + i\kappa\omega_k} \\ W_1 &= \frac{1}{2\varepsilon_q 2\varepsilon_r} \sum_{\sigma,\kappa} \frac{1}{\sigma\varepsilon_p + \varepsilon_q + \varepsilon_r + i\kappa\omega_k} \\ W_2 &= \frac{1}{2\varepsilon_r} \sum_{\sigma,\tau,\kappa} \frac{1}{\sigma\varepsilon_p + \tau\varepsilon_q + \varepsilon_r + i\kappa\omega_k}. \end{aligned} \tag{3.6}$$

Note that in general there would have been a term with three thermal factors but the coefficient of this term can be determined to be 0. This is in fact a generic feature of the thermal field theory in consideration; besides the vacuum contribution (which every diagram possesses), most[2] diagrams which are n particle irreducible will consist of a contribution with n thermal factors, one with $n-1$ thermal factors and so on. Moreover, we will find evidence that in the case of the sunset diagram those contributions may be obtained by "cutting" propagators of the original diagram and evaluating those "cut" diagrams in standard quantum field theory.

Reformulation of results When trying to do the angular integrations with the parts of W given in (3.6) one quickly runs into nasty difficulties stemming from the delta function in (3.4). It turns out that a far more tractable formulation is possible by rewriting (3.6) to exchange the 3−dimensional delta function with a 4−dimensional delta function. To derive the reformulation we consider the integral

$$I_0 = \int dp_4 dq_4 dr_4 \frac{1}{p_4^2 + p^2 + m^2} \frac{1}{q_4^2 + q^2 + m^2} \frac{1}{r_4^2 + r^2 + m^2} \delta\left(p_4 + q_4 + r_4 + k_4\right).$$

Using the integral representation of the delta function the above integral factorizes and the three integrals of the form

$$\int_{-\infty}^{+\infty} d\omega \frac{e^{i\omega l}}{\omega^2 + \varepsilon^2} = \Theta(l) \frac{2\pi i}{2i\varepsilon} e^{-l\varepsilon} + \Theta(-l) \frac{2\pi i}{2i\varepsilon} e^{+l\varepsilon}$$

can be calculated using contour integration. The final integration over l is trivial and we find the mentioned reformulation of W_0:

$$I_0 = \frac{1}{2\pi} \frac{2\pi}{2\varepsilon_p} \frac{2\pi}{2\varepsilon_q} \frac{2\pi}{2\varepsilon_r} \sum_{\kappa} \frac{1}{\varepsilon_p + \varepsilon_q + \varepsilon_r + i\kappa k_4} = (2\pi)^2 W_0(\omega_k = k_4).$$

For W_1 we calculate the integral

$$I_1 = \int dq_4 dr_4 \frac{1}{q_4^2 + q^2 + m^2} \frac{1}{r_4^2 + r^2 + m^2} \times \left(\delta\left(p_4 + q_4 + r_4 + \omega_k\right) + \delta\left(-p_4 + q_4 + r_4 + \omega_k\right)\right)\Big|_{p_4^2 = -\varepsilon_p^2}$$

which – using the same steps as above – becomes

$$\begin{aligned} I_1 &= \frac{2\pi}{2\varepsilon_q 2\varepsilon_r} \sum_{\kappa} \left(\frac{1}{\varepsilon_q + \varepsilon_r + i\kappa(p_4 + k_4)} + \frac{1}{\varepsilon_q + \varepsilon_r + i\kappa(-p_4 + k_4)} \right)\Bigg|_{p_4^2 = -\varepsilon_p^2} \\ &= \frac{2\pi}{2\varepsilon_q 2\varepsilon_r} \sum_{\sigma,\kappa} \frac{1}{\sigma\varepsilon_p + \varepsilon_q + \varepsilon_r + i\kappa\omega_k} = 2\pi W_1 \end{aligned}$$

[2] A notable exception to this interesting picture is however the double-scoop diagram already calculated in this section (due to the term D_0^2). Unfortunately, we do not know if double-scoop type diagrams are the only ones that cannot be separated in the suggested manner.

Finally, for W_2 a similar reformulation is possible. However, it turns out that the trivial modification

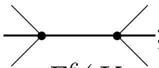
$$W_2 = \sum_{\sigma,\tau} \frac{1}{\varepsilon_r^2 - \left(i\omega_k + \sigma\varepsilon_p + \tau\varepsilon_q\right)^2}$$

is most convenient for the angular integrations. Using the reformulations $W_0 = I_0/(2\pi)^2$, $W_1 = I_1/(2\pi)$ in (3.5), Π_2^A becomes

$$\Pi_2^A = \frac{\lambda^2}{6}\left(J_0 + J_1 + J_2\right) \tag{3.7}$$

with J_i, $i = 0, 1, 2$ the contributions with zero, one and two thermal factors, respectively, defined by

$$\begin{aligned}
J_0 &= \int \frac{d^4P d^4Q d^4R}{(2\pi)^8} \frac{1}{P^2+m^2} \frac{1}{Q^2+m^2} \frac{1}{R^2+m^2} \delta^4\left(P+Q+R+K\right) \\
J_1 &= \int \frac{d^3p}{(2\pi)^7} \frac{3n(\varepsilon_p)}{2\varepsilon_p} \times \\
&\quad \times \int d^4Q d^4R \frac{1}{Q^2+m^2} \frac{1}{R^2+m^2} \left(\delta^4\left(P+Q+R+K\right) + \delta^4(-P+Q+R+K)\right)\Big|_{P^2=-m^2} \\
J_2 &= \int \frac{d^3p d^3q}{(2\pi)^6} \frac{3n(\varepsilon_p)n(\varepsilon_q)}{2\varepsilon_p 2\varepsilon_q} \sum_{\sigma,\tau} \frac{(-1)}{\left(i\omega_k + \sigma\varepsilon_p + \tau\varepsilon_q\right)^2 - \left|\mathbf{k}+\mathbf{p}+\mathbf{q}\right|^2 - m^2}.
\end{aligned}$$

Interpretation of the reformulation As mentioned above the contribution with zero thermal factors corresponds to the sunset diagram in $T = 0$ Euclidean $\lambda\phi^4$ theory. It is interesting however that $\lambda^2 J_1$ may be obtained by the following procedure: cutting one of the three lines of the sunset diagram (explaining the factor three) results in the "fish" diagram Γ_2^4 shown in figure 4.1. $\lambda^2 J_1$ is then given as the integral $3\int_0^\infty \frac{dp}{(2\pi)^3}\frac{n(\varepsilon_p)}{2\varepsilon_p}\mathcal{K}_1$ with $\mathcal{K}_1 = \sum_{\kappa=\pm}\Gamma_2^4(K+\kappa P)|_{P^2=-m^2}$ and $\Gamma_2^4(R)$ the fish diagram in $T = 0$ quantum field theory and external momentum R. Interestingly, also $\lambda^2 J_2$ can be described in this picture. Cutting two lines of the sunset diagram (of which there are again three possibilities) results in the diagram $\Gamma_2^6 =$ ; $\lambda^2 J_2$ is then given as $3\int_0^\infty \frac{dp}{(2\pi)^3}\int_0^\infty \frac{dq}{(2\pi)^3}\frac{n(\varepsilon_p)}{2\varepsilon_p}\frac{n(\varepsilon_q)}{2\varepsilon_q}\mathcal{K}_2$ with $\mathcal{K}_2 = \sum_{\kappa,\sigma=\pm}\Gamma_2^6(K+\kappa P+\sigma Q)|_{P^2=Q^2=-m^2}$. Therefore, the sunset diagram can be written as

$$\Pi_2^B = \frac{1}{6}\left(\Pi_2^{B,vac} + 3\int_0^\infty \frac{dp}{(2\pi)^3}\frac{n(\varepsilon_p)}{2\varepsilon_p}\mathcal{K}_1 + 3\int_0^\infty \frac{dp}{(2\pi)^3}\frac{dq}{(2\pi)^3}\frac{n(\varepsilon_p)}{2\varepsilon_p}\frac{n(\varepsilon_q)}{2\varepsilon_q}\mathcal{K}_2\right)$$

with $\mathcal{K}_1$ and $\mathcal{K}_2$ the sunset diagram with one and two lines cut and summed over the cut momenta set on mass shell. This scheme seems to apply to all diagrams that do not have a term D_0^2 in their algebraic structure. If a similar scheme could be found for those diagrams we would be left with a diagrammatic expansion of TQFT diagrams in terms of standard quantum field diagrams without having to do any thermal sums.

3.3 Evaluation of the integrals

We return to our main objective, the calculation of the contributions $J_{0,1,2}$. As we will see, the further evaluation of the integrals turns out to be rather involved from a technical point of view.

In general one would like to have an analytic expression for the sunset diagram in terms of the external variables m, $\mathbf{k}$ and k_0; this, however seems to be prohibited because the J_1 and J_2 contributions involve integrals over thermal factors which – in general – one does not know how to evaluate analytically. Furthermore it turns out that even the J_0 integral, which corresponds to the vacuum contribution of the sunset diagram, cannot – in contrast to the other diagrams we have calculated so far – be evaluated in closed form using standard functions[3]. On the other hand, trying to directly integrate the expressions J_0, J_1 and J_2 numerically (after having integrated over the delta functions) is also not feasible since

- it will turn out that both J_0 and J_1 do have ultraviolet divergent parts that have to be extracted before integrating the finite part
- the analytic continuation $i\omega_k \to k_0 + i\eta$ of the integrals is non-trivial (as opposed to the cases we have treated so far). It has to be done extremely carefully since naive continuation may lead to results that seem correct but on closer inspection turn out not to be (see e.g.[26], [27]).
- naive counting reveals we have to do 5 non-trivial integrations to evaluate J_2. Later analytic evaluation will show that the integrand possesses logarithmic divergences that render the numerical evaluation tedious and time-consuming.

Because of the above problems we will try to do as many integrals as possible analytically, and only later resort to numerical integration of the ones that remain.

3.3.1 Zero thermal factors

We will start by the evaluation of the J_0 contribution that will turn out to be more work than anticipated, given that it corresponds to the sunset diagram in standard quantum field theory for which we expected to find complete results in the literature. This turned out not to be the case, but we found it convenient to base our calculation on the method proposed in [4] and afterwards compare our analytical result numerically with [33][4] in a later chapter. After integrating over

[3] A closed expression using the dilogarithm (or Spence) function – according to [4] and [33] – is possible, however.

[4] The analytical result for the sunset diagram in this reference seems to have been printed rather carelessly and is thereby flawed. Nevertheless, their numerical results (which they base on another method) seem to be correct.

the delta function J_0 becomes

$$J_0 = \int_{P,Q} \frac{1}{P^2+m^2} \frac{1}{Q^2+m^2} \frac{1}{(P+Q+K)^2+m^2}$$

where we have used the shorthand notation $\int_P = \int \frac{d^4P}{(2\pi)^4}$. Using the identity

$$1 = \frac{1}{2D}\left(\frac{\partial P^\mu}{\partial P^\mu} + \frac{\partial Q^\mu}{\partial Q^\mu}\right)$$

with D the number of space-time dimensions we can rewrite the J_0 as

$$J_0 = -\frac{1}{2D}\int_{P,Q}\left(P^\mu\frac{\partial}{\partial P^\mu} + Q^\mu\frac{\partial}{\partial Q^\mu}\right)\frac{1}{P^2+m^2}\frac{1}{Q^2+m^2}\frac{1}{(Q+P+K)^2+m^2}$$

having integrated by parts. Differentiating and using the symmetries of the integrand $P \leftrightarrow Q$ as well as shifts $P + R \to P$ we obtain

$$J_0 = \frac{1}{D}\int_{P,Q}\frac{-K_\mu (P+Q+K)^\mu + 3(P+Q+K)^2}{(P^2+m^2)(Q^2+m^2)\left((P+Q+K)^2+m^2\right)^2}$$

which, when adding and subtracting m^2 in the numerator can be written as

$$\begin{aligned} J_0 &= \frac{1}{D}3J_0 - \frac{1}{D}\left(K_\mu I_A^\mu(K) + 3m^2 I_B(K)\right) \\ &= -\frac{1}{D-3}\left(K_\mu I_A^\mu(K) + 3m^2 I_B(K)\right) \end{aligned} \tag{3.8}$$

with

$$\begin{aligned} I_A^\mu(K) &= \int_{P,Q}\frac{(P+Q+K)^\mu}{(P^2+m^2)(Q^2+m^2)\left((P+Q+K)^2+m^2\right)^2} \\ I_B(K) &= \int_{P,Q}\frac{1}{(P^2+m^2)^2(Q^2+m^2)\left((P+Q+K)^2+m^2\right)}. \end{aligned}$$

In order to calculate I_A^μ and I_B we will introduce Feynman parameters, one loop at a time. As mentioned before this may be the cause of subtle problems in TQFT since the analytic continuation $i\omega_k \to k_0 + i\eta$ becomes quite difficult (in general there appear unphysical branch points in the complex k_0 plane which have to be cancelled by adding another function to the final result). We will show that in our case no such branch points appear.

Concentrating on I_B first we rewrite

$$I_B(K) = \int_{P,Q}\frac{1}{(P^2+m^2)^2}\int_0^1 dx \frac{1}{\left[Q^2+m^2+(P+K)^2 x(1-x)\right]^2}. \tag{3.9}$$

The integral over dQ is divergent and we will again use dimensional regularization in order to extract the pole. We get

$$I_B(K) = \int_P \frac{1}{(P^2+m^2)^2} \int_0^1 dx \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \frac{1}{\left[m^2 + (P+K)^2 x(1-x)\right]^{\varepsilon}}.$$

Extracting $x(1-x)$ from the denominator and using formula (A.11) we obtain

$$\begin{aligned} I_B(K) &= \frac{\Gamma(2+\varepsilon)}{(4\pi)^{2-\varepsilon}} \int_0^1 dx \int_0^1 dy \frac{y^{-1+\varepsilon}}{(x(1-x))^{\varepsilon}} (1-y) \times \\ & \int_P \frac{1}{\left[P^2 + K^2 y(1-y) + m^2\left(1-y+\frac{y}{x(1-x)}\right)\right]^{\varepsilon+2}}, \end{aligned}$$

which, after doing the dP integration using (A.12) becomes

$$I_B(K) = \frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \int_0^1 dx \int_0^1 dy \frac{y^{-1+\varepsilon}}{(x(1-x))^{\varepsilon}} (1-y)\, A(x,y)^{-2\varepsilon},$$

with

$$A(x,y) = K^2 y(1-y) + m^2\left(1-y+\frac{y}{x(1-x)}\right). \tag{3.10}$$

The evaluation of I_A^μ proceeds in a similar way: substituting $P \to -P-Q-K$ and introducing a Feynman parameter as in (3.9), then integrating over dQ gives

$$I_A^\mu = \int_P \frac{-P^\mu}{(P^2+m^2)^2} \int_0^1 dx \frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \frac{1}{\left[m^2 + (P+K)^2 x(1-x)\right]^{\varepsilon}}.$$

Introducing a second Feynman parameter and using

$$\frac{\partial}{\partial K_\mu}\left(\frac{1}{\left[(P+Ky)^2 + A(x,y)\right]^{\varepsilon+1}}\right) = \frac{2y\left(P^\mu + K^\mu y\right) + 2K^\mu y(1-y)}{\left[(P+Ky)^2 + A(x,y)\right]^{\varepsilon+2}}(-1-\varepsilon)$$

we can do the dP integration similar to I_B and obtain after a little bit of algebra

$$I_A^\mu = K^\mu \frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \int dx dy (1-y) \left(\frac{y}{x(1-x)A^2}\right)^{\varepsilon}.$$

Collecting the above results and using $D = 4-2\varepsilon$ as above we obtain

$$J_0 = -\frac{1}{1-2\varepsilon} \frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}} \int_0^1 dx \int_0^1 dy \left(\frac{y}{x(1-x)A^2}\right)^{\varepsilon} \left(K^2(1-y) + 3m^2 \frac{1-y}{y}\right). \tag{3.11}$$

Doing the analytical continuation $i\omega_k \to k_0 + i\eta$, $\eta \to 0$ results in $K^2 \to -(k_0 + i\eta)^2 + k^2$. Ignoring the uninteresting possibilities $x = 0$, $x = 1$, $y = 0$, the contribution J_0 only has poles for $A^2 = 0$. These are attained for

$$(k_0 + i\eta)^2 = \frac{m^2}{y(1-y)}\left(1 - y + \frac{y}{x\,(1-x)}\right) + k^2$$

which – since the r.h.s is manifestly positive – only allows real values of $k_0 + i\eta$. This is good news as obviously we did *not* pick up any unphysical branch points in our expression for J_0. No further correction terms have to be added and (3.11) may be continued to Minkowskian momenta $K_M = (k_0 + i\eta)^2 - k^2$ without restriction.

Extraction of the divergences and finite terms is quite lengthy and will not be presented here (the details can be found in appendix C). We obtain for $\lambda^2 J_0$

$$\lambda^2 J_0 = -\lambda''^2\left(\frac{1}{\varepsilon^2}\frac{3m^2}{2} + \frac{1}{\varepsilon}\left(\frac{3}{2}m^2\left(3 - 2\ln\hat{m}^2\right) - \frac{1}{4}K_M^2\right)\right) + \lambda^2 J_0^{fin}$$

with $\lambda^2 J_0^{fin}$ the finite contribution to $\lambda^2 J_0$ given in (C.8) and $\lambda'' = \frac{\lambda}{\mu^{2\varepsilon}(4\pi)^2}$.

3.3.2 One thermal factor

For the J_1 integral we first use the delta-functions to do the dR integration and get

$$J_1 = \int \frac{d^3p}{(2\pi)^7}\frac{3n(\varepsilon_p)}{2\varepsilon_p}\int d^4Q \frac{1}{Q^2+m^2}\left(\frac{1}{(Q+K+P)^2+m^2} + \frac{1}{(Q+K-P)^2+m^2}\right)\Bigg|_{P^2=-m^2}.$$

In order to do the dQ integration we will once again use the Feynman identity (A.11) and have to check at the end of the calculation that no singularities in the complex k_0 plane have appeared.

Using dimensional regularization in order to extract the divergent part of the dQ integral we obtain

$$\begin{aligned} J_1 \;=\; & \int \frac{d^3p}{(2\pi)^3}\frac{3n(\varepsilon_p)}{2\varepsilon_p}\frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}} \times \\ & \times \int_0^1 dx\left[\left(m^2 + x(1-x)(K+P)^2\right)^{-\varepsilon} + \left(m^2 + x(1-x)(K-P)^2\right)^{-\varepsilon}\right]_{P^2=-m^2}. \end{aligned}$$

Since the integral is evaluated at $P^2 = -m^2$ and we may use the symmetry of the integrand $\mathbf{p} \leftrightarrow -\mathbf{p}$ we write J_1 as

$$J_1 = \int \frac{d^3p}{(2\pi)^3}\frac{3n(\varepsilon_p)}{2\varepsilon_p}\frac{\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}\int_0^1 dx \sum_{\kappa=\pm}\left[m^2 + x(1-x)\left((\omega_k + i\kappa\varepsilon_p)^2 + (\mathbf{k}+\mathbf{p})^2\right)\right]^{-\varepsilon}.$$

To extract the divergent terms we expand everything in powers of ε and find – using the results from the J_0 calculation –

$$\lambda^2 J_1 = (4\pi)^2 \mu^{2\varepsilon} \lambda''^2 \int \frac{d^3p}{(2\pi)^3} \frac{3n(\varepsilon_p)}{2\varepsilon_p} \frac{1}{\varepsilon} \sum_\kappa \int_0^1 dx \left(1 - \varepsilon \ln \hat{B}_\kappa(x)\right)$$

with λ'' as above and

$$\hat{B}_\kappa(x) = \frac{e^\gamma}{4\pi\mu^2} \left[m^2 + x(1-x)\left((\omega_k + i\kappa\varepsilon_p)^2 + (\mathbf{k}+\mathbf{p})^2\right)\right].$$

Doing the analytic continuation $i\omega_k \to k_0 + i\eta$ we find that the only poles of J_1 are attained for $\hat{B}_\kappa = 0$, that is for

$$(k_0 - \kappa\varepsilon_p + i\eta)^2 = m^2 + x(1-x)\left(\mathbf{k}+\mathbf{p}\right)^2.$$

Again there are no poles in the complex k_0 plane and the analytical continuation of $K = (\mathbf{k}, \omega_k)$ to $K_M = (k_0, \mathbf{k})$ is straightforward. Since $\ln \hat{B}_\kappa(x)$ can be written as $\ln \hat{B}_\kappa(x) = \ln \hat{m}^2 + \ln \left[1 + x(1-x)\frac{4}{a_1}\right]$ with the parameter a_1 given as

$$\frac{4m^2}{-E_\kappa^2 + \left(\mathbf{k}+\mathbf{p}\right)^2} = a_1 \tag{3.12}$$

and $E_\kappa^2 = (k_0 - \kappa\varepsilon_p + i\eta)^2$ we can use (C.1) for the dx integration and obtain

$$\lambda^2 \mathrm{Re} J_1 = (4\pi)^2 \mu^{2\varepsilon} \lambda''^2 \int \frac{d^3p}{(2\pi)^3} \frac{3n(\varepsilon_p)}{2\varepsilon_p} \left(\frac{2}{\varepsilon} - 2\ln \hat{m}^2 + 4 - \sum_\kappa f_2(a_1)\right).$$

Substitution of $\left|\mathbf{p}+\mathbf{k}\right|^2 = u^2$ in order to do the angular integrals gives

$$\lambda^2 \mathrm{Re} J_1 = 6\mu^{2\varepsilon} \lambda''^2 \int_0^\infty p^2 dp \frac{n(\varepsilon_p)}{\varepsilon_p} \left(\frac{4}{\varepsilon} - 4\ln \hat{m}^2 + 8 - \sum_\kappa \frac{1}{pk} \int_{|p-k|}^{p+k} u du f_2(a_1)\right)$$

with $f_2(a_1)$ taking the form[5]

$$\begin{aligned}
f_2(a_1) &= \sqrt{\tfrac{E_\kappa^2 - M_u^2}{E_\kappa^2 - u^2}} \ \ln \frac{\sqrt{E_\kappa^2 - u^2} + \sqrt{E_\kappa^2 - M_u^2}}{\sqrt{E_\kappa^2 - u^2} - \sqrt{E_\kappa^2 - M_u^2}}, \quad u^2 < E_\kappa^2 - 4m^2 \\
&= 2\sqrt{\tfrac{M_u^2 - E_\kappa^2}{E_\kappa^2 - u^2}} \ \arctan \sqrt{\frac{E_\kappa^2 - u^2}{M_u^2 - E_\kappa^2}}, \quad E_\kappa^2 - 4m^2 < u^2 < E_\kappa^2 \\
&= \sqrt{\tfrac{M_u^2 - E_\kappa^2}{u^2 - E_\kappa^2}} \ \ln \frac{\sqrt{M_u^2 - E_\kappa^2} + \sqrt{u^2 - E_\kappa^2}}{\sqrt{M_u^2 - E_\kappa^2} - \sqrt{u^2 - E_\kappa^2}}, \quad E_\kappa^2 < u^2.
\end{aligned}$$

and $M_u^2 = 4m^2 + u^2$.

[5] Incidentally, this is the same function that appeared in [29].

3.3.3 Two thermal factors

Since for J_2 all the integrations over momenta are cut off by the Bose-Einstein distributions the integral is manifestly finite. We therefore do not have to regularize it and may directly continue the external Matsubara frequency $i\omega_k$ analytically to a Minkowskian momentum k_0. Therefore J_2 reads

$$\begin{aligned} J_2 &= \int \frac{d^3p d^3q}{(2\pi)^6} \frac{3n(\varepsilon_p)n(\varepsilon_q)}{2\varepsilon_p 2\varepsilon_q} \sum_{\sigma,\tau} \frac{(-1)}{E_{\sigma\tau}^2 - |\mathbf{k}+\mathbf{p}+\mathbf{q}|^2 - m^2} \\ &= \int \frac{p^2 dp q^2 dq}{(2\pi)^6} \frac{3n(\varepsilon_p)n(\varepsilon_q)}{2\varepsilon_p 2\varepsilon_q} \sum_{\sigma,\tau} I_2'(p,q,K_M) \end{aligned}$$

with $E_{\sigma\tau} = k_0 + \sigma\varepsilon_p + \tau\varepsilon_q + i\eta$ and $I_2'(p,q,K_m)$ containing the angular integrations which are in fact quite lengthy and shall be presented in detail in the appendix (see (C.9)). Taking directly result (C.19) for I_2' from the appendix we obtain

$$\mathrm{Re}\ J_2 = \frac{3}{8(2\pi)^4} \int_0^\infty pdpqdq \frac{n(\varepsilon_p)n(\varepsilon_q)}{\varepsilon_p\varepsilon_q k} f_4(p,q,K_M)$$

with

$$\begin{aligned} f_4(p,q,K_M) = \sum_{\sigma\tau} &(f_3(E_{\sigma\tau}, p+k+q) + f_3(E_{\sigma\tau}, p-k-q) \\ &-f_3(E_{\sigma\tau}, p+|k-q|) - f_3(E_{\sigma\tau}, p-|k-q|)) \end{aligned}$$

and f_3 defined in (C.18).

The result Summing up all the contributions in (3.7), the real part of the sunset diagram at finite temperature is given as

$$\begin{aligned} \mathrm{Re}\ \Pi_2^B = -\lambda''^2 \Bigg[&\frac{m^2}{4\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{m^2}{4}(3 - 2\ln\hat{m}^2) - \frac{K_M^2}{24} - 4\mu^{2\varepsilon}T_1(m)\right) \\ &+ \frac{m^2}{2}\left(\ln^2\hat{m}^2 - 3\ln\hat{m}^2 + \frac{\pi^2}{12} + \frac{7}{2}\right) \\ &-K_M^2\left(\frac{11}{48} - \frac{1}{12}\ln\hat{m}^2\right) - \frac{m^2}{6} I_C(-K_M^2/m^2) \\ &4(\ln\hat{m}^2 - 2)\mu^{2\varepsilon}T_1(m) + \mu^{2\varepsilon}\sum_\kappa \int_0^\infty pdp\frac{n(\varepsilon_p)}{\varepsilon_p}\frac{1}{k}\int_{|p-k|}^{p+k} uduf_2(a_1) \\ &-\mu^{4\varepsilon}\int_0^\infty pdpqdq\frac{n(\varepsilon_p)n(\varepsilon_q)}{\varepsilon_p\varepsilon_q k} f_4(p,q,K_M)\Bigg] \end{aligned} \quad (3.13)$$

3.4 Summary

In this chapter we calculated the double-scoop diagram and extracted its divergences using dimensional regularization. Then we evaluated the thermal sums of the sunset diagram and found that the diagram consists of 3 contributions, a vacuum part and two contributions with one and two thermal factors, respectively, for which we tried to give an interpretation. We extracted the divergences from these contributions and did as many analytic integrations as possible.

Chapter 4

Renormalization

In the previous chapters we have calculated the self-energies to first and second order in λ that follow from the original Lagrangian. We found that the respective contributions contain divergent terms that we have to get rid of in order to obtain a meaningful physical quantity. To achieve this, we will use the concept of renormalization, the basic idea of which comes from the observation that in one-loop graphs the divergences amount to shifts in the parameters of the action [7]. However, note that this shift in e.g. the mass can be cancelled by a change in the normalization of the original – or *bare* – mass (denoted by a subscript b); this would be sufficient to render the self-energy finite to first order, but for the second order we also have to renormalize the field and the coupling constant. To be more precise, let us consider the theory described by the original Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi_b\right)\left(\partial_\mu \phi_b\right) - \frac{1}{2}m_b^2\phi_b^2 - \frac{\lambda_b}{4!}\phi_b^4$$

and effect a rescaling of the normalization of the bare field ϕ_b, mass m_b and coupling constant λ_b by real and constant factors,

$$\begin{aligned} \phi_b &= \sqrt{Z_0}\phi_r \\ m_b &= Z_1 Z_0^{-1} m_r \\ \lambda_b &= Z_2 Z_0^{-2} \lambda_r \end{aligned}$$

where the subscript r denotes the *re*normalized quantities. In the limit of vanishing coupling constant $\lambda_r \to 0$ the theory is free and the divergences vanish; in this limit we therefore do not have to renormalize the theory [21] and consequently define

$$Z_i = 1 + \delta_i, \quad \lim_{\lambda_r \to 0} \delta_i = 0$$

with $i = 0, 1, 2$. Each quantity δ_i is therefore formally of order λ_r and the presence of these terms is characteristic of an interacting theory. It is therefore natural to express the Lagrangian $\mathcal{L}$ as a sum in terms of the renormalized quantities,

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi_r\right)\left(\partial_\mu \phi_r\right) - \frac{1}{2}m_r^2\phi_r^2 - \frac{\lambda_r}{4!}\phi_r^4 + \mathcal{L}_\delta = \mathcal{L}_{ren}$$

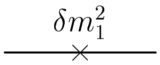
with

$$\begin{aligned}\mathcal{L}_\delta &= \frac{1}{2}\delta_0\left(\partial^\mu\phi_r\right)\left(\partial_\mu\phi_r\right) - \frac{1}{2}\delta_1 m_r^2\phi_r^2 - \frac{\delta_2\lambda_r}{4!}\phi_r^4 \\ &= \mathcal{L}_\phi + \mathcal{L}_m + \mathcal{L}_\lambda. \end{aligned} \tag{4.1}$$

The three constituents of $\mathcal{L}_\delta$ that are called *wave-function-*, *mass-* and *coupling-constant* counterterm, respectively, and will be treated as interactions. These counterterms may now be used to cancel the divergences in the self-energy order by order *at* $T=0$, thereby leaving us with a finite, renormalized result for the vacuum self-energy[1]. We will show that renormalizing the vacuum self-energy is sufficient to render the theory finite also at $T \neq 0$, that is, all the thermal divergences that we have found in the last chapter have to cancel automatically.

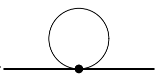
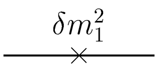
4.1 Renormalization of Π_1

We will start by recalculating the self-energy to first order in terms of the renormalized quantities; to render it finite, we will only need the mass counterterm $\frac{1}{2}\delta_1 m_r^2\phi_r^2$ and simply ignore the other two counterterms for the moment. It is useful to expand $\delta_1 m_r^2 = \delta m_1^2 + \delta m_2^2 + \ldots$ with δm_i^2 of ith order in λ_r, so that the result for Π_i determines δm_i^2 exactly. For (2.21) the divergences are only of first order in λ_r and we therefore only need the counterterm to first order given diagrammatically as



$= -\delta m_1^2,$

where we have fixed the Feynman rules for counterterms similar to the one of the previous section. The only difference is the treatment of the symmetry factor that – since there is a 2 instead of a 4! in the denominator – now becomes 1 for counterterm vertices. Diagrammatically, the renormalized self-energy to first order is then given as the sum of figure (2.1) and the counterterm,

$\Pi_1^{ren} = \frac{1}{2}$  $+$ 

where for the contribution from the tadpole diagram we can use (2.21), replacing bare by renormalized quantities. We now have to choose δm_1^2 in such a way that Π_1^{ren} is not divergent anymore. However, the choice of δm_1^2 is not unique since besides the divergent term (which is fixed by our method of regularization) it may contain finite terms as well. To avoid ambiguities in our result we therefore have

[1]Note that in general it is not guaranteed that a given theory is renormalizable at all; however, by restricting our Lagrangian to be of the form (2.1) we already have chosen a theory that is known to be renormalizable [1].

to choose a renormalization scheme at this point and stick to it throughout the rest of our calculations. We will opt for the scheme known as minimal subtraction (MS), which – as the name already suggests – is defined by subtracting only poles. From (2.21) and the above identity we therefore find

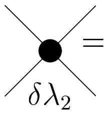
$$\delta m_1^2 = \frac{\lambda_r''}{2} m_r^2 \frac{1}{\varepsilon},$$

obtaining the renormalized, manifestly finite result

$$\Pi_1^{ren} = \frac{\lambda_r''}{2} m_r^2 \left(1 - \ln \hat{m_r}^2\right) - 4\lambda_r'' \mu^{2\varepsilon} T_1(m_r). \tag{4.2}$$

4.2 Renormalizing the 4-point Green function

In order to obtain a finite result for the self-energy to second order we first have to determine the coupling-constant counterterm to second order (there is no first order contribution since $\delta_2 \sim O(\lambda_r)$ and consequently $\delta_2\lambda_r \sim O(\lambda_r^2)$). Expressing $\delta_2\lambda_r = \delta\lambda_2 + \delta\lambda_3 + \dots$ similar to the mass counterterm, the second order contribution is then given diagrammatically as



$= -\delta\lambda_2.$

$\delta\lambda_2$

To determine $\delta\lambda_2$ we have to calculate divergent part of the 4-point Green function to second order[2]. The only contribution to the 4-point function to second order generated by the "classical" part of $\mathcal{L}_{ren}$ is the so-called "fish" diagram (figure 4.1) which in algebraic form is given by

$$\Gamma_{2,A}^{(4)} = \lambda_r^2 S_4 \sum\!\!\!\!\!\!\!\int_P \sum\!\!\!\!\!\!\!\int_Q (2\pi)^3 \delta^3(\mathbf{p}+\mathbf{q}+\mathbf{k}) \beta\delta(\omega_p + \omega_q + \omega_k) D_0(Q) D_0(P)$$

with the symmetry factor S_4 determined to be $S_4 = \frac{1}{2(4!)^2}(8.4.6.3.2 + 8.3.4.3.2) = \frac{3}{2}$.

Using (3.2) to do the thermal sums (3.3) we can write

$$\Gamma_{2,A}^{(4)} = \frac{3\lambda_r^2}{2} \int \frac{d^3p d^3q}{(2\pi)^3} \delta^3(\mathbf{p}+\mathbf{q}+\mathbf{k}) \left(W_0 + \frac{2n(\varepsilon_p)}{2\varepsilon_p} W_1\right)$$

with

$$\begin{aligned} W_0 &= \frac{1}{2\varepsilon_p 2\varepsilon_q} \sum_{\kappa=\pm} \frac{1}{\varepsilon_p + \varepsilon_q + i\kappa\omega_k} \\ W_1 &= \frac{1}{2\varepsilon_q} \sum_{\sigma,\kappa=\pm} \frac{1}{\sigma\varepsilon_p + \varepsilon_q + i\kappa\omega_k} \end{aligned}$$

[2]To first order, the 4-point Green function is trivially finite and therefore does not have to be renormalized.

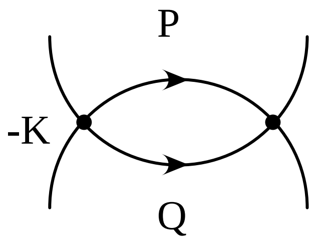


Figure 4.1: The fish diagram

and where we have used the symmetry $p \leftrightarrow q$ in the integrand. From the previous chapter we know how to rewrite W_0 as an integral over Euclidean momenta; doing the trivial sum over σ for W_1 we have

$$\begin{aligned} W_0 &= \frac{1}{2\pi}\int dp_4 dq_4 \frac{1}{P^2+m_r^2}\frac{1}{Q^2+m_r^2}\delta(p_4+q_4+\omega_k) \\ W_1 &= \sum_{\kappa=\pm}\frac{1}{\varepsilon_q^2-(i\omega_k+\kappa\varepsilon_p)^2}. \end{aligned}$$

As already anticipated earlier, the fish diagram can be written as a vacuum term and a term with one thermal factor

$$\begin{aligned} \Gamma_{2,A}^{(4)} &= \frac{3\lambda_r^2}{2}\int\frac{d^4P}{(2\pi)^4}\frac{1}{P^2+m_r^2}\frac{1}{(P+K)^2+m_r^2}+ \\ &\quad +\frac{3\lambda^2}{2}\int\frac{d^3p}{(2\pi)^3}\frac{2n(\varepsilon_p)}{2\varepsilon_p}\sum_\kappa\frac{1}{m_r^2+|\mathbf{p}+\mathbf{k}|^2-(i\omega_k+\kappa\varepsilon_p)^2}. \end{aligned}$$

Introducing a Feynman parameter and regularizing the integral using dimensional regularization as in the previous chapter, the real part of the first term becomes

$$\frac{3\lambda_r^2}{2}\int\frac{d^{4-2\varepsilon}P}{(2\pi)^{4-2\varepsilon}}\frac{1}{P^2+m_r^2}\frac{1}{(P+K)^2+m_r^2}=\frac{3}{2}\frac{\lambda_r'^2}{(4\pi)^2}\mu^{2\varepsilon}\left(\frac{1}{\varepsilon}+2-\ln\hat{m}_r{}^2-f_2\left(\frac{4m_r^2}{K^2}\right)\right).$$

Once again the analytic continuation is trivial since there are no branch points in the complex k_0 plane. Defining $E_\kappa=k_0+\kappa\varepsilon_p$ and substituting $|\mathbf{p}+\mathbf{k}|=u$ we can do the angular integrals

$$\begin{aligned} \int_{-1}^{+1} d(\cos\theta)\,d\phi\frac{1}{m_r^2+|\mathbf{p}+\mathbf{k}|^2-E_\kappa^2} &= \frac{1}{pk}\int_{|p-k|}^{p+k} du\frac{u}{m_r^2-E_\kappa^2+u^2} \\ &= \frac{1}{2pk}\ln\frac{|m_r^2-E_\kappa^2+(p+k)^2|}{|m_r^2-E_\kappa^2+(p-k)^2|} \end{aligned}$$

and find the following result for $\Gamma^{(4)}_{2,A}$

$$\begin{aligned}\Gamma^{(4)}_{2,A} &= \frac{3\lambda'^2_r}{32\pi^2}\mu^{2\varepsilon}\left(\frac{1}{\varepsilon}+2-\ln\hat{m}_r{}^2 - f_2\left(\frac{4m_r^2}{K^2}\right)\right)\\ &\quad +6\lambda''^2_r\int_0^\infty p^2dp\frac{2n(\varepsilon_p)}{2\varepsilon_p}\sum_\kappa\frac{1}{2pk}\ln\frac{|m_r^2-E_\kappa^2+(p+k)^2|}{|m_r^2-E_\kappa^2+(p-k)^2|}.\end{aligned}$$

The full contribution to the 4-point Green function to second order is then given by

$$\Gamma^{(4)}_2 = \Gamma^{(4)}_{2,A} - \delta\lambda_2$$

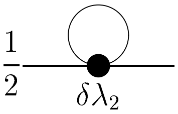
so that – conform to the MS scheme – we have to choose

$$\delta\lambda_2 = \frac{24\mu^{2\varepsilon}\lambda''^2_r\pi^2}{\varepsilon}$$

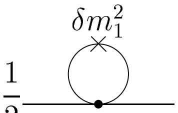
to make $\Gamma^{(4)}_2$ finite.

4.3 Renormalization of Π_2

We now have all the ingredients to recalculate the self-energy to second order in terms of the renormalized quantities. Besides the double-scoop and the sunset diagram generated from the "classical" part of $\mathcal{L}_{ren}$ we also get contributions from the counterterms. First note that coupling-constant counterterm to second order generates the diagram

$\frac{1}{2}$  $= -\frac{1}{2}\delta\lambda_2 \sum\!\!\!\!\!\!\!\int_P D_0(P)$

that contributes to the self-energy Π_2. Furthermore, we find that also the mass counterterm of first order that we determined in the first section of this chapter contributes

$\frac{1}{2}$  $= \frac{-\delta m_1^2}{2}(-\lambda)\sum\!\!\!\!\!\!\!\int_P D_0^2(P)$

to Π_2. Fortunately we already have calculated the relevant thermal sums in the previous chapters so that with the results from appendix B we obtain

$$\begin{aligned}-\frac{1}{2}\delta\lambda_2\sum\!\!\!\!\!\!\!\int_P D_0(P) &= \frac{3}{4}\lambda''^2_r\left[\frac{m_r^2}{\varepsilon^2}+\frac{1}{\varepsilon}\left(m_r^2\left(1-\ln\hat{m}_r{}^2\right)-8\mu^{2\varepsilon}T_1(m_r)\right)\right.\\ &\quad\left.+m_r^2\left(1+\frac{\pi^2}{12}-\ln\hat{m}_r{}^2+\frac{1}{2}\ln^2\hat{m}_r{}^2\right)\right] \qquad (4.3)\end{aligned}$$

and

$$\frac{-\delta m_1}{2}(-\lambda)\sum\!\!\!\!\!\!\int_p D_0^2(P) = \frac{m_r^2\lambda_r''^2}{4}\left[\frac{1}{\varepsilon^2}+\frac{1}{\varepsilon}\left(-\ln\hat{m}_r{}^2 - 8\mu^{2\varepsilon}\partial_{m_r^2}T_1(m_r)\right) + \frac{\pi^2}{12}+\frac{1}{2}\ln^2\hat{m}_r{}^2\right]. \tag{4.4}$$

If we now add these diagrams contributing to the self-energy to second order, that is (3.1), (3.13) and the two diagrams above coming from the counterterms we end up with the result

$$\begin{aligned}
&\lambda_r''^2\left[\frac{m_r^2}{2\varepsilon^2}-\frac{m_r^2}{4\varepsilon}+\frac{K_M^2}{24\varepsilon}+\frac{m_r^2}{4}\left(2\ln^2\hat{m}_r{}^2+5\ln\hat{m}_r{}^2-5\right)\right.\\
&+\frac{K_M^2}{6}\left(\frac{11}{8}-\frac{1}{2}\ln\hat{m}_r{}^2\right)+\frac{m_r^2}{6}I_C(-K_M^2/m_r^2)-(6\ln\hat{m}_r{}^2-8)\mu^{2\varepsilon}T_1(m_r)\\
&-\mu^{2\varepsilon}T_3(K_M)+\mu^{4\varepsilon}T_4(K_M)\\
&\left.-16\mu^{4\varepsilon}T_1(m_r)\partial_{m_r^2}T_1(m_r)-2\mu^{2\varepsilon}m_r^2(1-\ln\hat{m}_r{}^2)\partial_{m_r^2}T_1(m_r)\right],
\end{aligned}$$

where

$$\begin{aligned}
T_3(K_M) &= \frac{1}{k}\int_0^\infty pdp\frac{n(\varepsilon_p)}{\varepsilon_p}\sum_\kappa\int_{|p-k|}^{p+k}udu f_2\left(\frac{4m_r^2}{-E_\kappa^2+u^2}\right)\\
T_4(K_M) &= \frac{1}{k}\int_0^\infty pdp\quad qdq\frac{n(\varepsilon_p)n(\varepsilon_q)}{\varepsilon_p\varepsilon_q}f_4(p,q,K_M).
\end{aligned}$$

Note that the divergences involving thermal factors have all cancelled.

Now we add the final contributions coming from the counterterms; by choosing δm_2 accordingly, the terms $\frac{\lambda''^2m^2}{2\varepsilon^2}$ and $-\frac{\lambda''^2m^2}{4\varepsilon}$ are removed. To get rid of the divergent term $\frac{\lambda''^2K_M^2}{24\varepsilon}$, however, we have to use $\mathcal{L}_\phi$. Treating this term as an interaction we have to go through eq. (2.17) and (2.19) to find that it contributes

$$\overset{-\delta\phi_2}{\longrightarrow\!\!\!\!\!\!\!\times\!\!\!\longrightarrow} = \delta\phi_2K^2.$$

After doing the analytic continuation $K^2\to -K_M^2$ we have to choose $\delta\phi = \frac{\lambda''^2}{24\varepsilon}$ to cancel the last divergence of the self-energy. We therefore obtain the fully renormalized self-energy to second order in λ as

$$\begin{aligned}
\Pi_2^{ren} = \lambda_r''^2&\left[\frac{m_r^2}{4}\left(2\ln^2\hat{m}_r{}^2+5\ln\hat{m}_r{}^2-5\right)+\frac{K_M^2}{6}\left(\frac{11}{8}-\frac{1}{2}\ln\hat{m}_r{}^2\right)\right.\\
&+\frac{m_r^2}{6}I_C(-K_M^2/m_r^2)-(6\ln\hat{m}_r{}^2-8)\mu^{2\varepsilon}T_1(m_r)\\
&-\mu^{2\varepsilon}T_3(K_M)+\mu^{4\varepsilon}T_4(K_M)\\
&\left.-16\mu^{4\varepsilon}T_1(m_r)\partial_{m_r^2}T_1(m_r)-2\mu^{2\varepsilon}m_r^2(1-\ln\hat{m}_r{}^2)\partial_{m_r^2}T_1(m_r)\right].
\end{aligned}\tag{4.5}$$

4.4 Summary

In this chapter we introduced renormalization and reorganized the original Lagrangian in terms of the renormalized quantities, which gives rise to three counterterms which we called wave-function-, mass- and coupling-constant counterterms. We found that the mass counterterm to first order could be chosen in such a way as to exactly cancel the UV-divergences of the tadpole diagram, where we resolved the resulting ambiguities of the finite terms by sticking to the particular renormalization scheme MS. Thereby we obtained a manifestly finite result for the self-energy to first order. We then calculated the "fish-diagram" to determine the relevant coupling-constant counterterm, added all loop diagrams that contribute to the self-energy to second order, and finally obtained a finite result for the self-energy to second order by choosing the wave-function and mass counterterms accordingly to second order.

Chapter 5

Numerical results

In order to use the results we obtained in the previous chapters for the self energy we have to evaluate the behaviour of the remaining integrals in (4.5). Unfortunately, extraction of information on their form by analytic means is prohibitively difficult[1] and in order to analyze the behaviour of $\Pi^{ren}(K_M) = \Pi_1^{ren} + \Pi_2^{ren}(K_M)$ for *all* values of external Minkowski momenta K_M we therefore have to resort to numerical integration. We first will concentrate on the only non-trivial contribution to the vacuum part of the sunset diagram and then treat the thermal integrals.

For convenience, and to make results easier to read, we have dropped the subscript r of the renormalized quantities in the following.

5.1 The integral $I_C(-K_M^2/m^2)$

To obtain the real part of the contribution λJ_0^{fin} for all values of K_M^2 we have to calculate the integral $I_C(-K_M^2/m^2 = s)$ defined in (C.7). The numerical evaluation is straightforward and the result is plotted in figure 5.1 [19]. The change in the slope occurs for $s = -9$ which – as discussed in appendix C – corresponds to the threshold of real particle production. For $s < -9$ the integrand behaves in a rather non-trivial way which leads to loss of numerical precision; however, the absolute numerical error stays smaller than 10^{-3} [19].

To verify that no systematic error entered in our calculation we compared our result numerically to the one obtained in [33] which is valid for $K_M^2 < 0$. Since [33] chose a different regularization procedure (which amounted to neglecting all terms of order K_M^0 and K_M^2) we subtracted these contributions numerically from $I_C(s)$ to obtain $I'_C(s)$. Plotting $I'_C(s)$ and the result from [33] we obtain figure 5.2 where the two results are indistinguishable to the naked eye; therefore, we have plotted the relative error as a function of k in figure 5.3.

[1]For special cases, e.g. $\mathbf{k} = 0$ or $k_0 = 0$ analytic results for the thermal integrals as a power series in $m\beta$ are possible and indeed have been calculated for most thermal integrals.

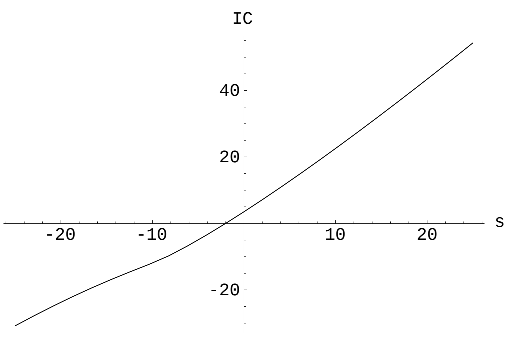


Figure 5.1: The integral $I_C(s)$ where $s = -K_M^2/m^2$.

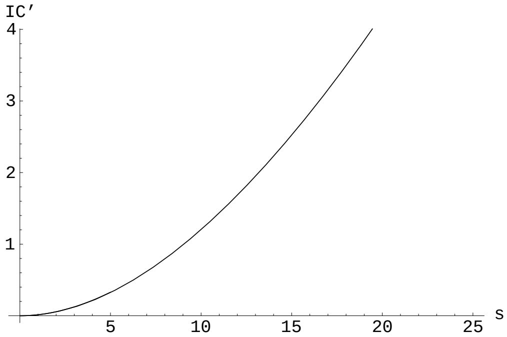


Figure 5.2: Comparison of our result to that of [33]: only the contributions to $I_C(s)$ of second or higher order in $s = -K_M^2/m^2$ (denoted I_C') are plotted.

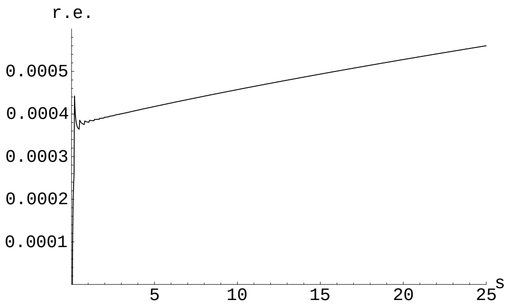


Figure 5.3: The relative error of our result compared to [33].

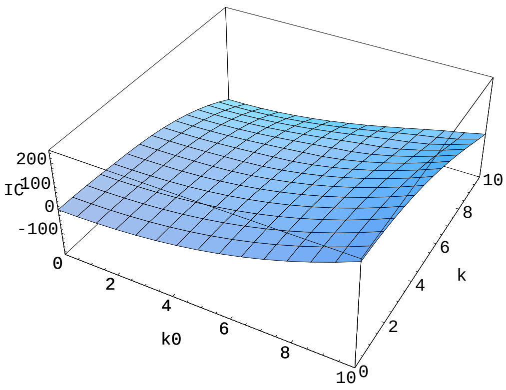


Figure 5.4: A 3-dimensional plot of $I_C(-K_M^2/m^2)$ as a function of $\tilde{k}_0$ and $\tilde{k}$ for $\tilde{m} = 1$.

Eventually we shall be interested in the value of Π^{ren} on mass shell; the corresponding value of $I_C(-1)$ is easily calculated analytically to be $\frac{7}{4}$.

Finally, in figure 5.4 we show a 3-dimensional plot of I_C as a function of k_0 and k.

5.2 The integral $T_1(m)$

We now turn our attention towards the thermal integrals. For a start, we will calculate the T_1 integral that we encountered in the second chapter, defined by

$$T_1(m) = \int_0^\infty dp \frac{p^2}{\varepsilon_p} n(\varepsilon_p).$$

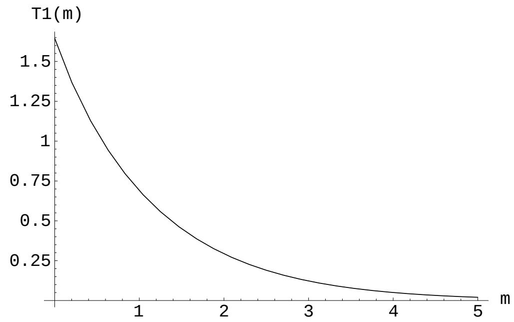


Figure 5.5: The integral $T_1(m)/T^2$ as a function of $\tilde{m}$.

By making the substitution $p = u\,T$ we can extract a leading factor T^2 so that

$$T_1(m) = T^2 \int_0^\infty du \frac{u^2}{\tilde{\varepsilon_u}} \tilde{n}(\tilde{\varepsilon_u}) \tag{5.1}$$

where

$$\begin{aligned} \tilde{\varepsilon_u} &= \sqrt{u^2 + \tilde{m}^2} \\ \tilde{m} &= \beta m \\ \tilde{n}(x) &= n(Tx) = \frac{1}{e^x - 1}. \end{aligned} \tag{5.2}$$

With this substitution $T_1(m)$ has become T^2 times a dimensionless function of $\tilde{m}$. The form of this function can easily be derived by numerical integration and the result is shown in figure 5.5. Note that the value for $\tilde{m} = 0$ is given by $\pi^2/6$, a result that we shall derive analytically in the next chapter.

5.3 The integral $T_3(K_M)$

The next thermal integral we want to evaluate is $T_3(K_M)$, given by

$$T_3(K_M) = \frac{1}{k} \int_0^\infty pdp \frac{n(\varepsilon_p)}{\varepsilon_p} \sum_\kappa \int_{|p-k|}^{p+k} udu f_2(a_1 = \frac{4m^2}{-E_\kappa^2 + u^2})$$

with $f_2(a)$ defined in (C.1) and a_1 in (3.12). We again do the substitution $p = \tilde{p}\,T$, $u = \tilde{u}\ T$ to extract the dimensionful leading factor of T^2 and then do the numerical integrations. Due to two reasons these computations are complicated and need much more time than e.g. $T_1(m)$ to evaluate. The first reason is that $T_3(K_M)$ contains two numerical integrations with an integrand that possesses logarithmic singularities (e.g. the ones hidden away in $f_2(a_1)$) that have to be handled with care numerically. Secondly, since $T_3(K_M)$ depends on the external Minkowski momentum as well as on m these logarithmic singularities are in

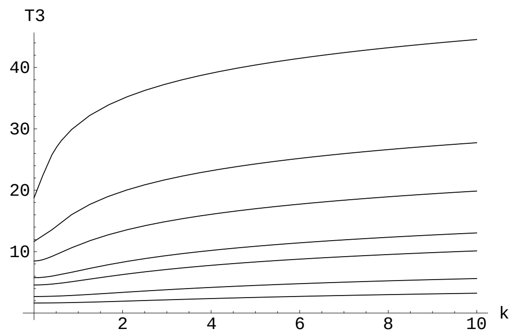


Figure 5.6: The integral $T_3(K_M)/T^2$ plotted for on-shell K_M as a function of $\tilde{k}$. The different curves correspond to $\tilde{m} = 0.1, 0.3, 0.5, 0.8, 1, 1.5$ and 2, in descending order.

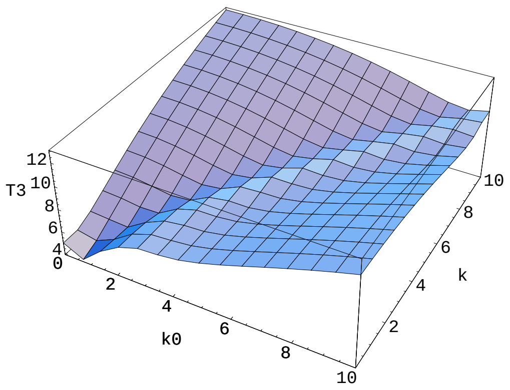


Figure 5.7: A 3-dimensional plot of the integral $T_3(K_M)/T^2$ for $\tilde{m} = 1$ as a function of $\tilde{k}_0$ and $\tilde{k}$.

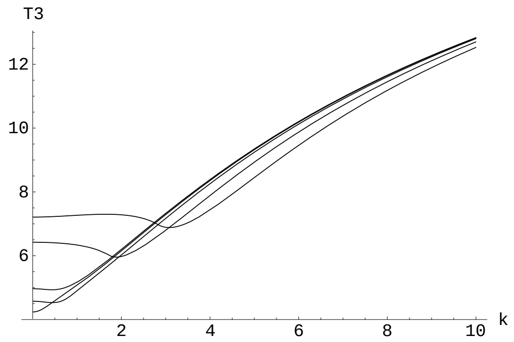


Figure 5.8: The integral $T_3(K_M)/T^2$ plotted for $\tilde{m} = 1$ as a function of $\tilde{k}$. The different curves correspond to $k_0 = 3, 2, 1, 0.5$ and 0.1, in descending order at $k = 0$.

general complicated functions of K_M and m and therefore not easily identified. Nevertheless the results obtained [19] seem to be correct to high precision in the case of $T_3(K_M)$ [48].

As can be seen from figure 5.6, the momentum dependence of $T_3(k_0 = \sqrt{m^2 + k^2}, k)$ for $\tilde{k} > 1$ seems to be close to $\tilde{k}^\tau$ with τ some positive fraction of 1.

5.4 The integral $T_4(K_M)$

The only integral we have not evaluated so far is $T_4(K_M)$ defined as

$$T_4(K_M) = \frac{1}{k} \int_0^\infty pdp \int_0^\infty qdq \frac{n(\varepsilon_p)n(\varepsilon_q)}{\varepsilon_p \varepsilon_q} f_4(p, q, K_M), \tag{5.3}$$

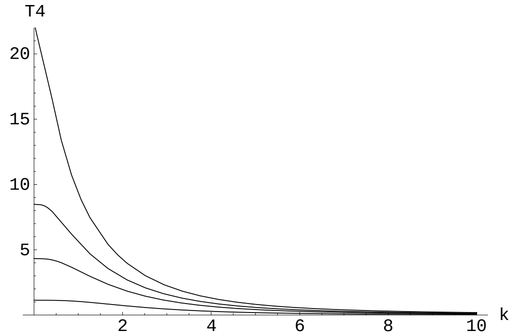


Figure 5.9: The integral $T_4(K_M)/T^2$ for on-shell K_M as a function of $\tilde{k}$. The different curves correspond to $\tilde{m} = 0.1, 0.3, 0.5$ and 1 in descending order.

with

$$f_4(p,q,K_M) = \sum_{\sigma\tau} \big(f_3(E_{\sigma\tau}, p+k+q) + f_3(E_{\sigma\tau}, p-k-q) \\ -f_3(E_{\sigma\tau}, p+|k-q|) - f_3(E_{\sigma\tau}, p-|k-q|)\big)$$

and f_3 defined in (C.18). After extracting the leading factor of T^2 we could do the numerical integrations using [19]. However, for $T_4(K_M)$ this seems not to be the best idea, given that the evaluation of figure 5.9 took about a week on a modern desktop computer. The reason for this seems again to be the logarithmic singularities of the integrand; however, these singularities are found to be curves in the k_0-k plane that can be calculated numerically. By taking advantage of this fact one can construct a computer program that does the calculation in a fraction of the time, arriving at the same results as shown in figure 5.9 [49].

Note that in contrast to the results for T_3, $T_4(k_0 = m, k)$ seems to be exponentially decreasing with growing k.

5.4.1 Momentum dependence of Π_2^{ren}

Since the only 3-momentum dependent contributions to Π_2^{ren} is the difference between the integrals T_3 and T_4 we have plotted this quantity in figure 5.10. Not surprisingly, the power law behaviour of T_3 wins out over the exponentially suppressed T_4 for large $\tilde{k}$. The small $\tilde{k}$ behaviour seems to be more complicated however.

5.5 Summary

In this chapter we did a numerical evaluation of the integrals in (4.5) that we have not been able to solve analytically. We found that we could do all the integrals

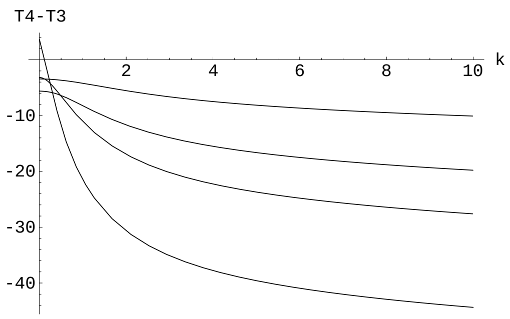


Figure 5.10: The difference $(T_4 - T_3)/T^2$ as a function of $\tilde{k}$ for on-shell K_M and $\tilde{m} = 1, 0.5, 0.3, 0.1$ in descending order for large $\tilde{k}$.

using [19], but the momentum dependent contributions took rather a long time to be evaluated. We finally plotted the 3-momentum dependence for Π_2^{ren} which seems to indicate that it grows like $\tilde{k}$ to some positive power for large $\tilde{k}$. S

Chapter 6

Analytic evaluation of thermal integrals

In the previous chapter we have obtained some information on the non-trivial constituents of the self-energy to second loop order by numerical means. However, it would still be of value to have an analytic handle on those integrals, even if only for special cases. There are in fact some special cases that – besides being tractable which already makes them interesting in themselves – are valuable as parts of the computation of physical objects. For example, the case $k_0 = 0$, $k^2 \to -m^2$ will contribute to the screening mass [38] and $k = 0$, $k_0^2 = m^2$ to the thermal or plasmon mass [32] (the terminology will become a little less mysterious in the next chapter). In the following we shall concentrate on the latter case and try to obtain an analytic expression for the contributing integrals as a high temperature expansion series (in powers of βm).

6.1 The integral $T_1(m)$

We begin with the simplest case, the integral $T_1(m)$. Substituting $p = u\, T$ in order to render the integration variable dimensionless extracts a factor T^2 and leaves us with eq.(5.1). Assuming $\tilde{m}$ small (we will specify exactly what is meant by small later on) we are interested in an expansion of $T_1(m)$ in powers of[1] $\tilde{m}$. Naively, one would expect that a simple Taylor expansion of the integrand would be sufficient to yield a correct result. However, it turns out that this only works for the lowest order (and – as we go on to tackle the more complicated integral T_4 – not even there). The problem is that since the integrand is a function of $\tilde{m}^2$ there is no way of making the Taylor expansion aware of the possibility that uneven powers or logarithms of $\tilde{m}$ might exists (the reader is invited to verify this!). If they do, however, the coefficients of the naive expansion in powers of $\tilde{m}^2$ turn out to be divergent as a consequence.

[1] Quantities with a tilde are defined in (5.2).

For the simple case of $T_1(m)$ there is a relatively easy fix to obtain the next order in the expansion by regularizing the divergent coefficient integral and extract a term linear in $\tilde{m}$ [30]. We will, however, adopt a much more elegant method due to [31] in order to find the power series expansion of $T_1(m)$.

Using the series expansion (B.6) for the Bose-Einstein factor in (5.1) we obtain for $T_1(m)$

$$T_1(m) = T^2 \sum_{k=1}^{\infty} \int_0^{\infty} u^2 du \frac{\exp\left[-k\tilde{\varepsilon}_u\right]}{\tilde{\varepsilon}_u}.$$

Substituting $1 + \left(\frac{u}{\tilde{m}}\right)^2 = v^2$ we get

$$T_1(m) = T^2 \sum_{k=1}^{\infty} \int_1^{\infty} dv \tilde{m}^2 \sqrt{v^2 - 1} e^{-k\tilde{m}v}$$

which – using the integral representation of the modified Bessel function of the second kind [13] 7.12 (19) –

$$K_n(x) = \frac{\pi^{1/2}}{\Gamma(1/2+n)} \left(\frac{x}{2}\right)^n \int_1^{\infty} dt e^{-xt} (t^2-1)^{n-1/2} \tag{6.1}$$

can be written as

$$T_1(m) = T^2 \sum_{k=1}^{\infty} \frac{\tilde{m}}{k} K_1\left(k\tilde{m}\right). \tag{6.2}$$

Low temperature expansion Having managed to write $T_1(m)$ as a sum over modified Bessel functions immediately provides us with the low temperature (large $\tilde{m}$) behaviour by using the known asymptotic form of K_1 [12] 7.13.1 (7),

$$K_1(x) \simeq \sqrt{\frac{\pi}{2x}} e^{-x}, \; x \gg 1.$$

The low temperature expansion of $T_1(m)$ is therefore given by

$$T_1(m) = T^2 \sqrt{\frac{\pi\tilde{m}}{2}} \mathrm{Li}_{3/2}(e^{-\tilde{m}}) \tag{6.3}$$

where we have introduced a "$\frac{3}{2}$-logarithm" function $\mathrm{Li}_{3/2}$ defined as

$$\mathrm{Li}_{3/2}(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^{3/2}} \tag{6.4}$$

similar to the definition of di- and trilogarithm functions in [16].

High temperature expansion For small $\tilde{m}$ (high temperature) this approach obviously does not work. However, to extract the high temperature expansion of a sum of the form (6.2) we can make use of a powerful identity that involves Mellin integral transformations. The Mellin transform $\mathcal{M}[f(x)](s)$ of a function $f(x)$ and its inverse, respectively, are defined [10] as

$$\begin{aligned}\mathcal{M}\left[f(x)\right](s) &= \int_0^\infty dx f(x) x^{s-1} \\ f(x) &= \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} x^{-s} \mathcal{M}\left[f(x)\right](s).\end{aligned}$$

The Mellin transform is analytic in a strip $a < \text{Re } s < b$ with a, b and c real numbers and c in the interval $[a, b]$. Using the definition of Riemann's zeta function $\zeta(s)$ (A.5) and acting with the operation $\int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} x^{\nu-s} \mathcal{M}\left[f(x)\right](s-\nu)$ on this definition we obtain [25]

$$\sum_{n=1}^{\infty} n^{-\nu} f(nx) = x^{\nu} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} x^{-s} \zeta(s) \mathcal{M}\left[f(x)\right](s-\nu) \tag{6.5}$$

with $\max(1, a+\nu) < c < b+\nu$ if $\mathcal{M}[f](s)$ is analytic on $a < s < b$. This result will prove to be a powerful tool to systematically extract the high temperature expansion coefficients from sums of the type found in (6.2). Applying (6.5) to (6.2) we find

$$T_1(m) = T^2 \tilde{m} \left(\tilde{m} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \tilde{m}^{-s} \zeta(s) \mathcal{M}[K_1(x)](s-1) \right). \tag{6.6}$$

The Mellin transform of $K_\nu(ax)$ is given [11] 6.8 (26) as

$$\mathcal{M}\left[K_\nu(ax)\right](s) = a^{-s} 2^{s-2} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu}{2}\right) \tag{6.7}$$

and is analytic for $\text{Re } s > |\text{Re } \nu|$. Therefore c in (6.6) must fulfill $c > 2$ and $T_1(m)$ becomes

$$T_1(m) = T^2 \tilde{m}^2 \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \tilde{m}^{-s} \zeta(s) 2^{s-3} \Gamma\left(\frac{s}{2} - 1\right) \Gamma\left(\frac{s}{2}\right). \tag{6.8}$$

Using the asymptotic behaviour of $\Gamma(x)$ and $\zeta(x)$ in appendix A we find that the integrand approaches

$$\left(\frac{\tilde{m}}{2\pi}\right)^{-s} \left(\frac{|s|\pi}{2}\right)^{3/2} \cos\frac{-\pi s}{2}$$

for $s \to -\infty$; the integral in (6.8) may therefore be evaluated by closing the contour to the left if $\tilde{m} < 2\pi$. On the one hand, this is a little surprising since the requirement "small" as used above then stands for $\tilde{m} < 2\pi$ instead of having the more usual meaning of $\tilde{m} \ll 1$. On the other hand however, note that at $m = 2\pi T$ the mass is equal to the only dimensionful scale of the theory, that of the first Matsubara frequency ω_1. The breakdown of the high temperature expansion therefore does not occur at some random number but precisely at the point when the mass touches this scale inherent to the theory.

Rewriting the integrand using $x\Gamma(x) = \Gamma(x+1)$ we find that it has a simple pole at $s = 2$, a simple pole at $s = 1$ coming from the ζ-function and finally double poles at $s = -2n,\ n = 0, 1, \ldots$ For the simple poles the residues are given by

$$\mathrm{Res}_{s=2}\left[\left(\frac{\tilde{m}}{2}\right)^{-s}\zeta(s)\frac{\Gamma^2(s/2)}{s/2-1}\right] = 2\left(\frac{\tilde{m}}{2}\right)^{-2}\frac{\pi^2}{6}$$

$$\mathrm{Res}_{s=1}\left[\left(\frac{\tilde{m}}{2}\right)^{-s}\zeta(s)\frac{\Gamma^2(s/2)}{s/2-1}\right] = \left(\frac{\tilde{m}}{2}\right)^{-1}1\frac{\pi(-2)}{1}$$

whereas for the double poles located at $s = -2n$, $n = 0, 1, \ldots$ we obtain

$$\begin{aligned}
\mathrm{Res}_{s=-2n}\left[\left(\frac{\tilde{m}}{2}\right)^{-s}\zeta(s)\frac{2\Gamma^2(\frac{s}{2})}{s-2}\right] &= -2\left(\frac{\tilde{m}}{2}\right)^{-s}\ln\frac{\tilde{m}}{2}\zeta(s)\frac{((s+2n)\Gamma(s/2))^2}{s-2} + \\
&\quad +2\left(\frac{\tilde{m}}{2}\right)^{-s}\zeta'(s)\frac{((s+2n)\Gamma(s/2))^2}{s-2} + \\
&\quad -2\left(\frac{\tilde{m}}{2}\right)^{-s}\zeta(s)\frac{((s+2n)\Gamma(s/2))^2}{(s-2)^2} + \\
&\quad + 4\left(\frac{\tilde{m}}{2}\right)^{-s}\zeta(s)\frac{(s+2n)\Gamma(\frac{s}{2})}{s-2}\left((s+2n)\Gamma(\frac{s}{2})\right)'\Bigg|_{s=-2n} .
\end{aligned}$$

Since $\zeta(-2n) = 0$ for all nonzero n the first term is non-vanishing only for $n = 0$ where it becomes $-2\ln\tilde{m}/2$. The second term is easily evaluated to be $2\ln 2\pi$ for $n = 0$ and

$$-\left(\frac{\tilde{m}}{2}\right)^{2n}\frac{\Gamma(2n+1)}{(2\pi)^{2n}}(-1)^n\zeta(2n+1)\frac{2}{(n!)^2(n+1)}$$

for $n \neq 0$ using the identities found in the appendix. With $\zeta(0) = -\frac{1}{2}$ the third and fourth terms are found to be $+1$ and $\frac{1}{2}4(-\gamma)$ for $n = 0$, respectively, and again vanish for $n \neq 0$ because of the zeta function. Collecting these result we find the following high temperature expansion for $T_1(m)$:

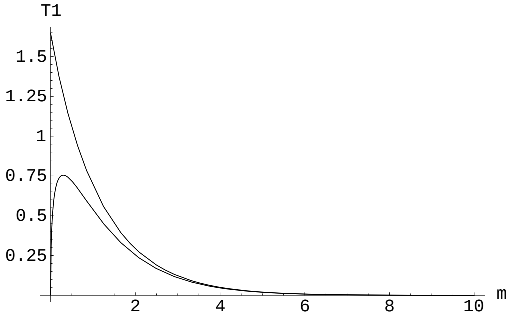


Figure 6.1: Comparing the low temperature expansion (6.3) to the exact numerical result: plotting $T_1(m)/T^2$ as a function of $\tilde{m}$ the expansion is valid for large values of $\tilde{m}$.

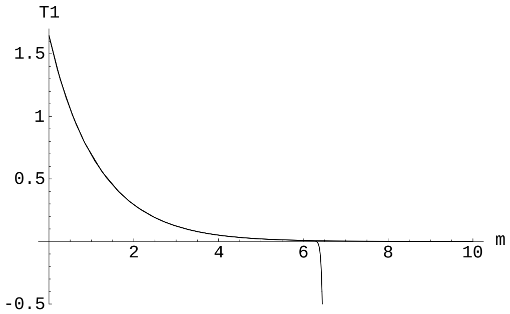


Figure 6.2: Comparing the high temperature expansion (6.9) to the exact numerical result: $T_1(m)/T^2$ is plotted against (6.9) truncated after the first 100 terms.

$$\begin{aligned} T_1(m) = T^2 \Bigg[& \frac{\pi^2}{6} - \frac{\tilde{m}}{2}\pi - \frac{1}{4}\tilde{m}^2 \left(\ln \frac{\tilde{m}}{4\pi} - \frac{1}{2} + \gamma \right) \\ & - \frac{1}{4}\tilde{m}^2 \sum_{n=1}^{\infty} \left(\frac{\tilde{m}}{4\pi} \right)^{2n} (-1)^n \frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+2)} \zeta(2n+1) \Bigg] . \end{aligned} \tag{6.9}$$

It is finally instructive to compare the low and high temperature expansions to the result obtained by numerical integration as done in figures 6.1 and 6.2. The high temperature expansion (6.9) breaks down completely for values of $\tilde{m}$ larger than 2π, as anticipated from our calculation. The relative errors with respect to the exact result are plotted in (6.4) for (6.9); when truncating (6.9) after the term linear in $\tilde{m}$ the relative error becomes large already for $\tilde{m} \ll 1$. For low temperatures the relative error of the analytic result (6.3) is shown in figure 6.3; since for $\tilde{m} = 2\pi$ it becomes $0.05496\ldots$ the low temperature expansion is already a rather good approximation to the exact result when the high temperature expansion breaks down.

6.2 The integral $T_3(K_M)$

The next thermal integral we want to treat is $T_3(K_M)$ defined as

$$T_3(K_M) = T^2 \int_0^\infty v dv \frac{\tilde{n}(\tilde{\varepsilon}_v)}{\tilde{\varepsilon}_v} \sum_{\kappa=\pm} \frac{1}{\tilde{k}} \int_{|v-\tilde{k}|}^{v+\tilde{k}} w dw f_2\left(\frac{4\tilde{m}^2}{-\tilde{E}_\kappa^2 + w^2}\right), \tag{6.10}$$

where we have extracted the temperature dependence by substituting $p = T\ v$, $u = T\ w$ and introduced $\tilde{k} = \beta k$, $\tilde{E}_\kappa = \beta k_0 - \kappa \tilde{\varepsilon}_v$. As before, for general K_M we

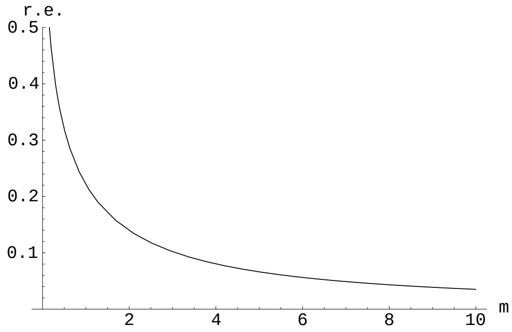


Figure 6.3: The relative error of the low temperature expansion with respect to the exact result.

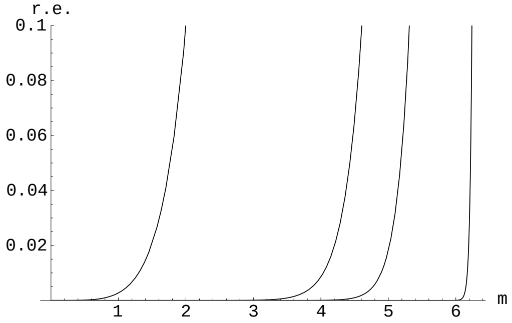


Figure 6.4: The relative error of the high temperature expansion to order $\tilde{m}^2$ and with the first $5, 10$ and 100 terms of the series in (6.9) with respect to the exact result.

are unable to do the integrals and – as anticipated – have to restrict ourselves to a special case for which the calculation becomes feasible.

6.2.1 Solution for a special case

The special case we want to treat is K_M on mass-shell with vanishing external 3-momentum $k = 0$, implying $k_0^2 = m^2$. From (6.10) it immediately becomes clear that for vanishing k the factor $1/\tilde{k}$ cancels the second integral and (6.10) becomes

$$T_3(m,0) = 2T^2 \int_0^\infty v^2 dv \frac{\tilde{n}(\tilde{\varepsilon}_v)}{\tilde{\varepsilon}_v} \sum_\kappa f_2 \left(\frac{4\tilde{m}^2}{-(m+\kappa\tilde{\varepsilon}_v)^2 + v^2} \right),$$

where $T_3(m,0)$ denotes the special case $T_3(k_0 = m, k = 0)$. The argument of f_2 becomes

$$\frac{4\tilde{m}^2}{-(\tilde{m}+\kappa\tilde{\varepsilon}_v)^2 + v^2} = \frac{-2}{1+\kappa\sqrt{1+(v/\tilde{m})^2}} \tag{6.11}$$

which lies in the interval $[-1, 0[$ and $]0, \infty[$ for $\kappa = +$ and $\kappa = -$, respectively. From the definition of f_2 in (C.1) we then know that for given κ we always stay on the same branch of f_2 for arbitrary $\tilde{m}$, making calculations much simpler. Considering $\sqrt{1+a_1}$ with a_1 the argument of f_2 given in (6.11) we find

$$\sqrt{1+a_1} = \sqrt{\frac{-1+\kappa\sqrt{1+\left(\frac{v}{\tilde{m}}\right)^2}}{1+\kappa\sqrt{1+\left(\frac{v}{\tilde{m}}\right)^2}}} = \frac{\sqrt{1+\left(\frac{v}{\tilde{m}}\right)^2}-\kappa}{\sqrt{\left(\frac{v}{\tilde{m}}\right)^2}} = \frac{\tilde{m}}{v}\left(\sqrt{1+\frac{v}{\tilde{m}})^2}-\kappa\right).$$

Collecting the above information and doing a little algebra we can rewrite $\sum_\kappa f_2(a_1)$ in a surprisingly simple form,

$$\sum_\kappa f_2(a_1) = \frac{2}{v}\tilde{\varepsilon}_v \ln\left[\frac{v}{\tilde{m}} + \sqrt{1+\left(\frac{v}{\tilde{m}}\right)^2}\right].$$

Inserting this simple result into $T_3(m,0)$ there are some other cancellations and we obtain

$$T_3(m,0) = 4T^2\int_0^\infty vdv\ \tilde{n}(\tilde{\varepsilon}_v)\ln\left[\frac{v}{\tilde{m}} + \sqrt{1+\left(\frac{v}{\tilde{m}}\right)^2}\right]. \tag{6.12}$$

Noting that $\ln\left(x+\sqrt{1+x^2}\right) = \operatorname{arsinh}(x)$ we are naturally led to the substitution $v = \tilde{m}\sinh x$ with which T_3 becomes

$$T_3(m,0) = 4T^2\tilde{m}^2\int_0^\infty dx\ \ \sinh x\cosh x\ \tilde{n}(\tilde{m}\cosh x)\ x.$$

Now we use (B.6) to develop the thermal factor into a series and upon noting that

$$\partial_{\tilde{m}} e^{-k\tilde{m}\cosh x} = -k\cosh x\ e^{-k\tilde{m}\cosh x}$$

we obtain

$$T_3(m,0) = 4T^2\tilde{m}^2\sum_{k=1}^\infty \frac{1}{k}\left(-\partial_{\tilde{m}}\right)\int_0^\infty dx\ e^{-k\tilde{m}\cosh x}\sinh x\ x.$$

Integrating by parts we find

$$\begin{aligned}
T_3(m,0) &= 4T^2\tilde{m}^2\left(-\partial_{\tilde{m}}\right)\sum_{k=1}^\infty \frac{1}{k^2\tilde{m}}\int_0^\infty dx\ e^{-k\tilde{m}\cosh x} \\
&= 4T^2\tilde{m}^2\left(-\partial_{\tilde{m}}\right)\sum_{k=1}^\infty \frac{1}{k^2\tilde{m}}K_0(k\tilde{m})
\end{aligned}$$

where $K_0(x)$ is again a modified Bessel of the second kind and we have used its representation given in [13] 7.12 (21).

Low temperature expansion As was the case for $T_1(m)$, the low temperature expansion is easily obtained by using the asymptotic behaviour of the Bessel function; the result is

$$T_3(m,0) = 4T^2\sqrt{\frac{\pi}{2}}\left(\frac{3}{2\sqrt{\tilde{m}}}\mathrm{Li}_{5/2}(e^{-\tilde{m}}) + \sqrt{\tilde{m}}\mathrm{Li}_{3/2}(e^{-\tilde{m}})\right) \tag{6.13}$$

where besides the now familiar "$\frac{3}{2}$-logarithm" we have introduced a "$\frac{5}{2}$-logarithm" function in complete analogy to (6.4).

High temperature expansion The small $\tilde{m}$ behaviour is obtained by recasting the sum into an integral representation using (6.5):

$$\sum_{k=1}^{\infty} \frac{1}{k^2} K_0(k\tilde{m}) = \tilde{m}^2 \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \tilde{m}^{-s} \zeta(s) \mathcal{M}[K_0](s-2),$$

where $\mathcal{M}[K_0]$ denotes the Mellin transform of the modified Bessel function of the second kind (6.7). The sum is then given by

$$\sum_{k=1}^{\infty} \frac{1}{k^2} K_0(k\tilde{m}) = \left(\frac{\tilde{m}}{4}\right)^2 \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \left(\frac{\tilde{m}}{2}\right)^{-s} \zeta(s) \Gamma^2(\frac{s}{2}-1)$$

with $c > 2$ from the analyticity condition of the Mellin transform. As was the case for $T_1(m)$ the contour can be closed to the left for $\tilde{m} < 2\pi$, encircling double poles at $s = -2n$, $n = -1, 0, 1, 2, ...$ as well as a single pole at $s = 1$. It remains to calculate the residues which for the double poles are given as

$$\begin{aligned}
\mathrm{Res}_{s=-2n}\left[\left(\frac{\tilde{m}}{2}\right)^{-s} \zeta(s)\Gamma^2(\frac{s}{2}-1)\right] \quad &= \quad -\left(\frac{\tilde{m}}{2}\right)^{-s} \ln\left(\frac{\tilde{m}}{2}\right) \zeta(s) \left[(s+2n)\Gamma(\frac{s}{2}-1)\right]^2 \\
&\quad + \left(\frac{\tilde{m}}{2}\right)^{-s} \zeta'(s) \left[(s+2n)\Gamma(\frac{s}{2}-1)\right]^2 \\
&\quad + \left(\frac{\tilde{m}}{2}\right)^{-s} \zeta(s) \frac{d}{ds}\left[(s+2n)\Gamma(\frac{s}{2}-1)\right]^2 \Bigg|_{s=-2n}.
\end{aligned}$$

Using the results of the previous section and those of the appendix we find the first term to be $-16\tilde{m}^{-2}\zeta(2)\ln\tilde{m}/2$, $2\ln\tilde{m}/2$ and 0 for $n = -1, 0$ and $n > 0$, respectively. The second term becomes $16\tilde{m}^{-2}\zeta'(2)$, $-2\ln 2\pi$ and

$$4\left(\frac{\tilde{m}}{2}\right)^{2n} \frac{\zeta'(-2n)}{\Gamma^2(n+2)}$$

for those values of n; finally the third term is found to be $-16\tilde{m}^{-2}\gamma\zeta(2)$, $2(\gamma-1)$ and 0 for $n = -1, 0$ and $n > 0$, respectively. Together with the single pole at $s = 1$ that possesses the residue $8\pi\tilde{m}^{-1}$ all the contributions add up to

$$\begin{aligned}
T_3(m,0) = 4T^2\tilde{m}^2 \Big[&\partial_{\tilde{m}}(-\tilde{m}^{-1})(-\zeta(2)\ln\tilde{m}/2 + \zeta'(2) - \gamma\zeta(2)) \\
&+\partial_{\tilde{m}}(-\tilde{m})\frac{1}{8}(\ln\frac{\tilde{m}}{4\pi} + \gamma - 1) \\
&+\partial_{\tilde{m}}(-\tilde{m})\frac{1}{8}\sum_{n=1}^{\infty}\left(\frac{\tilde{m}}{4\pi}\right)^{2n} \frac{(-1)^n\Gamma(2n+1)\zeta(2n+1)}{\Gamma^2(n+2)}\Big],
\end{aligned}$$

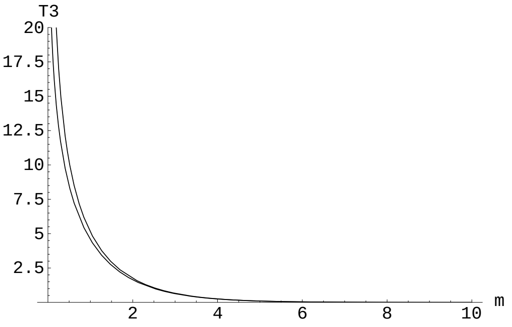


Figure 6.5: Comparing the low temperature expansion (6.13) to the exact numerical result: the analytic result (upper curve) is very close to the exact value, even for small values of $\tilde{m}$.

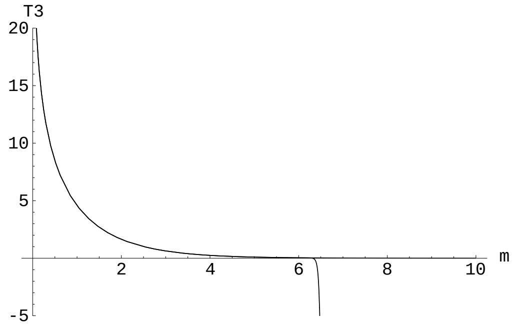


Figure 6.6: Comparing the high temperature expansion (6.14) to the exact numerical result: the analytic result truncated after the first 100 terms is indistinguishable from the exact result and again breaks down sharply at $\tilde{m} = 2\pi$.

where the contribution from the pole at $s = 1$ drops out because of the derivative. Differentiating with respect to $\tilde{m}$ then gives the final result for the high temperature expansion:

$$\begin{aligned} T_3(m,0) = T^2 \Bigg[&\frac{2\pi^2}{3}(-\ln\frac{\tilde{m}}{2} + \frac{\zeta'(2)}{\zeta(2)} - \gamma + 1) \\ &+\tilde{m}^2\frac{1}{2}(-\ln\frac{\tilde{m}}{4\pi} - \gamma) \\ &-\frac{\tilde{m}^2}{2}\sum_{n=1}^{\infty}\left(\frac{\tilde{m}}{4\pi}\right)^{2n}\frac{(-1)^n\Gamma(2n+2)\zeta(2n+1)}{\Gamma^2(n+2)}\Bigg] \end{aligned} \tag{6.14}$$

Evaluating the coefficients it can be checked that the first line actually corresponds to the result found in [32] by numerical means. Comparing our results for the high and low temperature expansions to the exact results obtained by numerical integration of (6.12) we find the plots shown in figures 6.5 and 6.6. Finally we have plotted the relative errors with respect to the exact results in figures 6.7 and 6.8

6.3 The integral $T_4(K_M)$

The last thermal integral for which we would like to find the high- and low temperature behaviour (at least for a special case) is $T_4(K_M)$, defined in (5.3). Already in the last chapter we found that this integral needs much time to be evaluated numerically, which is reflected in the analytical approach in the sense

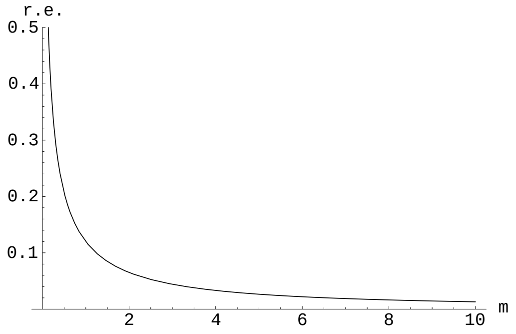


Figure 6.7: The relative error of the low temperature expansion with respect to the exact result.

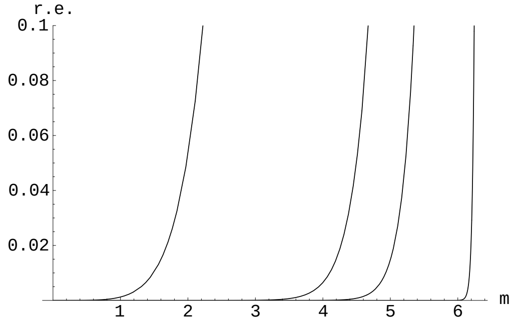


Figure 6.8: The relative error of the high temperature expansion to order $\tilde{m}^2$ and with the first $5, 10$ and 100 terms of the series in (6.14).

that the path to the result is everything but straightforward, and we will have to confront many mathematical technicalities to finally extract a minimum of information, even in the special case we are interested in.

6.3.1 Approximate solution to a special case

Again we want to treat the case $k = 0$ and $k_0^2 = m^2$; to save some unnecessary work it is convenient to express

$$f_4(p, q, K_M) = \frac{2kpq}{(2\pi)^2} \sum_{\sigma,\tau} I_2'(p, q, K_M)$$

and use the result (C.17) rather than (C.19) for I_2', since in the former the limit $k = 0$ is evident; the term $\frac{1}{k} f_4(p, q, k_0 = m, k = 0)$ then becomes

$$\frac{1}{k} f_4(p, q, k_0 = m, k = 0) = 2 \sum_{\sigma,\tau} \ln \left| \frac{E_{\sigma\tau}^2 - \varepsilon_{p+q}^2}{E_{\sigma\tau}^2 - \varepsilon_{p-q}^2} \right|,$$

where $E_{\sigma\tau} = m + \sigma\varepsilon_p + \tau\varepsilon_q$. Denoting $T_4(k_0 = m, k = 0)$ as $T_4(m, 0)$ as for T_3 we have

$$T_4(m, 0) = 2 \int_0^\infty pdp \int_0^\infty qdq \frac{n(\varepsilon_p)n(\varepsilon_q)}{\varepsilon_p \varepsilon_q} \sum_{\sigma,\tau} \ln \left| \frac{E_{\sigma\tau}^2 - \varepsilon_{p+q}^2}{E_{\sigma\tau}^2 - \varepsilon_{p-q}^2} \right|.$$

Using the sum over the signs σ and τ we may rewrite the logarithm as

$$\sum_{\sigma,\tau} \ln \left| \frac{E_{\sigma\tau}^2 - \varepsilon_{p+q}^2}{E_{\sigma\tau}^2 - \varepsilon_{p-q}^2} \right| = \sum_{\sigma,\tau} \ln \left| \frac{m^2 - (\varepsilon_{p+q} + \sigma\varepsilon_p + \tau\varepsilon_q)^2}{m^2 - (\varepsilon_{p-q} + \sigma\varepsilon_p + \tau\varepsilon_q)^2} \right|;$$

the argument of the logarithm can then be further simplified by using the identities

$$\begin{aligned} m^2 - (\varepsilon_{p+q} + \varepsilon_p + \varepsilon_q)^2 = & \quad -2(m^2 + p^2 + q^2 + pq + \varepsilon_p \varepsilon_q) \\ (m^2 - (\varepsilon_{p+q} + \varepsilon_p + \varepsilon_q)^2)(m^2 - (\varepsilon_{p+q} - \varepsilon_p - \varepsilon_q)^2) = & \quad -8pq(m^2 + \varepsilon_p \varepsilon_q - pq) \end{aligned}$$

(and similarly for the other combinations of signs) to obtain

$$\sum_{\sigma,\tau} \ln \left| \frac{m^2 - (\varepsilon_{p+q} + \sigma\varepsilon_p + \tau\varepsilon_q)^2}{m^2 - (\varepsilon_{p-q} + \sigma\varepsilon_p + \tau\varepsilon_q)^2} \right| = 2 \ln \left| \frac{p+q}{p-q} \right|,$$

a result that was already known [32]. The task would now be to find a high temperature expansion for the double thermal integral

$$T_4(m, 0) = 4 \int_0^\infty pdp \int_0^\infty qdq \frac{n(\varepsilon_p) n(\varepsilon_q)}{\varepsilon_p \varepsilon_q} \ln \left| \frac{p+q}{p-q} \right|, \tag{6.15}$$

which, unfortunately, we were not able to produce (at least not to all orders). This comes not without reason since trying to use the same techniques as in the previous section – notably to rewrite the integrals as sums over modified Bessel functions – fails utterly. In fact, by noting that the logarithm in (6.15) is equal to two times the atanh function of p/q or q/p, respectively, and expanding the latter, one eventually finds a double sum over *incomplete* modified Bessel functions for which we have not been able to find any mathematical treatment in the literature. Using (B.5) on the other hand, the integrals become manageable, but unfortunately contain divergences of different orders; even though one might try to use a regularization procedure as in [30], the analytic continuation to vanishing regularization parameter is far from being trivial and in fact is even made worse by the appearance of a double infinite sum of the Epstein zeta function type [18].

Facing these severe problems of extracting all the orders of the high temperature expansion we will limit ourselves to calculating only contributions up to order[2] $\tilde{m}$. In fact, the contributions to order $\tilde{m}^0$ can be extracted comparatively easily by separating the integral into soft (small p/T) and hard (large p/T) momentum contributions (as done in [32]). However, albeit this method can be extended for the soft momentum regime to all orders by using (B.4), the introduced cutoff dependence further complicates the calculation of the hard contributions, making this technique not applicable for our purpose.

6.3.2 Analytic integration and extraction of linear terms

To tackle (6.15) we begin by substituting p and q by the dimensionless variables $u = p/T$ and $v = p/T$, thereby extracting a factor T^2 out of the integral. Next

[2]This order is a relevant contribution to the plasmon mass to order $\lambda^{5/2}$ as will become clear in the next chapter.

we use the symmetry of the integrand to extract a factor of 2, reducing the upper limit of the dv integration to u. So far we have

$$T_4(m,0) = 8T^2 \int_0^\infty udu \int_0^u vdv \frac{\tilde{n}(\tilde{\varepsilon}_u)\tilde{n}(\tilde{\varepsilon}_v)}{\tilde{\varepsilon}_u\tilde{\varepsilon}_v} \ln \frac{u+v}{u-v}.$$

We now substitute $\frac{v}{u} = t$ and rewrite the logarithm as $\operatorname{atanh}(t)$, obtaining

$$T_4(m,0) = 16T^2 \int_0^1 dt\ t\ \operatorname{atanh}(t) \int_0^\infty du\ u^3 \frac{\tilde{n}(\tilde{\varepsilon}_u)\tilde{n}(\tilde{\varepsilon}_{ut})}{\tilde{\varepsilon}_u\tilde{\varepsilon}_{ut}}. \tag{6.16}$$

In order to do one of the integrations we use (B.5) for $\tilde{n}(\tilde{\varepsilon}_{ut})$, separating T_4 into two parts

$$T_4(m,0) = 16T^2 \left(T_{4,1} + \sum_n T_{4,2}(n) \right)$$

with

$$\begin{aligned} T_{4,1} &= -\frac{1}{2} \int_0^1 dt\ t\ \operatorname{atanh}(t) \int_0^\infty du\ u^3 \frac{\tilde{n}(\tilde{\varepsilon}_u)}{\tilde{\varepsilon}_u} \frac{1}{\sqrt{(ut)^2 + \tilde{m}^2}} \qquad (6.17) \\ T_{4,2}(n) &= \int_0^1 dt\ t\ \operatorname{atanh}(t) \int_0^\infty du\ u^3 \frac{\tilde{n}(\tilde{\varepsilon}_u)}{\tilde{\varepsilon}_u} \frac{1}{(ut)^2 + \tilde{m}^2 + 4\pi^2 n^2}. \end{aligned}$$

First contribution To calculate the relevant contributions from $T_{4,1}$ we expand

$$\frac{1}{\sqrt{(ut)^2 + \tilde{m}^2}} = \frac{1}{\tilde{m}} \sum_{k=0}^\infty \binom{-\frac{1}{2}}{k} \left(\frac{u}{\tilde{m}}\right)^{2k} t^{2k}$$

and – using the series expansion of $\operatorname{atanh} t$ – we obtain for the dt integration

$$\int_0^1 dt\ t^{2k+1} \operatorname{atanh}(t) = \sum_{n=0}^\infty \frac{1}{2n+1} \frac{1}{2n+2k+3}. \tag{6.18}$$

The resulting double sum can be done [19] and gives

$$\begin{aligned} \sum_{n=0}^\infty \sum_{k=0}^\infty \binom{-\frac{1}{2}}{k} \left(\frac{u}{\tilde{m}}\right)^{2k} \frac{1}{4n^2 + 4n(k+2) + 2k + 3} &= \frac{\tilde{m}}{u^2} \Bigg(u\ \operatorname{arsinh}\Big(\frac{u}{\tilde{m}}\Big) \\ &\quad - \tilde{\varepsilon}_u \ln \sqrt{1 + \left(\frac{u^2}{\tilde{m}^2}\right)} \Bigg) \end{aligned}$$

so that we find for $T_{4,1}$:

$$T_{4,1} = -\frac{1}{2} \int_0^\infty du\ u \frac{\tilde{n}(\tilde{\varepsilon}_u)}{\tilde{\varepsilon}_u} \left(u\ \operatorname{arsinh}\Big(\frac{u}{\tilde{m}}\Big) - \tilde{\varepsilon}_u \ln \sqrt{1 + \left(\frac{u^2}{\tilde{m}^2}\right)} \right).$$

Integrating by parts using (B.7) we obtain

$$T_{4,1} = \frac{1}{2}\int_0^\infty du \ln\left(1 - e^{-\tilde{\varepsilon}_u}\right)\left(\operatorname{arsinh}\Big(\frac{u}{\tilde{m}}\Big) - \frac{u}{\tilde{\varepsilon}_u}\ln\sqrt{1 + \Big(\frac{u^2}{\tilde{m}^2}\Big)}\right).$$

To extract the constant and first order terms in $\tilde{m}$ we do a Taylor expansion of the above result. The constant contribution is easily evaluated to be

$$\lim_{\tilde{m}\to 0} T_{4,1} = \frac{1}{2}\int_0^\infty du \ln\left(1 - e^{-u}\right)\ln 2 = -\frac{\pi^2 \ln 2}{12}$$

while the first order coefficient is given by

$$\lim_{\tilde{m}\to 0}\frac{\partial}{\partial\tilde{m}} T_{4,1} = \frac{1}{2}\lim_{\tilde{m}\to 0}\left[\int_0^\infty du\ \tilde{n}(\tilde{\varepsilon}_u)\frac{\tilde{m}}{\tilde{\varepsilon}_u}\left(\operatorname{arsinh}\Big(\frac{u}{\tilde{m}}\Big) - \frac{u}{\tilde{\varepsilon}_u}\ln\sqrt{1 + \Big(\frac{u^2}{\tilde{m}^2}\Big)}\right)\right.$$
$$\left. + \int_0^\infty du\ \ln\left(1 - e^{-\tilde{\varepsilon}_u}\right)\frac{u\tilde{m}}{\tilde{\varepsilon}_u^3}\left(\ln\sqrt{1 + \Big(\frac{u^2}{\tilde{m}^2}\Big)} - 1\right)\right].$$

It should be pointed out that we cannot do the limit under the integral since the factor of $\tilde{m}$ would give zero even for divergent integrals. The only correct way is therefore to do the integration first and then evaluate the limit $\tilde{m} \to 0$. Fortunately, the two contributions are UV-finite with the Bose-Einstein distribution function and the logarithm acting as cut-off for large momenta. We can therefore approximate these functions as

$$\begin{aligned}\tilde{n}(\tilde{\varepsilon}_u) &\simeq \frac{1}{\tilde{\varepsilon}_u}\\ \ln\left(1 - e^{-\tilde{\varepsilon}_u}\right) &\simeq \ln\tilde{\varepsilon}_u,\end{aligned}$$

and eventually do the integrations by substituting $u = \tilde{m}\sinh w$ and $\tilde{\varepsilon}_u = \tilde{m}\ x$, respectively. We find [19]

$$\begin{aligned}\int_0^\infty dw \frac{w}{\cosh w} &= 2\mathcal{C}\\ \int_1^\infty dx \frac{-\ln x}{x^2} &= -1\\ \int_1^\infty dx \frac{\ln^2 x}{x^2} &= 2\end{aligned}$$

with $\mathcal{C}$ being Catalan's constant with a numerical value of $0.915966\ldots$. Numerically these results are indistinguishable from the integrals without approximation. Collecting our results we find

$$\lim_{\tilde{m}\to 0}\frac{\partial}{\partial\tilde{m}} T_{4,1} = \frac{1}{2}\lim_{\tilde{m}\to 0}(2\mathcal{C} - 1 + 1) = \mathcal{C}.$$

The constant and first order part of $T_{4,1}$ are then given as

$$T_{4,1} = -\frac{\pi^2}{12} \ln 2 + \tilde{m}\, \mathcal{C}. \tag{6.19}$$

Second contribution for $n \neq 0$ To calculate the contributions that are due to $T_{4,2}(n)$ we expand

$$\frac{1}{(ut)^2 + \tilde{m}^2 + 4\pi^2 n^2} = \frac{1}{\tilde{m}^2 + 4\pi^2 n^2} \sum_{k=0}^{\infty} (-1)^k t^{2k} \left(\frac{u^2}{\tilde{m}^2 + 4\pi^2 n^2} \right)^k$$

and make use of eq.(6.18) to do the dt integration. Evaluating the double sum [19] then leaves us with the result

$$\int_0^1 dt\; t\; \mathrm{atanh}(t) \frac{1}{(ut)^2 + \tilde{m}^2 + 4\pi^2 n^2} = \frac{1}{2u^2} \arctan^2 \frac{u}{\sqrt{\tilde{m}^2 + 4\pi^2 n^2}}.$$

The contribution $T_{4,2}(n)$ then reads

$$T_{4,2}(n) = \frac{1}{2} \int_0^\infty du\; u \frac{\tilde{n}(\tilde{\varepsilon}_u)}{\tilde{\varepsilon}_u} \arctan^2 \frac{u}{\sqrt{\tilde{m}^2 + 4\pi^2 n^2}},$$

which after integration by parts becomes

$$T_{4,2}(n) = -\int_0^\infty du \ln\left(1 - e^{-\tilde{\varepsilon}_u}\right) \frac{\sqrt{\tilde{m}^2 + 4\pi^2 n^2}}{u^2 + \tilde{m}^2 + 4\pi^2 n^2} \arctan \frac{u}{\sqrt{\tilde{m}^2 + 4\pi^2 n^2}}. \tag{6.20}$$

We will show that $T_{4,2}(n)$ does not possess $\tilde{m}^1$ contributions for any $n \neq 0$, so that for our purposes it is sufficient to evaluate the constant part given by

$$\lim_{\tilde{m} \to 0} T_{4,2}(n) = -\int_0^\infty du \ln\left(1 - e^{-u}\right) \frac{2\pi n}{u^2 + 4\pi^2 n^2} \arctan \frac{u}{2\pi n},\; n \neq 0. \tag{6.21}$$

If $T_{4,2}(n)$, $n \neq 0$ had a contribution of order $\tilde{m}$, its coefficient would be given by

$$\lim_{\tilde{m} \to 0} \frac{\partial}{\partial \tilde{m}} T_{4,2}(n),\; n \neq 0,$$

with the partial derivative calculated to be

$$\begin{aligned}\partial_{\tilde{m}} T_{4,2}(n) = -\tilde{m} \int_0^\infty du \Bigg[&\frac{\tilde{n}(\tilde{\varepsilon}_u)}{\tilde{\varepsilon}_u} \frac{\sqrt{\tilde{m}^2 + 4\pi^2 n^2}}{u^2 + 4\pi^2 n^2} \arctan \frac{u}{\sqrt{\tilde{m}^2 + 4\pi^2 n^2}} \\ + \ln\left(1 - e^{-\tilde{\varepsilon}_u}\right) \Bigg(&-\frac{1}{(u^2 + \tilde{m}^2 + 4\pi^2 n^2)^2} + \frac{(\tilde{m}^2 + 4\pi^2 n^2)^{-3/2}}{u^2 + \tilde{m}^2 + 4\pi^2 n^2} \arctan \frac{u}{\sqrt{\tilde{m}^2 + 4\pi^2 n^2}} \\ &- \frac{2\sqrt{\tilde{m}^2 + 4\pi^2 n^2}}{(u^2 + \tilde{m}^2 + 4\pi^2 n^2)^2} \arctan \frac{u}{\sqrt{\tilde{m}^2 + 4\pi^2 n^2}} \Bigg) \Bigg].\end{aligned}$$

To show that the coefficient is zero, it is sufficient to show that the above integrals (without the leading factor of $\tilde{m}$) diverge logarithmically at most for $\tilde{m} \to 0$. For all but one of the above integrals this is trivial to prove since

$$\int_0^\infty du \lim_{\tilde{m}\to 0} \ln\left(1 - e^{-\tilde{\varepsilon}_u}\right) \left(-\frac{1}{(u^2+m^2+4\pi^2n^2)^2}\right.$$
$$\left. + \frac{(m^2+4\pi^2n^2)^{-3/2}}{u^2+m^2+4\pi^2n^2} \arctan\frac{u}{\sqrt{m^2+4\pi^2n^2}} - \frac{2\sqrt{m^2+4\pi^2n^2}}{(u^2+m^2+4\pi^2n^2)^2} \arctan\frac{u}{\sqrt{m^2+4\pi^2n^2}}\right)$$
$$= const.,$$

for any $n \neq 0$. It remains to be shown that the only remaining integral is also logarithmically divergent at most: first observe that $\tilde{n}(\tilde{\varepsilon}_u)$ again acts as cut-off for higher momenta and we can use the approximation from above to extract the leading term. The integral then reads

$$\int_0^\infty du \frac{1}{\tilde{\varepsilon}_u^2} \arctan\frac{u}{\sqrt{\tilde{m}^2+4\pi^2n^2}} \frac{\sqrt{\tilde{m}^2+4\pi^2n^2}}{u^2+\tilde{m}^2+4\pi^2n^2}$$

and we can extract another constant contribution by using

$$\frac{1}{u^2+\tilde{m}^2}\frac{1}{u^2+\tilde{m}^2+4\pi^2n^2} = \frac{1}{4\pi^2n^2}\left(\frac{1}{u^2+\tilde{m}^2} - \frac{1}{u^2+\tilde{m}^2+4\pi^2n^2}\right)$$

and

$$\int_0^\infty du \frac{\sqrt{\tilde{m}^2+4\pi^2n^2}}{u^2+\tilde{m}^2+4\pi^2n^2} \arctan\frac{u}{\sqrt{\tilde{m}^2+4\pi^2n^2}} = \frac{\pi^2}{8}.$$

To calculate the remaining integral

$$\int_0^\infty du \frac{1}{\tilde{\varepsilon}_u^2} \arctan\frac{u}{\sqrt{\tilde{m}^2+4\pi^2n^2}}$$

we divide the integration domain into two regions by introducing an arbitrary parameter Λ with $\tilde{m} \ll \Lambda < 2\pi$. Approximating the arctan function by its argument in the domain $0 < u < \Lambda$ and setting $\tilde{m} = 0$ for $u > \Lambda$ (which is a valid approximation since the integral is UV finite and the IR behaviour is cut off by Λ) the resulting integrals are

$$\begin{aligned}
\int_0^\Lambda du \frac{1}{\tilde{\varepsilon}_u^2}\frac{u}{\sqrt{\tilde{m}^2+4\pi^2n^2}} &= \frac{1}{4\pi n}\ln\frac{\Lambda^2}{\tilde{m}^2} \\
\int_\Lambda^\infty du \frac{1}{u^2}\arctan\frac{u}{2\pi n} &= \frac{1}{2\pi n}\left(1 - \ln\Lambda + \frac{1}{2}\ln\Lambda^2 + 1\right),
\end{aligned}$$

where we have used $\tilde{m} \ll \Lambda \ll 2\pi$. The potentially dangerous terms involving $\ln\Lambda$ cancel, and we may safely send $\Lambda \to 0$ without picking up any more divergent contributions. Collecting the above results we find that

$$\partial_{\tilde{m}} T_{4,2}(n) = -\tilde{m}\frac{1}{4\pi^2n^2}\left(-\ln m + const.\right)$$

for $n \neq 0$, so that $\lim_{\tilde{m}\to 0} \partial_{\tilde{m}} T_{4,2} = 0$, $n \neq 0$, which completes the proof. For our purposes it is therefore sufficient to calculate the sum over the constant part (6.21) which after the substitution $u = 2\pi n v$ is given by

$$-2\sum_{n=1}^{\infty}\int_0^{\infty} dv \ln\left(1-e^{-2\pi n v}\right)\frac{1}{1+v^2}\arctan v = 2(0.046\ldots) \tag{6.22}$$

where the sum and the integral have been evaluated numerically [19].

Second contribution for $n = 0$ The only remaining integral that we have to calculate is given by (6.20) for $n = 0$,

$$T_{4,2}(0) = -\int_0^{\infty} du \ln\left(1-e^{-\tilde{\varepsilon}_u}\right)\frac{\tilde{m}}{u^2+\tilde{m}^2}\arctan\frac{u}{\tilde{m}}.$$

Despite its simple appearance it is nonetheless a little bit tricky to extract the $\tilde{m}$ contribution from it. The difficulty stems mainly from the fact that the usual series expansions for $\arctan(x)$ are exclusively valid in the regions $-1 \leq x \leq 1$ and $1 \leq x < \infty$. However, by using the algebraic identity

$$\arctan(x) = \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right)$$

which is valid for $0 \leq x < \infty$ and observing that

$$\arcsin(y) = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-1)^n\frac{y^{2n+1}}{2n+1}$$

we find a series expansion of $\arctan(x)$ for the whole region $0 \leq x < \infty$, given by

$$\arctan(x) = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-1)^n\frac{1}{2n+1}\frac{x^{2n+1}}{(1+x^2)^{n+1/2}}.$$

Inserting this into the above expression for $T_{4,2}(0)$ we find

$$T_{4,2}(0) = -\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{1}{2n+1}\int_0^{\infty} du \ln\left(1-e^{-\tilde{\varepsilon}_u}\right)\frac{\tilde{m}}{u^2+\tilde{m}^2}\left(\frac{u}{\tilde{\varepsilon}_u}\right)^{2n+1}.$$

Substituting $\tilde{\varepsilon}_u = \tilde{m}\ v$ and using

$$(v^2-1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k v^{2n-2k}$$

the integral becomes

$$T_{4,2}(0) = -\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{1}{2n+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\int_0^{\infty} du \ln\left(1-e^{-mv}\right)\frac{1}{v^{2k+2}}.$$

Expanding finally the logarithm using

$$\ln\left(1-e^{-x}\right) = -\sum_{j=1}^{\infty}\frac{e^{-jx}}{j},$$

$T_{4,2}(0)$ is given as a threefold sum over the generalized exponential integrals $E_n(x)$ (defined in (A.9)),

$$T_{4,2}(0) = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{1}{2n+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\sum_{j=1}^{\infty}\frac{1}{j}E_{2k+2}(j\tilde{m}).$$

To calculate the high temperature expansion we use (6.5) so that

$$\sum_{j=1}^{\infty}\frac{1}{j}E_{2k+2}(j\tilde{m}) = \tilde{m}\int_{1-i\infty}^{1+i\infty}\frac{ds}{2\pi i}\tilde{m}^{-s}\zeta(s)\frac{\Gamma(s-1)}{2k+s}$$

where we have use the Mellin transform of the $E_n(x)$ given in (A.10). The integrand is convergent for $s\to-\infty$ and $\tilde{m}<2\pi$ so we can evaluate the integral by closing the contour to the left. For our purposes it is sufficient to include the residues at the poles $s=1$ (double) and $s=0$ (double for $k=0$, else single), which – using the techniques of the previous sections – are easily evaluated to be $-\frac{\ln\tilde{m}}{2k+1}-\frac{1}{(2k+1)^2}$ as well as $\tilde{m}\left(-\frac{1}{2}\ln\tilde{m}+1-\frac{\gamma}{2}\right)\delta_{k,0}$ and $\tilde{m}\left(\frac{1}{4k}\right)$ for $k\neq 0$, respectively. It now remains to calculate the double sums over k and n for these contributions, for which we find [19]

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{1}{2n+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{2k+1} &= \frac{\pi^2}{8}\\
\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{1}{2n+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{1}{(2k+1)^2} &= \frac{\pi^2}{8}\ln 2+\frac{7}{16}\zeta(3)\\
\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{1}{2n+1} &= \frac{\pi}{2}\\
\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n\frac{1}{2n+1}\sum_{k=1}^{n}\binom{n}{k}(-1)^k\frac{1}{k} &= -4\mathcal{C}+\pi\ln 2.
\end{aligned}$$

The first, third and fourth result is straightforward, whereas to obtain the second line we made use of the identity between harmonic numbers and the digamma-function found in the appendix, as well as $\psi(\frac{3}{2}+n)-\psi(\frac{1}{2}+n)=\frac{2}{2n+1}$. Collecting the above results, we find for $T_{4,2}(0)$:

$$T_{4,2}(0) = -\frac{\pi^2}{8}\ln\frac{\tilde{m}}{2}-\frac{7}{16}\zeta(3)+\frac{m}{T}\left[\frac{\pi}{4}(-\ln\frac{\tilde{m}}{2}+2-\gamma)-\mathcal{C}\right]$$

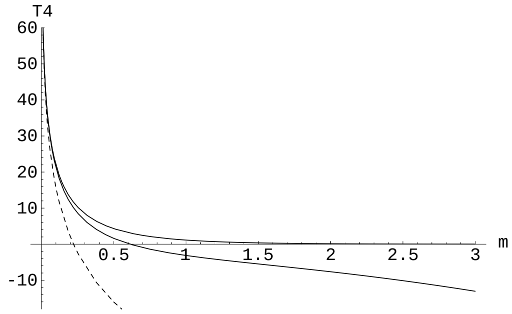


Figure 6.9: Comparing numerical and analytical results (upper, lower and dashed curves respectively) for $T_4(m,0)$. The analytic result (6.23) remains close to the exact result until $\tilde{m} = 0.613\ldots$, where it drops below zero. The dashed line corresponds to the leading ($O(\tilde{m}^0)$) order term in the high temperature expansion.

The analytic result for $T_4(m,0)$ to first order in $\tilde{m}$ is therefore given as

$$T_4(m,0) = 8T^2 \left(-\frac{\pi^2}{4} \ln \tilde{m} - c_1 + \tilde{m} \left(-\frac{\pi}{2} \ln \tilde{m} + c_2 \right) \right) \tag{6.23}$$

with c_1 and c_2 numerical constants given by

$$\begin{aligned} c_1 &= \pi^2 \ln 2 \left(\frac{1}{4} + \frac{1}{6} \right) + \frac{7}{8}\zeta(3) - 4(0.046\ldots) = 3.718\ldots \\ c_2 &= \frac{\pi}{2}(2 - \gamma + \ln 2) = 3.3237\ldots. \end{aligned}$$

Note that the $\mathcal{C}$s have cancelled; the only contribution to the above result that we were not able to calculate analytically for lack of time is the sum over the constant part of the contribution $T_{4,2}(n)$ given in (6.22). Nevertheless we have determined all the other contributions analytically, so the coefficient of the linear term in the above high temperature expansion is exact.

Comparing (6.23) with the exact result obtained by integrating (6.15) numerically we obtain the plot shown in figure 6.9. In the figures 6.10 and 6.11 we have plotted the relative and absolute error of (6.23) with respect to the exact result: the full line corresponds to (6.23) while the dashed line gives the error of the result previously known [32], that is, without the order $\tilde{m}$ correction term. From these plots it is apparent that the addition of this first order correction term improves the lowest order result dramatically.

6.4 Summary

In this chapter we calculated the high- and low temperature expansion for the thermal integrals $T_1(m)$ and $T_3(K_M)$ for the special case of $k_0 = m$, $k = 0$ using

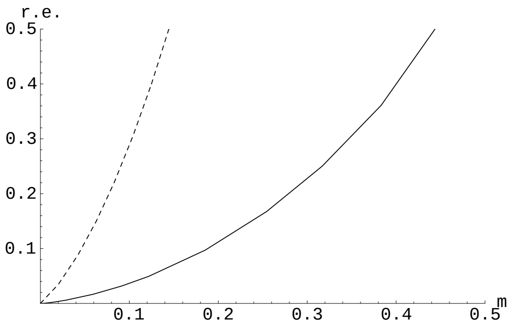


Figure 6.10: The relative error of the high temperature expansion (6.23) with respect to the exact result. The dashed line corresponds to the result without the linear term in $\tilde{m}$.

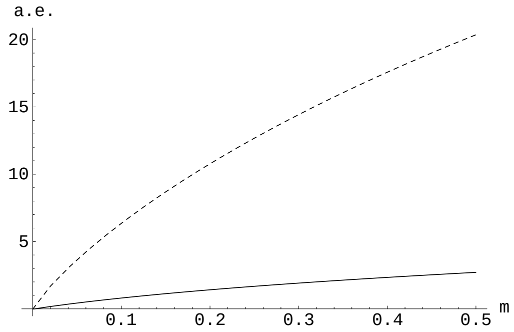


Figure 6.11: The absolute error of the high temperature expansion (6.23) with respect to the exact result. The dashed line corresponds to the result without the linear term in $\tilde{m}$.

the asymptotic behaviour of the involved Bessel functions and Mellin integral transforms, respectively. For the thermal integral T_4 we were unable to derive the full high temperature expansion but succeeded to obtain it to first order. The respective analytical results were compared to the exact results found by numerical integration, and the analytic prediction of the breakdown of the high temperature expansion for $\tilde{m} > 2\pi$ for T_1 and T_3 was confirmed.

Chapter 7

HTL resummation and plasmon mass

In the last chapter we derived a high-temperature expansion of $T_1(m)$ which essentially provides us with the leading temperature dependent contribution to the self-energy,

$$\Pi_1^\beta \simeq -\frac{\lambda T^2}{24},$$

where we have neglected higher orders. As we raise the temperature to higher and higher values we eventually enter a regime where $\tilde{m} = m/T \ll 1$ so that the intrinsic particle mass becomes negligible with respect to the temperature. In this regime the leading contribution to the pole of the full propagator is given by $\simeq \frac{\lambda^{1/2}T}{\sqrt{24}}$ rather than m, acting as the relevant cutoff for IR singularities in loop diagrams. Higher loop diagrams will then accumulate powers of $\lambda^{1/2}$ in their denominators which spoils the usual power counting since these cancel against the usual factors of λ in the numerator coming from the Feynman rules. Therefore, to consistently calculate a quantity to a given order in the coupling constant one has to take into account all – usually infinitely many – higher loop diagrams contributing to this order in λ. This eventually leads to the breakdown of standard perturbation theory at some order of λ ([39], [32], [40]).

There is, however, a way out of this misery: the perturbation theory may be resummed by constructing an effective theory that makes higher order calculations possible [41]. Although for scalar field theories as the one we consider the resummation is straightforward it becomes more delicate in the case of gauge theories, where contradictory results indicated that physical quantities apparently came out gauge dependent [23]. However, the advent of a technique called HTL ("hard thermal loop") resummation [42] finally settled these problems and restored gauge-independence [43].

In this chapter we will combine the HTL resummation technique with the results of standard perturbation theory obtained in the previous chapters to calculate the thermal or plasmon mass[1] of the neutral scalar field at finite temperature. For convenience, we will follow the step-by-step approach taken in [32] rather than determining the mass by a gap equation as in [44]. In order to make use of independent results of the previous chapters we will treat the mass in those results as a parameter that may be set to arbitrary values.

7.1 Plasmon mass to leading order

We begin by considering a massless theory at finite temperature[2] for which we calculate the self energy to first loop order. Since we evaluated this quantity for arbitrary mass m in chapter 2 the result is simply given by (4.2) with m set to zero,

$$\Pi_1 = -4\lambda'' \mu^{2\varepsilon} T_1(0).$$

From (6.9) we easily find $T_1(0) = \frac{T^2\pi^2}{6}$ so that to first order in λ the pole of the full propagator is (after analytic continuation) located at

$$K_M^2 = -\Pi_1 = \frac{\lambda T^2}{24},$$

as anticipated at the beginning of this chapter. Therefore, at a temperature where the intrinsic ($T = 0$) mass of the theory can be neglected the thermal corrections to the vacuum theory play the role of a dynamical mass that can be attributed to quasiparticles called plasmons. Whereas to lowest order this plasmon or thermal mass is just given by the above value,

$$m_T^2 = \frac{\lambda T^2}{24} \tag{7.1}$$

we have to decide for a definition of higher order corrections since – as was found in the previous chapters – higher loop diagrams are both momentum dependent and contain in general an imaginary part. The usual use of language in the literature [50] is such that the full momentum dependent quantity is referred to as thermal mass whereas the plasmon mass is defined as the real part of the pole of the full, on-shell Minkowski (analytically continued) propagator at zero external 3-momentum; we will stick to the latter one which was also considered in [32].

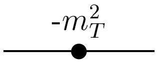
To include the effect of this new, dynamical mass into the theory and to calculate its next-to-leading-order (NLO) correction we have to construct an effective, HTL-resummed theory. This is done [32] by adding and subtracting a

[1] The term will be defined shortly.

[2] Note that this is equivalent to a massive theory at a sufficiently high temperature so that $m/T \ll 1$.

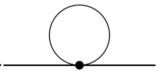
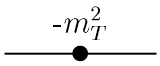
term $\frac{1}{2}m_T^2\phi^2$ to the original Lagrangian $\mathcal{L}$, obtaining

$$\mathcal{L}_1 = (\mathcal{L} - \frac{1}{2}m_T^2\phi^2) + \frac{1}{2}m_T^2\phi^2 \tag{7.2}$$

as the new effective Lagrangian. The subscript 1 on $\mathcal{L}_1$ is used to remind us that it now describes an effective theory with a tree-level mass m_T and an extra 2-point interaction term  that can be used to recalculate the self-energy. Again we find that we have already done all the hard work: we simply have to take the result for the self-energy of the massive theory calculated in chapter 2, set the mass parameter to m_T and add the contribution m_T^2 coming from the correction term:

$$\Pi_1(m_T) = \frac{\lambda''}{2}m_T^2\left[\frac{1}{\varepsilon} + 1 - \ln \hat{m_T}^2\right] - 4\lambda''\mu^{2\varepsilon}T_1(m_T) + m_T^2. \tag{7.3}$$

From chapter 4 we know that by using renormalization we can get rid of the divergent term in Π_1; however, since m_T is now temperature dependent it seems that we would have to introduce a temperature dependent mass counterterm into the theory, which would spoil renormalizability. We will, in fact, argue in the next section that this is not the case; for the moment, however, note that if we ignore all contributions to Π_1 that are of order λ^2 or higher we do not have to introduce any counterterms at all! The reason for this is that m_T^2 is of order λ so that the divergent term in (7.3) is of order λ^2, which we ignore. The self-energy is then simply given as

$$\Pi_1(m_T) = \frac{1}{2}$$  $$+$$ 

$$= \lambda T\frac{m_T}{8\pi} + O(\lambda^2) \tag{7.4}$$

where we have used our analytic result for the high temperature expansion of $T_1(m_T)$ given in (6.9) and ignored all contributions which are of order λ^2 or higher. It is interesting to see that $\Pi_1^{ren}(m_T)$ is formally of order $\lambda^{3/2}$, the unaccustomed power of 3/2 coming from the resummation of the perturbative theory.

We may ask ourselves if there are any other diagrams in the effective theory (7.2) that contribute to order $\lambda^{3/2}$ to the self-energy. The unfortunate answer is yes, there are in fact *infinitely* many such diagrams contributing to this order. However, it has been shown in [32] that although each of these diagrams contribute to order $\lambda^{3/2}$ individually, their sum is only of order λ^2. Therefore the self-energy of the effective theory to $\lambda^{3/2}$ is completely given by (7.4) and – by virtue of our definition – gives rise to a correction to the plasmon mass of this order. To close the cycle we include the effects of this corrected plasmon mass

$$m_T'^2 = m_T^2 - m_T\frac{\lambda T}{8\pi} + O(\lambda^2) \tag{7.5}$$

by again shifting the mass term in $\mathcal{L}_1$ to obtain a new effective Lagrangian

$$\mathcal{L}_2 = (\mathcal{L} - \frac{1}{2} m_T'^2 \phi^2) + \frac{1}{2} m_T'^2.$$

7.1.1 The problem of imaginary masses

The result (7.5) for the corrected mass is unsatisfactory insofar as for large coupling this mass will become imaginary. To see this, rewrite (7.5) in the form

$$m_T'^2 = m_T^2 \left(1 - \frac{3m_T}{\pi T}\right)$$

where it becomes apparent that for $\beta m_T > \frac{\pi}{3}$ the result becomes unphysical. However, there exists another way to see the above equation, namely as a first order approximation of an infinite series; ignoring all order λ^2 corrections, the corrected plasmon mass squared is given as

$$m_T'^2 = \frac{m_T^2}{1 + \frac{3m_T}{\pi T}} + O(\lambda^2)$$

which is also valid for larger λ and referred to as Padé approximation. Clearly, the above approximation is not unique and although one easily can think of simple alternatives the Padé approximation has been shown to be an improvement over the "strictly perturbative" result (7.5) when calculating physical quantities such as e.g. the entropy [45].

7.2 Plasmon mass to order λ^2

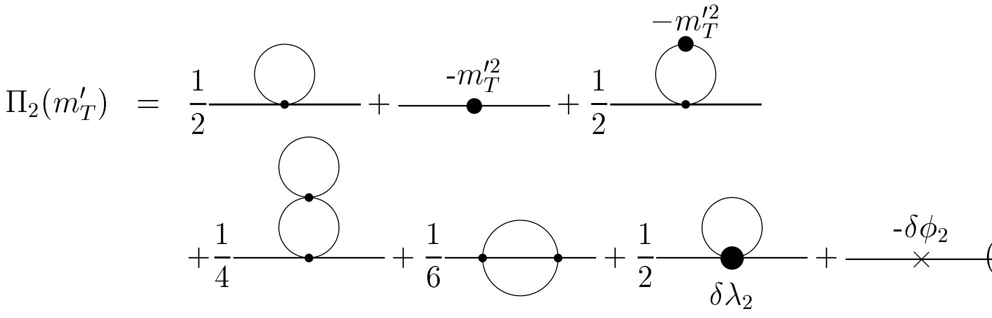
Fortunately, this will be the last resummation we have to do in order to obtain the corrected plasmon mass to order λ^2 (and even $\lambda^{5/2}$). Nevertheless we once again have to recalculate the self-energy contribution from the set of diagrams in (7.4) with this new effective theory; now, however, we are also interested in their contribution λ^2 so we will have to renormalize the theory in order to get rid of the divergent term in (7.3). By rescaling the original quantities as in chapter 4 we obtain for the Lagrangian

$$\mathcal{L} = \mathcal{L}_2 = \left(\frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) - \frac{1}{2} m_T'^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + \mathcal{L}_\phi + \mathcal{L}_m + \mathcal{L}_\lambda\right) + \frac{1}{2} m_T'^2 \phi^2 - \mathcal{L}_m = \mathcal{L}_2^{ren},$$

where we have dropped the index r of the renormalized quantities. Note that because of the interaction term, there are now *two* mass counterterms, equal in value but opposite in sign, so they just cancel and leave us with

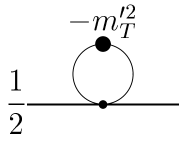
$$\mathcal{L}_2^{ren} = \left(\frac{1}{2} (\partial^\mu \phi) (\partial_\mu \phi) - \frac{1}{2} m_T'^2 \phi^2 - \frac{\lambda}{4!} \phi^4 + \mathcal{L}_\phi + \mathcal{L}_\lambda\right) + \frac{1}{2} m_T'^2 \phi^2.$$

Determining the coupling-constant counterterm as in chapter 4 there are the following set of diagrams that contribute to the self-energy up to second order in λ:



(7.6)

where we have used the notation of the coupling-constant and wave-function counterterms as in chapter 4. Before we start evaluating the diagrams, however, we have to pose ourselves the question if there are any other, higher loop diagrams that contribute to the self-energy to order λ^2. In [32] it has been shown that the sum of all the potentially "dangerous" diagrams – although individually of lower order – only contribute to order λ^3 or $\lambda^3 \ln \lambda$. For our calculations they are therefore completely irrelevant and, for the self-energy to order λ^2, (7.6) contains all the contributing diagrams. The first two diagrams in (7.6) are nothing else but Π_1 from the previous section (7.3) with m_T replaced by the corrected mass m'_T; the contribution of the last diagram in the first line is algebraically given by

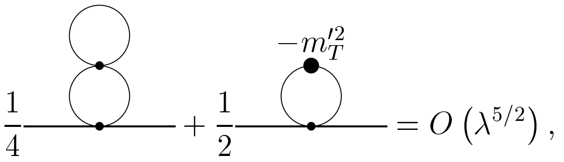


$$= -\lambda \frac{m'^2_T}{2} \sum\!\!\!\!\!\!\!\int_P D_0^2(P) \tag{7.7}$$

for which we can use the result (B.11) in the appendix. The diagrams in the second line of (7.6) have all been calculated for arbitrary mass parameter already in the previous chapters, so that using (3.1), (3.13) and (4.3) we find for (7.6)

$$\begin{aligned}
\Pi_2(m'_T) = &+\frac{\lambda''}{2}\left[m'^2_T\left(1-\ln \hat{m}'^2_T\right) - 4\lambda''\mu^{2\varepsilon}T_1(m'_T) + m'^2_T\right] \\
&+\lambda''^2\left[\frac{K_M^2}{6}\left(\frac{11}{8}-\frac{1}{2}\ln \hat{m}'^2_T\right) + \frac{m'^2_T}{6}I_C(-K_M^2/m'^2_T)\right. \\
&\left. -(4\ln \hat{m}'^2_T - 8)\mu^{2\varepsilon}T_1(m'_T) - \mu^{2\varepsilon}T_3(m'_T,K_M) + \mu^{4\varepsilon}T_4(m'_T,K_M)\right] \\
&+\frac{\lambda''}{2}\Big[\ln \hat{m}'^2_T\left(m'^2_T - 4\mu^{2\varepsilon}\lambda''T_1(m'_T)\right) + 8\left(m'^2_T - 4\mu^{4\varepsilon}\lambda''T_1(m'_T)\right)\partial_{m'^2_T}T_1(m'_T) \\
&+4\mu^{2\varepsilon}m'^2_T\lambda''(\ln \hat{m}'^2_T - 1)\partial_{m'^2_T}T_1(m'_T)\Big] + \frac{2\lambda''^2}{\varepsilon}m'^2_T\partial_{m'^2_T}T_1(m'_T) + O(\lambda^3).
\end{aligned} \tag{7.8}$$

There are several things to note from this result: first notice that we have arranged the result in such a way that the first line corresponds to the finite part of (7.3), the second and third line represent the finite contribution from the sunset diagram and finally the last two lines the contributions from



that cancel up to order $\lambda^{5/2}$. Secondly, all divergent terms but one have either cancelled or are formally of order λ^3 or higher, the only exception being the last term in the last line of (7.8) which comes from the product of the divergent part of the tadpole times the finite part of the bottom loop in the double-scoop diagram; restricting ourselves to calculating the self-energy to order λ^2, however, we can drop this term. Furthermore, since the plasmon mass is defined at $K_M^2 = m_T'^2$ and $k = 0$ the second line in (7.8) is also of order λ^3 and therefore does not contribute either. Finally, by observing that

$$\begin{aligned} m_T'^2 &= 4\lambda'' \mu^{2\varepsilon} T_1(m_T) + O(\lambda^2) \\ &= 4\lambda'' \mu^{2\varepsilon} T_1(m_T') + O(\lambda^2) \end{aligned}$$

the reason for writing the last lines of (7.8) in such a peculiar way becomes apparent: in the last two lines of (7.8) all leading orders of λ cancel and one is left with a contribution of order $\lambda^{5/2}$ which can be neglected. The correction of the plasmon mass to order λ^2 is therefore given as the sum of the first and the third line of (7.8),

$$m_T''^2 - m_T'^2 = \lambda'' m_T^2 \left[\frac{3}{2} \ln \frac{T^2}{4\pi\mu^2} + 2 \ln \frac{m_T^2}{T^2} + \alpha \right] + O(\lambda^{5/2}) \tag{7.9}$$

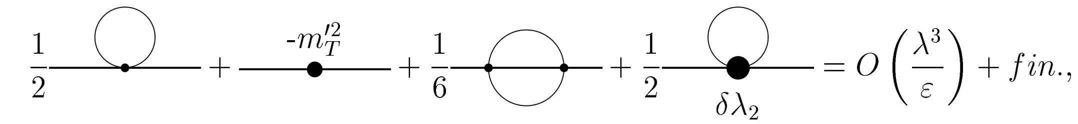
with $\alpha = 2 - \frac{\gamma}{2} + \frac{\zeta'(2)}{\zeta(2)} + \ln 8\pi + \frac{12}{\pi^2} c_1 \simeq 8.8865$ the numerical constant that was already found in [32]. To obtain this result we have used the high temperature expansion of T_1, T_3 and T_4 given in (6.9), (6.14) and (6.23).

7.3 Plasmon mass to order $\lambda^{5/2}$

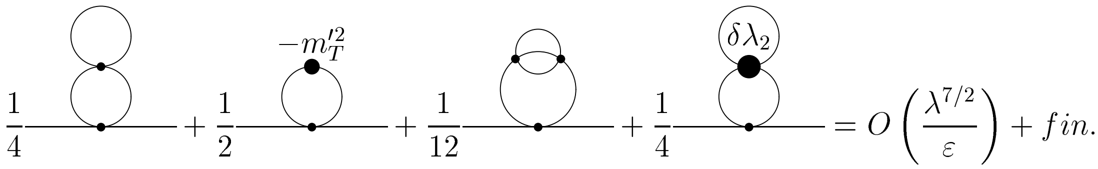
Scanning through the results of the previous section we find that (7.8) is actually valid to order $\lambda^{5/2}$. Furthermore, since we worked so hard on it in the previous chapter, we know the high temperature expansion of all the three thermal integral involved to order $\lambda^{5/2}$ as well. Finally, the diagrams thought of as dangerous to order λ^2 were shown to sum to λ^3, therefore not contributing anything to $\lambda^{5/2}$. Might we therefore get the plasmon mass to this order for free, without having to consider other diagrams?

The answer, of course, is no. There *has* to be at least one diagram that does not contribute to the plasmon mass to order λ^2, but to order $\lambda^{5/2}$, since

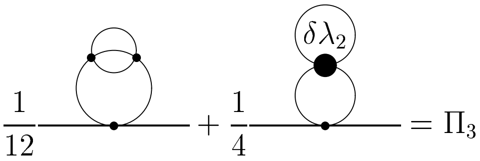
(7.8) contains a divergent term of this order. If there was no such diagram that contributes at this order, the plasmon mass to order $\lambda^{5/2}$ would be divergent! A simple way to find these missing diagrams is to go through the details of the algebraic contributions in (7.6) and find the relation



where we have neglected finite contributions for the moment. From our previous analysis we know that the divergent term in (7.8) comes from the tadpole plus mass-interaction diagram with a bottom loop attached; attaching such a bottom loop to all the above diagrams we find



The missing diagrams[3] can therefore be identified to be



where we have christened the first one to the name "gimp" diagram. The gimp diagram possesses the sunset diagram as subdiagram and its contribution to the self-energy will be dominated by the $n = 0$ term in the frequency sum of the bottom loop. Since the relevant IR cutoff is provided by the plasmon mass one can also estimate the leading contribution by simple power counting [32]: there is a factor $O(\lambda)$ coming from the bottom loop vertex, a factor $O(\lambda^2)$ from the sunset diagram and a factor $O(\lambda^{-1/2})$ coming from the IR-singularity of the bottom loop being cut off by the plasmon mass. Therefore, besides being needed to render the plasmon mass finite at $O(\lambda^{5/2})$, the gimp diagram will also contribute to the finite part of the self-energy to this order.

However, although this diagram is momentum independent the extraction of these finite contributions to order $\lambda^{5/2}$ is not trivial, and in fact there is strong reason to believe that published results on this matter [38] are not correct. Possibly the only way to extract the relevant contribution will be by numerical means.

We would have liked to present this last contribution to the plasmon mass to order $\lambda^{5/2}$ in this work but for lack of time we shall not be able to do so. However, with some more hard work the plasmon mass to order $\lambda^{5/2}$ will soon be known...

[3] There are again an infinite number of other diagrams that – although individually of order $\lambda^{5/2}$ – sum to a contribution of order λ^3 or higher; the only additional contribution to the self-energy of order $\lambda^{5/2}$ is therefore given by Π_3.

7.4 Summary

In this chapter we argued that standard perturbation theory at high temperature breaks down at some order of the coupling constant because higher-loop diagrams accumulate powers of λ in the denominator as a consequence of the λ dependence of the relevant IR cutoff, the plasmon mass. We found that a possible solution to include these non-perturbative aspects is HTL resummation of the theory and went on to calculate the plasmon mass to order λ^2 in this framework relying heavily on results from previous chapters. We indicated that our results are easily extendible to order $\lambda^{5/2}$ and the only unknown contribution to the plasmon mass to this order is a three-loop diagram that we did not have the time to evalute.

Chapter 8

Conclusions

In the present work we have studied the full momentum dependent real part of the self-energy in a thermal $\lambda\phi^4$ quantum field theory up to two loop order. We found that the key contribution to this quantity is the so-called sunset diagram that we evaluated step-by-step. The ensuing calculation showed that the result was dependent on 4 integrals that we could not solve analytically, one that is already present at $T = 0$ and three integrals that contain thermal factors and therefore only contribute at nonzero temperatures. However, numerically we were able to evaluate these integrals for arbitrary external momentum with a precision of 2 to 3 significant digits [48]. Moreover, in the special case of on-shell K_M and vanishing 3-momentum we found the exact high temperature expansion for three of these integrals, and the leading terms for the fourth one, which in total extends previously known results by one order of the series.

These analytical results permitted us to calculate the mass of the thermal quasiparticle excitations to second order in λ and also provide the full $\lambda^{5/2}$ contribution that is due to diagrams up to two-loop order. However, to determine the full contribution to this order we would have needed to calculate another 3-loop diagram that we were not able to evaluate due to lack of time.

The numerical results in turn were found to predict that the absolute value of self-energy contribution that is due to the two-loop diagrams rises as a positive power of the external 3-momentum for large k. The two-loop contribution therefore will exceed the one-loop contribution at some large external 3-momentum, whatever the value of the coupling constant, leading to a breakdown of perturbation theory. With the help of our results we will eventually be able to determine quantitatively at which values of k and λ this breakdown occurs.

However, more work needs to be done. First of all, it is imperative to determine the leading order contribution of the three-loop diagram identified in the last chapter which will provide the only missing contribution to the thermal mass to order $\lambda^{5/2}$. Secondly, from an aesthetic point of view it is also unsatisfactory that neither the full high-temperature expansion nor the low-temperature expansion of the thermal contribution T_4 for the special case $k = 0$ and on-shell K_M

is known. Thirdly, from looking at the numerical results of the thermal integrals involved it does not seem to be impossible to extract some analytical information out of the full-momentum dependent integrals, despite their appearance. At least for other special cases, e.g. large external 3-momentum or vanishing k_0, some results should come up in the foreseeable future.

Finally, we also hope that the techniques that we have used to treat the thermal integrals in this work show up in some form or the other also in more realistic (gauge) theories, such as QCD. There, however, calculations are complicated by the richer structure of the theory, notably gauge-dependencies, non-zero chemical potentials and a more involved resummation of the strictly perturbative expansion; nevertheless there has been much interest in this field recently and we hope that some of the aspects in this work will be helpful in the study of two-loop calculations in QCD at finite temperatures and densities.

Appendix A

Definitions and mathematical functions

A.1 Notations and conventions

Here we give a collection of formulas and conventions we have used in the main body of our work.

A.1.1 Metric

Throughout this work we will use the metric signature (+,-,-,-).

A.1.2 Vectors

We denote the usual vectors in Euclidean 3-space by fat symbols, e.g. $\mathbf{p}$, and their absolute value by lower case letters, $|\mathbf{p}| = p$. Furthermore, upper case letters will denote Euclidean 4-vectors, $P = (\mathbf{p}, p_4)$ with $P^2 = p^2 + p_4^2$. Finally, we have used Minkowskian 4-vectors that we indicate by a subscript M, e.g. $P_M = (p_0, \mathbf{p})$, $P_M^2 = p_0^2 - p^2$.

A.1.3 Infinite integrals

Unless otherwise stated all integrals without boundaries are to be understood as infinite integrals

$$\int dk = \int_{-\infty}^{+\infty} dk$$

A.1.4 Fourier transforms

We defined the Fourier transform of the function $f(x)$ as

$$\tilde{f}(k) = \int dx e^{ikx} f(x)$$

and – unless stated explicitly – have denoted the Fourier transform by a tilde. The inverse transform is given by

$$f(x) = \int \frac{dk}{2\pi} e^{-ikx} \tilde{f}(k)$$

Note that for periodic functions we use the definition of the Fourier series as given in [15].

A.1.5 Functional calculus

Functional differentiation is defined by analogy with ordinary differentiation (see [6]):

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\varepsilon \to 0} \frac{F[f(x) + \varepsilon \delta(x-y)] - F[f(x)]}{\varepsilon}. \tag{A.1}$$

Functional integrals can be done [3] by generalizing the Gaussian formula

$$\int \frac{d^n x}{(2\pi)^{n/2}} \exp\left[-\frac{1}{2}(x, Ax)\right] = (\det A)^{-1/2} \tag{A.2}$$

which is valid for any real symmetric matrix (or operator) A to an infinite-dimensional function space ϕ possessing an inner product $(.,.)$:

$$\int \mathcal{D}\phi e^{-\frac{1}{2}(\phi, A\phi)} = (\det A)^{-1/2}.$$

A.2 Mathematical functions

In this section we give a short collection of the properties of the functions that were used in the main text.

A.2.1 The Heavyside-function

We will use the following definition of the Heavyside (or step-) function:

$$\Theta(x) = \begin{cases} 1, & x > 0 \\ \frac{1}{2}, & x = 0 \\ 0, & x < 0 \end{cases}$$

possessing the integral representation

$$\Theta(\pm t) = \pm i \int \frac{dx}{2\pi} \frac{e^{-ixt}}{x \pm i\varepsilon}.$$

A.2.2 The Dirac delta function

Every sequence of smooth functions $\delta_n(x)$ for which

$$\lim_{n\to\infty} \int \delta_n(x)\phi(x)dx = \phi(0)$$

is a representation of the Dirac delta function [17]. In short one can write

$$\lim_{n\to\infty} \delta_n(x) = \delta(x).$$

An often used representation is given by

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{\pi} \frac{1}{a^2+\varepsilon^2} = \delta(a).$$

A particular important representation of the delta function is

$$\delta(p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dl \exp(ilp).$$

A.2.3 The function $\Gamma(x)$

The Gamma function is defined as

$$\Gamma(\alpha+1) = \int_0^\infty dx\ x^\alpha e^{-x}$$

for $\mathrm{Re}\alpha > -1$. For integer values of α it is given by

$$\Gamma(\alpha+1) = \alpha! = \alpha(\alpha-1)(\alpha-2)....$$

Analytic continuation to arbitrary values of α is done by splitting up the integral

$$\begin{aligned}\Gamma(\alpha+1) &= \int_0^1 dx\ x^\alpha e^{-x} + \int_1^\infty dx\ x^\alpha e^{-x} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(\alpha+n+1)} + \int_1^\infty dx\ x^\alpha e^{-x} \qquad \text{(A.3)}\end{aligned}$$

where it can be seen that $\Gamma(\alpha+1)$ has simple poles at $\alpha+1=-n$ with residues $(-1)^n/n!$.

Asymptotic form For $|\alpha| \gg 1$ the asymptotic behaviour of $\Gamma(\alpha+1)$ is given by

$$\Gamma(\alpha+1) = \sqrt{2\pi\alpha}\alpha^\alpha e^{-\alpha}\left(1+O(\alpha^{-1})\right)$$

for $|\arg(\alpha)| < \pi$.

Derivations It is useful to consider the following function

$$\psi(\alpha) = \frac{d \ln \Gamma(\alpha)}{d\alpha} = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}$$

also referred to as digamma function in the literature. From [12] 1.7 (2) we obtain the expansion

$$\psi(\alpha) = -\gamma + (\alpha - 1) \sum_{n=0}^{\infty} \frac{1}{(n+1)(\alpha+n)}$$

which also proves the recurrence formula $\psi(\alpha+1) = \psi(\alpha) + \frac{1}{\alpha}$.

It is interesting to see that with the help of the digamma function the analytic continuation of the so-called harmonic numbers $H(n)$ defined as

$$H(n) = \sum_{i=0}^{n} \frac{1}{i}$$

is possible. For this purpose, note that

$$H(n) = \psi(1+n) + \gamma = \frac{\Gamma'(1+n)}{\Gamma(1+n)}$$

and use the unique analytic continuation of the Gamma function.

Miscellaneous results The series expansion of $\Gamma(\varepsilon)$ for $0 <\in\ll 1$ can be found to be[1]

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + \frac{\varepsilon}{2}(\gamma^2 + \frac{\pi^2}{6}). \tag{A.4}$$

Sometimes one also has to expand $\Gamma(x+\varepsilon)$ for $x > 0$ in powers of ε. This is easily done by using the function ψ:

$$\Gamma(x+\varepsilon) = \Gamma(x)(1 + \varepsilon\psi(x) + \dots)$$

As it will be frequently used in the text we will furthermore calculate the term

$$\frac{d}{ds}\Gamma(\frac{s}{2} - 1)(s + 2n)\bigg|_{s=-2n}.$$

We have

$$\Gamma(\frac{s}{2} - 1) = \frac{\Gamma(\frac{s}{2} + n)}{(\frac{s}{2} - 1)(\frac{s}{2}) \dots (\frac{s}{2} + n - 1)}$$

[1] By example, by using [19]

The derivative of $\Gamma(\frac{s}{2}-1)(s+2n)$ is then given by

$$\Big[\Gamma(\frac{s}{2}-1)(s+2n)\Big]' = \frac{\Gamma(\frac{s}{2}+n+1)}{(\frac{s}{2}-1)(\frac{s}{2})\dots(\frac{s}{2}+n-1)}\Big(\psi(\frac{s}{2}+n+1) \\ -\frac{1}{\frac{s}{2}-1}-\frac{1}{\frac{s}{2}}-\dots-\frac{1}{\frac{s}{2}+n-1}\Big)$$

which – when using the above recurrence formula for $\psi(x)$ – can be written for $s=-2n$ as

$$\frac{d}{ds}\Gamma(\frac{s}{2}-1)(s+2n)\Big|_{s=-2n} = \frac{(-1)^n\psi(n+2)}{\Gamma(n+2)}.$$

A.2.4 The function $\zeta(x)$

Riemann's Zeta function is defined as

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \mathrm{Re}s>1 \tag{A.5}$$

for which an integral representation can be found quite easily by observing

$$\int_0^\infty x^{s-1}e^{-nx}dx = n^{-s}\Gamma(s), \quad \mathrm{Re}s>0$$

and summing both sides so that

$$\Gamma(s)\zeta(s)=\int_0^\infty dx\frac{x^{s-1}}{e^x-1}, \quad \mathrm{Re}s>1.$$

The definition can be used to continue $\zeta(s)$ analytically for general s with $\mathrm{Re}s>0$ as

$$\zeta(s)=1+\frac{2^{1-s}}{s-1}+\frac{1}{\Gamma(s)}\int_0^\infty dx\left(\frac{1}{1-e^{-x}}-\frac{1}{x}\right)e^{-2x}x^{s-1} \tag{A.6}$$

and – with the use of Riemann's functional relationship –

$$\pi^s\zeta(1-s)=2^{1-s}\Gamma(s)\zeta(s)\cos\frac{\pi s}{2} \tag{A.7}$$

also for $\mathrm{Re}s<0$. It can be concluded that $\zeta(s)$ has a simple pole at $s=+1$ with the residue $+1$.

Asymptotic form From (A.6) we extract the asymptotic behaviour

$$\lim_{s\to\infty}\zeta(s)=1+O(s^{-1}).$$

Series expansion From [12] 1.12 (17) we have the identity

$$\zeta(s) = (s-1)^{-1} + \gamma + \sum_{n=1}^{\infty} \gamma_n (s-1)^n,$$

with γ_n some real numbers for which the exact form can be found in the above reference.

Useful values

$$\begin{aligned} \zeta(0) &= -\frac{1}{2} \\ \zeta(-2n) &= 0, \quad n = 1, 2, ... \end{aligned}$$

Furthermore, it's interesting to look at the first derivative of $\zeta(s)$. Deriving (A.7) with respect to s we obtain

$$\begin{aligned} \zeta'(s) = \frac{1}{2}(2\pi)^s \ln 2\pi \frac{\zeta(1-s)}{\Gamma(s)\cos \pi s/2} + \frac{1}{2}(2\pi)^s \left(\frac{-\Gamma'(s)}{\Gamma^2(s)}\right) \frac{\zeta(1-s)}{\cos \pi s/2} \\ + \frac{1}{2}(2\pi)^s \frac{\sin \pi s/2}{\cos^2 \pi s/2} \frac{\pi}{2} \frac{\zeta(1-s)}{\Gamma(s)} - \frac{1}{2}(2\pi)^2 \frac{1}{\Gamma(s)\cos \pi s/2} \zeta'(1-s) \end{aligned}$$

For $s \to -2n$, $n = 1, 2, ...$ only the second term survives since the Gamma function $\Gamma(s)$ has poles at all negative integers $s = -2n$. The only term that contributes is the second for which we need the derivative of $1/\Gamma(s)$. Using (A.3) for $s \to -2n$ we find

$$\lim_{s \to -2n} \frac{-\Gamma'(s)}{\Gamma^2(s)} = -\lim_{s \to -2n} \frac{\frac{-1}{(2n)!(s+2n)^2} + O(s+2n)^0}{\frac{1}{((2n)!)^2 (s+2n)^2} + O(s+2n)^{-1}} = (2n)!$$

and therefore

$$\zeta'(-2n) = \frac{1}{2}(2\pi)^{-2n}(-1)^n \Gamma(2n+1)\zeta(2n+1), \quad n = 1, 2, ...$$

For $n = 0$ we quote the result of [14] where it is derived:

$$\zeta'(0) = -\frac{1}{2}\ln 2\pi.$$

A.2.5 The Beta function

The Beta function is defined by the integral

$$\int_0^1 t^{x-1}(1-t)^{y-1} dt = B(x, y)$$

with $x, y > 0$. Another very useful representation is given by

$$B(x, y) = b^x \int_0^\infty t^{x-1}(1+bt)^{-x-y} dt$$

with $x, y, b > 0$ found in [12] 1.5.(12). The Beta function can be expressed via the Gamma function as

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

A.2.6 The generalized hypergeometric function

The standard hypergeometric function $F(a, b, c, z)$ can be generalized to p parameters of the nature a, b and q parameters of the nature c as in [12] 4.1 (1). We will only make use of the case $p = 2$ and $q = 1$ for which the generalized hypergeometric function is defined to be

$$ {}_2F_1(a, b, c, z) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} \tag{A.8}$$

with $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, $(a)_0 = 1$.

A.2.7 The generalized exponential integrals

In the main text there appear generalized exponential integrals of the type

$$E_n(x) = \int_1^\infty du\ e^{-xu} u^{-n}, \tag{A.9}$$

which are related to the incomplete Gamma functions as ([13] 9.2)

$$E_n(x) = x^{n-1}\Gamma(1-n, x).$$

It is easy to derive the Mellin transform of these functions,

$$\begin{aligned} \mathcal{M}[E_n(x)](s) &= \int_0^\infty dx\ x^{s-1} E_n(x) \\ &= \int_1^\infty du\ u^{-n-s}\Gamma(s) \\ &= \frac{\Gamma(s)}{n+s-1} \end{aligned} \tag{A.10}$$

which is valid for $s > 1 - n$.

A.3 Formulas from standard field theory

A very useful formula is

$$\frac{1}{D_1^{a_1} D_2^{a_2} .... D_k^{a_k}} =$$
$$\frac{\Gamma(a_1 + a_2 + ... + a_k)}{\Gamma(a_1)\Gamma(a_2)...\Gamma(a_k)} \int_0^1 ... \int_0^1 dx_1...dx_k \frac{\delta(1 - x_1 - ... - x_k) x_1^{a_1 - 1} ... x_k^{a_k - 1}}{(D_1 x_1 + ... + D_k x_k)^{a_1 + ... + a_k}} \tag{A.11}$$

due to Feynman and found in this form in [4].

In order to calculate dimensionally reduced integrals the following formula is very convenient (found in [21])

$$\int \frac{d^n P}{(2\pi)^n} \frac{1}{(P^2 + A)^\alpha} = \frac{1}{(4\pi)^{n/2}} \frac{\Gamma(\alpha - \frac{n}{2})}{\Gamma(\alpha)} A^{n/2 - \alpha}. \tag{A.12}$$

Appendix B

Manipulations of thermal quantities

B.1 Bose-Einstein distribution

The Bose-Einstein distribution $n(z)$ is defined as

$$n(x) = \frac{1}{e^{\beta x} - 1},$$

with $\beta = 1/T$.

Useful identities The following identities are easily proved

$$\begin{aligned} n(x)n(y) &= n(x+y)\,(1 + n(x) + n(y)) && \text{(B.1)} \\ n(-x) &= -1 - n(x) && \text{(B.2)} \\ n(x) &= \frac{1}{2}\coth\frac{\beta x}{2} - \frac{1}{2}. && \text{(B.3)} \end{aligned}$$

Series expansions Very useful expansions of $n(x)$ are given in [12] 1.13 (1) and [30]

$$\begin{aligned} xn(x) &= \sum_{n=0}^{\infty} B_n \frac{(\beta x)^n}{n!}, \quad x < 2\pi && \text{(B.4)} \\ n(x) &= -\frac{1}{2} + \frac{1}{\beta x} + 2\sum_{n=1}^{\infty} \frac{1}{\beta} \frac{x}{x^2 + \frac{4\pi^2 n^2}{\beta^2}} && \text{(B.5)} \\ n(x) &= \sum_{n=1}^{\infty} e^{-n\beta x} && \text{(B.6)} \end{aligned}$$

where B_n are the Bernoulli numbers.

Integrals The only basic integral is

$$\int dx n(x) = \frac{1}{\beta} \ln\left(1 - e^{-\beta x}\right) + const. \tag{B.7}$$

B.2 Thermal sums

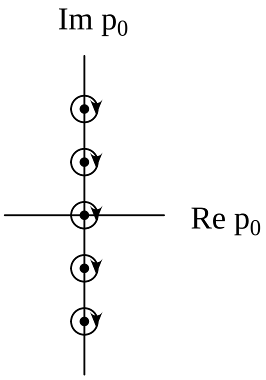
We shall present here [1] a technique to calculate infinite sums of the form

$$S = T \sum_{n=-\infty}^{\infty} f(p_0 = i\omega_n = 2\pi n T i).$$

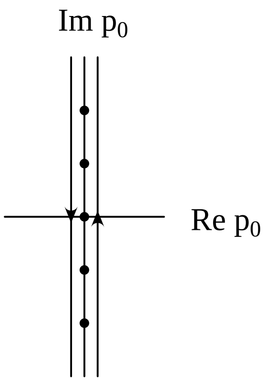
Using the fact the $\coth\left(\frac{1}{2}\beta p_0\right)$ has poles at $p_0 = 2\pi n T i$ with residues $2T$ and is bounded and analytic everywhere else we can rewrite the above sum as contour integral

$$S = \frac{T}{2\pi i} \oint dp_0 f(p_0) \frac{\beta}{2} \coth\left(\frac{\beta}{2} p_0\right)$$

as long as p_0 itself does not have poles at $p_0 = 2\pi n T i$. The contour is given by



and provided that $f(p_0)$ does not contain *any* poles on the imaginary axis it may be deformed into



Using the above identities between the cotangent hyperbolicus and the Bose-Einstein factor as well as a substitution $p_0 \to -p_0$ we obtain

$$\begin{aligned} S &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp_0 \frac{1}{2}\left[f(p_0) + f(-p_0)\right] \\ &+ \frac{1}{2\pi i}\int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} dp_0 \left[f(p_0) + f(-p_0)\right] n(p_0). \end{aligned} \tag{B.8}$$

For some thermal sums $f(p_0)$ decays rapidly enough for $p_0 \to \infty$ to close the above integrals to the right and evaluate them using the residue theorem. Easily derived are

$$\begin{aligned} T\sum_n \frac{1}{\omega_n^2 + \alpha^2} &= \frac{1}{2\alpha}\left(1 + 2n(\alpha)\right) \\ T\sum_n \frac{1}{\left(\omega_n^2 + \alpha^2\right)^2} &= -\frac{1}{2\alpha}\frac{\partial}{\partial\alpha} T\sum_n \frac{1}{\omega_n^2 + \alpha^2}. \end{aligned}$$

A thermal sum of a special kind shows up in section 3: the function $f(p_0)$ is given by $e^{p_0\theta} D_0(P)$ for which – using (B.8) and the symmetries of $D_0(P)$ – we may close the contour to the right also for the vacuum part [28], obtaining

$$T\sum_n \frac{e^{\omega_n\theta}}{\omega_n^2 + \alpha^2} = \frac{1}{2\pi i}\oint dp_0 \left(e^{-p_0\theta} + (e^{-p_0\theta} + e^{p_0\theta}) n(p_0)\right) \frac{1}{\omega_n^2 + \alpha^2}.$$

Using the residuum theorem the sum is easily found to be

$$T\sum_n \frac{e^{\omega_n\theta}}{\omega_n^2 + \alpha^2} = \frac{\cosh\left((\theta - \beta/2)\alpha\right)}{2\alpha\sinh\left(\beta\alpha/2\right)}. \tag{B.9}$$

B.3 Collection of sum-integrals

In the main text we derive some non-trivial sum integrals, regularized with dimensional regularization, which are not stated in closed form. Here we shall reproduce some basic results from chapter 2 and 3:

$$\lambda \sum\!\!\!\!\!\!\!\int_P D_0(P) = -\lambda'' m^2 \left(\frac{4\pi\mu^2}{m^2}\right)^{\varepsilon} \frac{\Gamma(\varepsilon)}{1-\varepsilon} + \lambda'' \mu^{2\varepsilon} 8 T_1(m) \tag{B.10}$$

which – when expanded to first order in ε is given by

$$\lambda \sum\!\!\!\!\!\!\!\int_P D_0 = -\lambda'' m^2 \left(\frac{1}{\varepsilon} + 1 - \ln\hat{m}^2 + \varepsilon\left[1 + \frac{\pi^2}{12} - \ln\hat{m}^2 + \frac{1}{2}\ln^2\hat{m}^2\right]\right) + 8\lambda''\mu^{2\varepsilon} T_1(m).$$

From this result we also can easily derive another sum-integral by using the identity

$$\lambda \sum\!\!\!\!\!\!\int_P D_0^2(P) = -\frac{\partial}{\partial m^2} \lambda \sum\!\!\!\!\!\!\int_P D_0(P) \tag{B.11}$$

which – again to first order in ε – is

$$\lambda \sum\!\!\!\!\!\!\int_P D_0^2(P) = \lambda'' \left(\frac{1}{\varepsilon} - \ln \hat{m}^2 + \varepsilon \left[\frac{\pi^2}{12} + \frac{1}{2} \ln^2 \hat{m}^2 \right] \right) - 8\lambda'' \mu^{2\varepsilon} \partial_{m^2} T_1(m).$$

Appendix C

Integrals

C.1 The integral $\int_0^1 dx \ln\left[1+\frac{4}{a}x(1-x)\right]$

This integral is often found when having introduced a Feynman parameter to do the momentum integrations. Since it is perhaps the most often used integral in this work we have calculated it here in detail.

Using integration by parts we obtain

$$\begin{aligned}
&\int_0^1 dx \ln\left[1+\frac{4}{a}x(1-x)\right] \\
&\quad = x\ln\left(1+\frac{4}{a}x(1-x)\right)\Bigg|_0^1 - \int_0^1 dx \quad x\frac{1}{1+\frac{4}{a}x(1-x)}\frac{4}{a}\left(1-x-x\right) \\
&\qquad = -\int_0^1 dx\ 4\frac{x(1-2x)}{a+4x(1-x)} \\
&\qquad\quad = -4\int_0^1 dx\left[\frac{1}{2}-\frac{1}{2}\frac{a+2x}{a+4x-4x^2}\right].
\end{aligned}$$

Separating by parts for $a \neq -1$ we can do the integrations

$$\begin{aligned}
\int_0^1 dx\frac{a+2x}{a+4x-4x^2} &= -\int_0^1 dx\left[\frac{1}{2}\frac{1+\sqrt{1+a}}{2x-1-\sqrt{1+a}}+\frac{1}{2}\frac{1-\sqrt{1+a}}{2x-1+\sqrt{1+a}}\right] \\
&= \frac{1}{4}\sqrt{1+a}\ln\left(\frac{1+\sqrt{1+a}}{1-\sqrt{1+a}}\right)^2,
\end{aligned}$$

obtaining therefore

$$\int_0^1 dx \ln\left[1+\frac{4}{a}x(1-x)\right] = -2+\frac{1}{2}\sqrt{1+a}\ln\left(\frac{1+\sqrt{1+a}}{1-\sqrt{1+a}}\right)^2$$

for $a \neq -1$. It is useful to split the result into real and imaginary part. For $a > 0$

the integrand is certainly real and therefore we have, for $a > 0$

$$\int_0^1 dx \ln\left[1 + \frac{4}{a}x(1-x)\right] = -2 + \sqrt{1+a}\ln\left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right).$$

On the other hand, for $a < -1$ the integrand also has to be real since

$$\max\left(4x(1-x)\right) = 1, \quad x\varepsilon[0,1]$$

Using the formula

$$ix\ln\left(\frac{1+ix}{1-ix}\right) = -2x\arctan(x)$$

we obtain

$$\int_0^1 dx \ln\left[1 + \frac{4}{a}x(1-x)\right] = -2 + \sqrt{|1+a|}\left(\pi - 2\arctan\sqrt{|1+a|}\right)$$

for $a < -1$. Using the identity

$$\arctan\frac{1}{x} = \frac{\pi}{2} - \arctan x$$

we rewrite the above result as

$$\int_0^1 dx \ln\left[1 + \frac{4}{a}x(1-x)\right] = -2 + 2\sqrt{|1+a|}\arctan\frac{1}{\sqrt{|1+a|}}.$$

For $a \in [-1, 0[$ the integrand is negative for some value of x and therefore the result will involve some imaginary part. The real part of the integral is given as

$$\mathrm{Re}\int_0^1 dx \ln\left[1 + \frac{4}{a}x(1-x)\right] = -2 + \sqrt{1+a}\ln\left(\frac{1+\sqrt{1+a}}{1-\sqrt{1+a}}\right).$$

To obtain the imaginary part we look at the values of x for which the argument of the logarithm vanishes. This happens for $x_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{a}{4}}$ which we use to construct the behaviour of the argument as shown in figure C.1. Depending on which cut of the logarithm we find ourselves we therefore obtain for the imaginary part

$$\mathrm{Im}\int_0^1 dx \ln\left[1 + \frac{4}{a}x(1-x)\right] = (2n+1)\pi\sqrt{1+a}$$

with $a \in [-1, 0[$ and $n \in \mathbb{N}$. Collecting our results we find

$$\int_0^1 dx \ln\left[1 + \frac{4}{a}x(1-x)\right] = -2 + f_2(a) + i(2n+1)\pi\sqrt{1+a}\Theta(a+1)\Theta(-a) \tag{C.1}$$

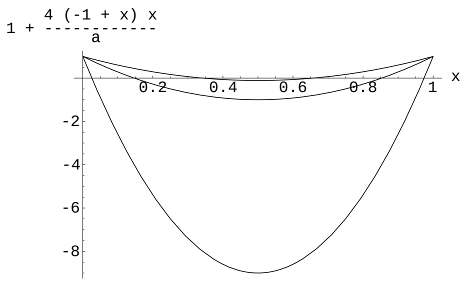


Figure C.1: The argument $1+\frac{4x(1-x)}{a}$ as a function of x for $a=-0.1,-0.5$ and -0.9 respectively.

with $f_2(a)$ given as

$$\begin{aligned} f_2(a) &= \sqrt{1+a}\ln\left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right), && a>0 \\ &= \sqrt{1+a}\ln\left(\frac{1+\sqrt{1+a}}{1-\sqrt{1+a}}\right), && 0>a>-1 \\ &= 2\sqrt{|1+a|}\arctan\frac{1}{\sqrt{|1+a|}} && a<-1. \end{aligned}$$

C.2 The J_0 integral

Here we calculate the J_0 integral corresponding to the vacuum sunset diagram which in (3.11) was defined as

$$\lambda^2 J_0 = -\left(\lambda'\right)^2 \frac{\mu^{4\varepsilon}}{1-2\varepsilon}\frac{\Gamma(2\varepsilon)}{(4\pi)^{4-2\varepsilon}}\int_0^1 dxdy\left(\frac{y}{x(1-x)A(x,y)^2}\right)^{\varepsilon}\left(K^2(1-y)+3m^2\frac{1-y}{y}\right)$$

with $\lambda'=\lambda\mu^{2\varepsilon}$. For later convenience we insert $1=e^{-2\varepsilon\gamma}e^{+2\varepsilon\gamma}$ so that after rearranging we obtain

$$\lambda^2 J_0 = -\lambda''^2\Gamma(2\varepsilon)\frac{e^{2\varepsilon\gamma}}{1-2\varepsilon}\int_0^1 dxdy\frac{y^{\varepsilon}}{(x(1-x))^{\varepsilon}}\hat{A}^{-2\varepsilon}(x,y)\left(K^2(1-y)+3m^2y^{-1}(1-y)\right)$$

with $\hat{A}(x,y)=\frac{e^{\gamma}A(x,y)}{4\pi\mu^2}$ and $\lambda''=\frac{\lambda'}{(4\pi)^2}=\frac{\lambda}{\mu^{2\varepsilon}(4\pi)^2}$. Now we split up J_0 into two parts

$$\lambda^2 J_0 = -\lambda''^2\Gamma(2\varepsilon)\frac{e^{2\varepsilon\gamma}}{1-2\varepsilon}\left(I'_A+I'_B\right) \tag{C.2}$$

with the integrals[1] I'_A and I'_B given by

$$\begin{aligned} I'_A &= K^2\int_0^1 dxdy(1-y)\frac{\hat{A}^{-2\varepsilon}(x,y)\quad y^{\varepsilon}}{(x(1-x))^{\varepsilon}} \\ I'_B &= 3m^2\int_0^1 dxdy(1-y)\frac{\hat{A}^{-2\varepsilon}(x,y)\quad y^{-1+\varepsilon}}{(x(1-x))^{\varepsilon}}. \end{aligned}$$

[1]These correspond to the integrals $K_\mu I^\mu_A$ and $3m^2 I_B$ from (3.8) without a leading factor $-\lambda''^2\Gamma(2\varepsilon)e^{2\varepsilon\gamma}/(1-2\varepsilon)$.

The integral I'_B Concentrating on I'_B first we use the identity

$$y^{-1+\varepsilon} = \frac{1}{\varepsilon}\frac{d}{dy}y^{\varepsilon}$$

to integrate by parts[2] and obtain

$$I'_B = 3m^2\frac{1}{\varepsilon}\int_0^1 dxdy\frac{y^{\varepsilon}}{(x(1-x))^{\varepsilon}}\left(\hat{A}(x,y)\right)^{-2\varepsilon}\left(1+2\varepsilon(1-y)\frac{\partial}{\partial y}\ln\hat{A}(x,y)\right).$$

Including the leading factor $\Gamma(2\varepsilon)$ from J_0 the contribution of I'_B diverges as ε^{-2}, $\varepsilon \to 0$. In order to obtain finite terms we therefore have to expand I'_B until second order in ε. Using

$$\left(\frac{y}{(x(1-x))\hat{A}^2}\right)^{\varepsilon} \simeq 1+\varepsilon\ln\left(\frac{y}{(x(1-x))\hat{A}^2}\right)+\varepsilon^2\frac{1}{2}\ln^2\left(\frac{y}{(x(1-x))\hat{A}^2}\right)+O\left(\varepsilon^3\right).$$

I'_B can be separated into contributions of order ε^0, ε^1 and ε^2:

$$I'_B = \frac{3m^2}{\varepsilon}(I_{B,1}+\varepsilon I_{B,2}+\varepsilon^2 I_{B,3}) \tag{C.3}$$

with

$$\begin{aligned} I_{B,1} &= 1 \\ I_{B,2} &= \int_0^1 dxdy\left(\ln\frac{y}{(x(1-x))\hat{A}^2}+2(1-y)\frac{\partial}{\partial y}\ln\hat{A}\right) \\ I_{B,3} &= \int_0^1 dxdy\left(\frac{1}{2}\ln^2\frac{y}{(x(1-x))\hat{A}^2}+2(1-y)\ln\frac{y}{(x(1-x))\hat{A}^2}\frac{\partial}{\partial y}\ln\hat{A}\right). \end{aligned}$$

The integral $I_{B,2}$ can be done fairly easily using integration by parts of the expression $\frac{d}{dy}\ln\hat{A}$ and the symmetries of the integrand. It turns out to be

$$I_{B,2} = 1-2\ln\hat{m}^2$$

with $\hat{m}^2 = \frac{e^{\gamma}m^2}{4\pi\mu^2}$. The integral $I_{B,3}$ cannot be done in closed form; however, we can greatly enhance its appearance by doing some cosmetics: beginning to separate the contributions which contain $\hat{A}$ and those that do not we get

$$\begin{aligned} I_{B,3} = \int_0^1 dxdy\Big[&\ln^2\frac{y}{x(1-x)}-2\ln\frac{y}{x(1-x)}\ln\hat{A}+2\ln^2\hat{A}+ \\ &+2(1-y)\ln\frac{y}{x(1-x)}\frac{\partial}{\partial y}\ln\hat{A}-4(1-y)\ln\hat{A}\frac{\partial}{\partial y}\ln\hat{A}\Big]. \end{aligned}$$

[2]Note that since the limit $\varepsilon \to 0$ has to be done only after integration we have $y^{\varepsilon}|_{y=0} = 0$.

Note that $\ln \hat{A}\frac{\partial}{\partial y}\ln \hat{A} = \frac{1}{2}\frac{\partial}{\partial y}\ln^2 \hat{A}$. Integrating by parts cancels the other term $\ln^2 \hat{A}$ and gives a total contribution $2\ln^2 \hat{m}^2$ to $I_{B,3}$. The other terms that contain $\hat{A}$ also possess a factor $\ln \frac{y}{x(1-x)}$ that we split up as $\ln y - \ln x(1-x)$ before again integrating the term $\partial_y \ln \hat{A}$ by parts, obtaining the surface contribution $2\ln x(1-x)\ln \hat{m}^2$. With these contributions we get for $I_{B,3}$

$$\begin{aligned} I_{B,3} &= 2\ln^2 \hat{m}^2 + 2\int_0^1 dx \ln x(1-x)\ln \hat{m}^2 + \int_0^1 dxdy \frac{1}{2}\ln^2 \frac{y}{x(1-x)} \\ &\quad + \int_0^1 dxdy\left(2\ln y \frac{\partial}{\partial y}\ln \hat{A} + 2\ln \hat{A}\right). \end{aligned}$$

Using

$$\begin{aligned} \int_0^1 dx \ln x \ln(1-x) &= 2 - \frac{\pi^2}{6} \\ \int_0^1 dx \ln^2 x &= 2 \end{aligned}$$

and the symmetry $x \leftrightarrow 1-x$ we can evaluate the terms depending only on x and y, obtaining

$$I_{B,3} = 2\ln^2 \hat{m}^2 - 4\ln \hat{m}^2 + 3 - \frac{\pi^2}{6} + \int_0^1 dxdy\left(2\ln y\frac{\partial}{\partial y}\ln \hat{A} + 2\ln \hat{A}\right).$$

We can do one more integration: by noting

$$\ln \hat{A} = \ln \frac{\hat{m}^2 y}{x(1-x)} + \ln\left(1 + x(1-x)(1-y)\left(\frac{K^2 y + m^2}{m^2 y}\right)\right) \tag{C.4}$$

we can make use of the result (C.1) to integrate the second term. Comparison yields the parameter a to be

$$\frac{4m^2 y}{(1-y)\left(K^2 y + m^2\right)} = a_0 \tag{C.5}$$

where we have denoted it as a_0 in order to distinguish it from other parameters of this form. The result for the dx integration is then provided by (C.1) and some trivial part so we obtain

$$\int_0^1 dx \ln \hat{A} = \ln \hat{m}^2 + \ln y + f_2(a_0) + i(2n+1)\pi\sqrt{1+a_0}\,\Theta(a_0+1)\Theta(-a_0).$$

$I_{B,3}$ is therefore given by

$$\begin{aligned} I_{B,3} &= 2\ln^2 \hat{m}^2 + 2\ln \hat{m}^2 + 1 - \frac{\pi^2}{6} \\ &\quad + \int_0^1 dy\left(\ln y \frac{\partial}{\partial y}\left(2\ln y + 2f_2(a_0)\right) + 2f_2(a_0)\right) \\ &\quad + i(2n+1)2\pi\int_0^1 dy\sqrt{1+a_0}\,\Theta(a_0+1)\Theta(-a_0). \end{aligned}$$

Using still another integration by parts and $\ln y\,(\ln y + f_2(a_0)) \to 0$ for $y \to 1$ and $y \to 0$ respectively, (which is actually not obvious for $y \to 0$ but nevertheless correct) we obtain

$$\begin{aligned} I_{B,3} &= 2\ln^2 \hat{m}^2 + 2\ln \hat{m}^2 + 1 - \frac{\pi^2}{6} + 2\int_0^1 dy \left(-\frac{\ln y + (1-y) f_2(a_0)}{y}\right) \\ &\quad + i(2n+1)2\pi \int_0^1 dy \sqrt{1+a_0}\,\Theta(a_0+1)\Theta(-a_0). \end{aligned}$$

It should be mentioned that – despite appearance – the integrand does not diverge like $\ln\varepsilon/\varepsilon$ for $\varepsilon \to 0$ but rather like $\ln\varepsilon$ (as can be easily checked).

The integral I'_A The contribution – including the leading $\Gamma(2\varepsilon)$ – from I'_A to J_0 diverges as ε^{-1}, $\varepsilon \to 0$ so we only need to expand I'_A up to first order in ε,

$$I'_A = K^2 \left(\frac{1}{2} + \varepsilon I_{A,2}\right) \tag{C.6}$$

with

$$I_{A,2} = \int_0^1 dx dy\,(1-y) \left(\ln \frac{y}{x(1-x)\hat{A}^2}\right).$$

Most of the integrations can be done using the methods outlined for I'_B leaving us with the following result for $I_{A,2}$:

$$\begin{aligned} I_{A,2} &= +\frac{7}{4} - \ln \hat{m}^2 - 2\int_0^1 dy (1-y) f_2(a_0) \\ &\quad -2i(2n+1)\pi \int_0^1 (1-y)\sqrt{1+a_0}\,\Theta(a_0+1)\Theta(-a_0) \end{aligned}$$

with a_0 defined as in (C.5) and having used

$$\int_0^1 dy (1-y) \ln y = -\frac{3}{4}.$$

Collection of terms Using finally the expansion of the Gamma function (A.4) which – up to order ε^2 – can be written as $\Gamma(\varepsilon) = \left(\frac{1}{2\varepsilon} + \varepsilon\frac{\pi^2}{6}\right)\left(e^{-\gamma}\right)^{2\varepsilon} + \mathcal{O}(\varepsilon^2)$ we can express the leading factor for $\lambda^2 J_0$ in (C.2) as

$$\Gamma(2\varepsilon)\frac{2e^{2\varepsilon\gamma}}{1-2\varepsilon} = \frac{1}{2\varepsilon} + 1 + 2\varepsilon + \varepsilon\frac{\pi^2}{6} + \mathcal{O}(\varepsilon^2).$$

Together with (C.3) and (C.6) we therefore obtain for $\lambda^2 J_0$

$$\begin{aligned}
\lambda^2 J_0 &= -\lambda''^2 \left(\frac{1}{\varepsilon^2}\frac{3m^2}{2} + \frac{1}{\varepsilon}\left(\frac{3}{2}m^2\left(3 - 2\ln\hat{m}^2\right) + \frac{1}{4}K^2\right)\right) \\
&\quad -\lambda''^2 \left(3m^2\left(\frac{I_{B,3}}{2} + \frac{\pi^2}{6} + I_{B,2} + 2\right) + K^2\left(\frac{I_{A,2}}{2} + \frac{1}{2}\right)\right),
\end{aligned}$$

which when fully expanded reads

$$\begin{aligned}
\lambda^2 J_0 = -\lambda''^2 &\left(\frac{1}{\varepsilon^2}\frac{3m^2}{2} + \frac{1}{\varepsilon}\left(\frac{3}{2}m^2\left(3 - 2\ln\hat{m}^2\right) + \frac{1}{4}K^2\right)\right) \\
&- \lambda''^2 3m^2 \left(\ln^2\hat{m}^2 - 3\ln\hat{m}^2 + \frac{\pi^2}{12} + \frac{7}{2}\right) \\
&- \lambda''^2 K^2 \left(\frac{11}{8} - \frac{1}{2}\ln\hat{m}^2\right) + \lambda''^2 m^2 I_C(K^2/m^2) \\
&- i\lambda''^2(2n+1)\pi \int_0^1 dy(3m^2 - K^2(1-y))\sqrt{1+a_0}\,\Theta(a_0+1)\Theta(-a_0)
\end{aligned}$$

with

$$I_C(K^2/m^2) = \int_0^1 dy \left\{\frac{3\ln y + (3 + y(K^2/m^2))(1-y)f_2(a_0)}{y}\right\}. \tag{C.7}$$

Analytic continuation To obtain a physical result we have to continue the external momentum $K = (\mathbf{k}, \omega_n)$ to a physical, Minkowskian momentum $K_M(k_0, \mathbf{k})$. We do this by the special continuation $i\omega_n \to k_0 + i\eta$ that we have already presented in the introduction. The only non-trivial part for J_0 lies in the imaginary part since we have to decide on which branch of the logarithm we find ourselves to determine n. Using (C.5) we find that for $k_0 > 0$ and $k_0 < 0$ we have $n = 0$ and $n = -1$ respectively. To further determine the imaginary part, we will use the Θ functions by noting that for $K_M^2 < m^2$ we always have $a_0 \geq 0$ since $y \in [0, 1]$ (compare (C.5)). For $K_M^2 > m^2$ we have $a_0 > 0$ if $-K_M^2 y + m^2 > 0$ and $a_0 < 0$ for $-K_M^2 y + m^2 < 0$. In the second case we may ask ourselves for which $K_M^2 > m^2/y$ we have $a_0 < -1$:

$$\begin{aligned}
\frac{4m^2 y}{(1-y)\left(-K_M^2 y + m^2\right)} &< -1 \\
m^2\left(3y+1\right) &> K_M^2 y(1-y) \\
m^2\frac{3y+1}{y(1-y)} &> K_M^2.
\end{aligned}$$

Since $\frac{3y+1}{y(1-y)} > \frac{1}{y}$ we have for $K_M^2 > m^2$:

$$
\begin{aligned}
y < m^2/K_M^2 & \quad\Rightarrow\quad a_0 > 0 \\
y > m^2/K_M^2, \quad K_M^2 < m^2\frac{3y+1}{y(1-y)} & \quad\Rightarrow\quad a_0 < -1 \\
y > m^2/K_M^2, \quad K_M^2 > m^2\frac{3y+1}{y(1-y)} & \quad\Rightarrow\quad a_0 \in [-1, 0[.
\end{aligned}
$$

The minimum of $\frac{3y+1}{y(1-y)}$ for $y \in [0,1]$ is obtained for $y = \frac{1}{3}$ and is

$$
\min \frac{3y+1}{y(1-y)} = 9
$$

This in turn means that for all $K_M^2 < 9m^2$ we always have either $a_0 > 0$ or $a_0 < -1$. Therefore we obtain an imaginary part only for $K_M^2 > 9m^2$, which can be interpreted as the threshold for the production of three real particles of mass m.

The final result for $\lambda^2 J_0$ is therefore given by

$$
\lambda^2 J_0 = -\lambda''^2 \left(\frac{1}{\varepsilon^2}\frac{3m^2}{2} + \frac{1}{\varepsilon}\left(\frac{3}{2}m^2\left(3 - 2\ln\hat{m}^2\right) - \frac{1}{4}K_M^2\right)\right) + \lambda^2 J_0^{fin}
$$

where J_0^{fin} denotes the finite part of J_0 given as

$$
\begin{aligned}
\lambda^2 J_0^{fin} = & -\lambda''^2 3m^2\left(\ln^2\hat{m}^2 - 3\ln\hat{m}^2 + \frac{\pi^2}{12} + \frac{7}{2}\right) \\
& + \lambda''^2 K_M^2\left(\frac{11}{8} - \frac{1}{2}\ln\hat{m}^2\right) + \lambda''^2 m^2 I_C(-K_M^2/m^2) \\
& - i\mathrm{sgn}(k_0)\lambda''^2\pi\int_{m^2/K_M^2}^{1} dy(3m^2 + K_M^2(1-y))\sqrt{1+a_0}\times \\
& \qquad \Theta(\frac{K_M^2}{m^2} - \frac{3y+1}{y(1-y)})\Theta(K_M^2 - 9m^2) \qquad \text{(C.8)}
\end{aligned}
$$

C.3 The J_2 integral

The J_2 integral involves angular integrals that are very difficult to tackle. It has taken the author several weeks to find all tricks necessary to arrive at the final result, all of which shall be presented here in the hope that they might be useful in further calculations.

In order to calculate the integral

$$
I_2'(p,q,K_m) = \int d\Omega_p d\Omega_q \frac{(-1)}{E_{\sigma\tau}^2 - |\mathbf{k}+\mathbf{p}+\mathbf{q}|^2 - m^2} \qquad \text{(C.9)}
$$

we define our coordinate system in such a way that

$$\begin{aligned}
\mathbf{k}\cdot\mathbf{p} &= kp\cos\theta_p \\
\mathbf{k}\cdot\mathbf{q} &= kp\cos\theta_p \\
\mathbf{p}\cdot\mathbf{q} &= pq\left(\sin\theta_p\sin\theta_q\cos\left(\phi_p-\phi_q\right)+\cos\theta_p\cos\theta_q\right).
\end{aligned}$$

First integration Then one of the $d\phi$-integrations becomes trivial and the remaining averaging over angles reads

$$I_2' = 2\pi\int d(\cos\theta_p)d(\cos\theta_q)\int_{-\pi}^{+\pi} d(\phi_p-\phi_q)\frac{1}{\beta+\gamma\cos\left(\phi_p-\phi_q\right)}$$

with

$$\begin{aligned}
\beta &= -E^2_{\sigma,\tau}+m^2+k^2+p^2+q^2+2\mathbf{k}\cdot\mathbf{p}+2\mathbf{k}\cdot\mathbf{q}+2pq\cos\theta_p\cos\theta_q \\
\gamma &= 2pq\sin\theta_p\sin\theta_q.
\end{aligned}$$

For $\beta^2>\gamma^2$ the integration over $(\phi_p-\phi_q)$ is straightforward ([15], integral 337) and yields

$$\int_{-\pi}^{+\pi} d(\phi_p-\phi_q)\frac{1}{\beta+\gamma\cos\left(\phi_p-\phi_q\right)} = \frac{2\pi}{\sqrt{\beta^2-\gamma^2}}\mathrm{sgn}(\beta).$$

For $\beta^2<\gamma^2$ the integrand has a pole at $\cos\left(\phi_p-\phi_q\right)=-\beta/\gamma$. Since the prescription of how to circumvent the pole is given by the term $i\eta$ in $E_{\sigma\tau}$ there is no ambiguity concerning the integration limits and we may again use the integration formula [15] 337 to obtain

$$\mathrm{Re}\int_{-\pi}^{+\pi} d(\phi_p-\phi_q)\frac{1}{\beta+\gamma\cos\left(\phi_p-\phi_q\right)} = 0$$

for $\beta^2<\gamma^2$. The result therefore is

$$\mathrm{Re}I_2' = (2\pi)^2\int d(\cos\theta_p)d(\cos\theta_q)\frac{1}{\sqrt{\beta^2-\gamma^2}}\Theta\left(\beta^2-\gamma^2\right)\mathrm{sgn}(\beta),$$

where

$$\begin{aligned}
\beta^2-\gamma^2 &= \frac{\left(N\cos\theta_p+A\right)^2+B}{N} \\
N &= 4p^2\left|\overrightarrow{k}+\overrightarrow{q}\right|^2 \\
A &= 2p\left(k+q\cos\theta_q\right)\left(S_{\sigma\tau}+2kq\cos\theta_q\right) \\
B &= 4p^2q^2\sin^2\theta_q\left[\left(S_{\sigma\tau}+2kq\cos\theta_q\right)^2-N\right] \\
S_{\sigma\tau} &= p^2+q^2+k^2+m^2-E^2_{\sigma\tau}
\end{aligned} \tag{C.10}$$

Second integration We already have written the term $\beta^2-\gamma^2$ in a convenient form to do the $\cos\theta_p$ integration. Substituting $N\cos\theta_p + A = \chi$ it involves the integral

$$\int_{-1}^{+1} d\cos\theta_p \frac{1}{\sqrt{\beta^2-\gamma^2}}\Theta\left(\beta^2-\gamma^2\right)\operatorname{sgn}(\beta) = \int_{-N+A}^{+N+A}\frac{d\chi}{\sqrt{N}}\frac{\Theta\left(\chi^2+B\right)}{\sqrt{\chi^2+B}}\operatorname{sgn}(\beta)$$

which, depending on the sign of B can be written as

$$\begin{aligned}\int_{-N+A}^{+N+A}\frac{d\chi}{\sqrt{N}}\frac{\Theta\left(\chi^2+B\right)}{\sqrt{\chi^2+B}}\operatorname{sgn}(\beta) &= \Theta\left(B\right)\int_{-N+A}^{+N+A}\frac{d\chi}{\sqrt{N}}\frac{\operatorname{sgn}(\beta)}{\sqrt{\chi^2+B}} \\ &\quad +\Theta\left(-B\right)\int_{-N+A}^{+N+A}\frac{d\chi}{\sqrt{N}}\frac{\Theta\left(\chi^2-|B|\right)}{\sqrt{\chi^2-|B|}}\operatorname{sgn}(\beta).\end{aligned}$$

In order to do the integrals we have to decide whether the sign function $\operatorname{sgn}(\beta)$ changes for χ inside the integration limits. For the first part of the integral $B>0$ and therefore $\left(S_{\sigma\tau}+2kq\cos\theta_q\right)^2 > N$. Since $N > \left|2\mathbf{k}\cdot\mathbf{p}+2pq\cos\theta_p\cos\theta_q\right|^2$ the sign function may be written as

$$\operatorname{sgn}\left(\beta\right)\Theta\left(B\right) = \operatorname{sgn}\left(\beta,+\right)\Theta\left(B\right)$$

where we denoted

$$\operatorname{sgn}\left(S_{\sigma\tau}+2kq\cos\theta_q \pm \left(2kp+2pq\cos\theta_q\right)\right) = \operatorname{sgn}\left(\beta,\pm\right)$$

for compact notation. Note that $\operatorname{sgn}\left(\beta,\pm\right)$ does not depend on χ. The first integral then becomes elementary and is given by

$$\Theta\left(B\right)\int_{-N+A}^{+N+A}\frac{d\chi}{\sqrt{N}}\frac{\operatorname{sgn}\left(\beta\right)}{\sqrt{\chi^2+B}} = \Theta\left(B\right)\frac{1}{\sqrt{N}}\ln\frac{N+A+\sqrt{\left(N+A\right)^2+B}}{-N+A+\sqrt{\left(N-A\right)^2+B}}\operatorname{sgn}\left(\beta,+\right). \tag{C.11}$$

For later use it will be convenient to write the logarithm in the form $\ln x = \frac{1}{2}\ln x^2\frac{B}{B}$ and using

$$\frac{B}{B} = \frac{\left(\sqrt{\left(N-A\right)^2+B}+N-A\right)\left(\sqrt{\left(N-A\right)^2+B}-\left(N-A\right)\right)}{\left(\sqrt{\left(N+A\right)^2+B}+N+A\right)\left(\sqrt{\left(N+A\right)^2+B}-\left(N+A\right)\right)}$$

so that

$$\ln\frac{N+A+\sqrt{\left(N+A\right)^2+B}}{-N+A+\sqrt{\left(N-A\right)^2+B}} = \operatorname{artanh}\frac{N+A}{\sqrt{\left(N+A\right)^2+B}}+\operatorname{artanh}\frac{N-A}{\sqrt{\left(N-A\right)^2+B}}. \tag{C.12}$$

For the second part $B<0$ and things are a bit trickier. We first take a closer look at the integration limits. Since $B<0 \Leftrightarrow N>\left(S_{\sigma\tau}+2kq\cos\theta_q\right)^2$ we can prove

$$\begin{aligned} N &> |A| \\ 2p\left|\mathbf{k}+\mathbf{q}\right|^2 &> \left|\left(k+q\cos\theta_q\right)\left(S_{\sigma\tau}+2kq\cos\theta_q\right)\right| \end{aligned}$$

which is true since $|\mathbf{k}+\mathbf{q}|>\left|k+q\cos\theta_q\right|$. Therefore the χ integration goes from $-N+A<0$ to $-|B|$ and from $|B|$ to $N+A>0$. Now we consider the sign function: first note that for $\left|S_{\sigma\tau}+2kq\cos\theta_q\right|>\left|2p\left(k+q\cos\theta_q\right)\right|$ we can repeat the argument for $\Theta(B)$ and get $\operatorname{sgn}(\beta)\,\Theta(-B)=\operatorname{sgn}(\beta,+)\,\Theta(-B)$. Unfortunately for $\left|S_{\sigma\tau}+2kq\cos\theta_q\right|<\left|2p\left(k+q\cos\theta_q\right)\right|$ the sign function will change its value for some $\chi\in[-N+A;N+A]$. Since $\cos\theta_p=\frac{\chi-A}{N}$ the value of χ where $\operatorname{sgn}(\beta)$ changes occurs at $\chi=\chi_0$ with

$$\chi_0=-\frac{2pq^2\sin^2\theta_q\left(S_{\sigma\tau}+2kq\cos\theta_q\right)}{\left(k+q\cos\theta_q\right)}.$$

Fortunately it can be shown that $|\chi_0|<|B|$ and therefore – again – the sign function changes outside the integration limits (although for the present case it will differ for the regions $[-N+A;-|B|]$ and $[|B|;N+A]$, respectively). The integral for $B<0$ can be split into

$$\begin{aligned} \int_{-N+A}^{+N+A}\frac{d\chi}{\sqrt{N}}\frac{\Theta\left(\chi^2-|B|\right)}{\sqrt{\chi^2-|B|}}\operatorname{sgn}(\beta) &= \int_{|B|}^{N+A}\frac{d\chi}{\sqrt{N}}\frac{\operatorname{sgn}(\beta,+)}{\sqrt{\chi^2-|B|}}+ \\ & \int_{|B|}^{N-A}\frac{d\chi}{\sqrt{N}}\frac{\operatorname{sgn}(\beta,+)}{\sqrt{\chi^2-|B|^2}}f\left(S_{\sigma\tau}\right) \end{aligned} \tag{C.13}$$

with

$$\begin{aligned} f\left(S_{\sigma\tau}\right) &= \Theta\left(\left|S_{\sigma\tau}+2kq\cos\theta_q\right|-\left|2p\left(k+q\cos\theta_q\right)\right|\right) \\ & -\Theta\left(-\left|S_{\sigma\tau}+2kq\cos\theta_q\right|+\left|2p\left(k+q\cos\theta_q\right)\right|\right). \end{aligned}$$

Substituting $1-\frac{|B|}{\chi^2}=\psi^2$, $\psi>0$ and determining the new limits of the integral we find

$$\begin{aligned} \int_{|B|}^{N\pm A}d\chi\frac{\Theta\left(\chi^2-|B|\right)}{\sqrt{\chi^2-|B|}} &= \Theta\left(\left(N\pm A\right)^2-|B|\right)\int_0^{\sqrt{1-\frac{|B|}{(N\pm A)^2}}}d\psi\frac{1}{1-\psi^2} \\ &= \Theta\left(\left(N\pm A\right)^2-|B|\right)\operatorname{artanh}\sqrt{1-\frac{|B|}{\left(N\pm A\right)^2}}. \end{aligned}$$

Note that

$$\left(N\pm A\right)^2+B=4p^2\left(k^2+q^2+2kq\xi\right)\left(S_{\sigma\tau}+2kq\cos\theta_q\pm 2kp\pm 2pq\cos\theta_q\right)^2 \tag{C.14}$$

so that

$$\sqrt{(N \pm A)^2 + B} = \sqrt{N} \left(S_{\sigma\tau} + 2kq\cos\theta_q \pm 2kp \pm 2pq\cos\theta_q\right) sgn\left(\beta, \pm\right)$$

Because of (C.14) the $\Theta-$functions $\Theta\left((N \pm A)^2 + B\right)$ go away since the argument is always positive. Using $\mathrm{artanh}x = \frac{1}{2}\ln\frac{1+x}{1-x}$ we may furthermore write

$$\begin{aligned}
&\mathrm{artanh}\sqrt{1 + \frac{B}{(N \pm A)^2}} \\
&\quad = \mathrm{sgn}\left(\beta, \pm\right)\frac{1}{2}\ln\frac{N \pm A + \sqrt{N}\left(S_{\sigma\tau} + 2kq\cos\theta_q \pm 2kp \pm 2pq\cos\theta_q\right)}{N \pm A - \sqrt{N}\left(S_{\sigma\tau} + 2kq\cos\theta_q \pm 2kp \pm 2pq\cos\theta_q\right)} \\
&= \mathrm{sgn}\left(\beta, \pm\right)\frac{1}{2}\ln\frac{\left(\sqrt{N} + S_{\sigma\tau} + 2kq\cos\theta_q\right)\left(\sqrt{N} \pm 2kp \pm 2pq\cos\theta_q\right)}{\left(\sqrt{N} - S_{\sigma\tau} - 2kq\cos\theta_q\right)\left(\sqrt{N} \mp 2kp \mp 2pq\cos\theta_q\right)}, \quad \text{(C.15)}
\end{aligned}$$

having used (C.10). We now have all the pieces together to write down the solution to the $\cos\theta_p$ integration in a simple way:

For $B > 0 \Leftrightarrow |S_{\sigma\tau} + 2kq\cos\theta_p| > N$ we have $\mathrm{sgn}\left(\beta, +\right) = \mathrm{sgn}\left(\beta, -\right)$ and – using (C.11), (C.12) and (C.15) – we find that the sign functions and part of the logarithms cancel. The solution to the χ integration is then given by

$$\int_{-N+A}^{+N+A}\frac{d\chi}{\sqrt{N}}\frac{\Theta\left(\chi^2 + B\right)}{\sqrt{\chi^2 + B}}\mathrm{sgn}(\beta) = \frac{1}{\sqrt{N}}\frac{1}{2}\ln\left(\frac{\sqrt{N} + S_{\sigma\tau} + 2kq\cos\theta_q}{\sqrt{N} - S_{\sigma\tau} - 2kq\cos\theta_q}\right)^2. \quad \text{(C.16)}$$

For $N > |S_{\sigma\tau} + 2kq\cos\theta_q| > |2kp + 2pq\cos\theta_q|$ again $\mathrm{sgn}\left(\beta, +\right) = \mathrm{sgn}\left(\beta, -\right)$ and – using (C.13) and (C.15) – we arrive at precisely the same result (C.16).

Finally for $|S_{\sigma\tau} + 2kq\cos\theta_q| < |2kp + 2pq\cos\theta_q|$ we have $\mathrm{sgn}(\beta, -) = -\mathrm{sgn}\left(\beta, +\right)$ and – again using (C.13) and (C.15) – we see that this sign cancels the relative sign picked up in this case therefore reproducing once again the result (C.16). The equation (C.16) is therefore the general solution of the χ integration. We have

$$I_2' = (2\pi)^2\int d(\cos\theta_q)\frac{1}{\sqrt{N}}\frac{1}{2}\ln\left(\frac{\sqrt{N} + S_{\sigma\tau} + 2kq\cos\theta_q}{\sqrt{N} - S_{\sigma\tau} - 2kq\cos\theta_q}\right)^2.$$

Third integration We still have to do the final angular integration of (C.16) over $\cos\theta_q$, which – due to our effort of writing the result of the χ integration in a simple form – becomes elementary when making the substitution $N = 4p^2\left(k^2 + 2kq\cos\theta_q + q^2\right) = 4p^2u^2$, $u > 0$. Then $d\cos\theta_q = \frac{u}{kq}du$ and the remaining integral reads

$$\mathrm{Re}I_2' = (2\pi)^2\frac{1}{2kpq}\int_{|k-q|}^{k+q}du\ln\left|\frac{2pu + p^2 + u^2 + m^2 - E_{\sigma\tau}^2}{2pu - p^2 - u^2 - m^2 + E_{\sigma\tau}^2}\right| \quad \text{(C.17)}$$

which – when distinguishing between the cases $m^2 > E_{\sigma\tau}^2$ and $m^2 < E_{\sigma\tau}^2$ – becomes elementary and can be written as

$$\mathrm{Re}I_2' = (2\pi)^2 \frac{1}{2kpq} \left(f_3(E_{\sigma\tau}, p+u) + f_3(E_{\sigma\tau}, p-u)\right)\Big|_{|k-q|}^{k+q}$$

with

$$\begin{aligned} f_3(E,r) &= r \ln\left(\left|E^2 - m^2 - r^2\right|\right) + \sqrt{E^2 - m^2} \ln \frac{\left|\sqrt{E^2 - m^2} + r\right|}{\left|\sqrt{E^2 - m^2} - r\right|}, \quad E^2 > m^2 \\ &= r \ln\left(m^2 - E^2 + r^2\right) + 2\sqrt{m^2 - E^2} \arctan \frac{r}{\sqrt{m^2 - E^2}}, \quad E^2 < m^2 \end{aligned} \tag{C.18}$$

Collecting the above results we therefore obtain

$$\begin{aligned} \mathrm{Re}I_2' &= \int d\Omega_p d\Omega_q \frac{(-1)}{E_{\sigma\tau}^2 - |\mathbf{k} + \mathbf{p} + \mathbf{q}|^2 - m^2} \\ &= (2\pi)^2 \frac{1}{2kpq} \left(f_3(E_{\sigma\tau}, p+k+q) + f_3(E_{\sigma\tau}, p-k-q)\right. \\ &\quad \left. - f_3(E_{\sigma\tau}, p + |k-q|) - f_3(E_{\sigma\tau}, p - |k-q|)\right). \end{aligned} \tag{C.19}$$

Note that the function $f_3(E,r)$ corresponds to the function f_3 in [29] with the sole exception of an added absolute value in the numerator of the logarithm for $E^2 > m^2$ (which was not necessary in [29] because there $r > 0$). However, here this absolute value is in fact important since there exist values p, k, q so that $r = p - k - q + \sqrt{E^2 - m^2} < 0$.

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