Special Relativity Recap I

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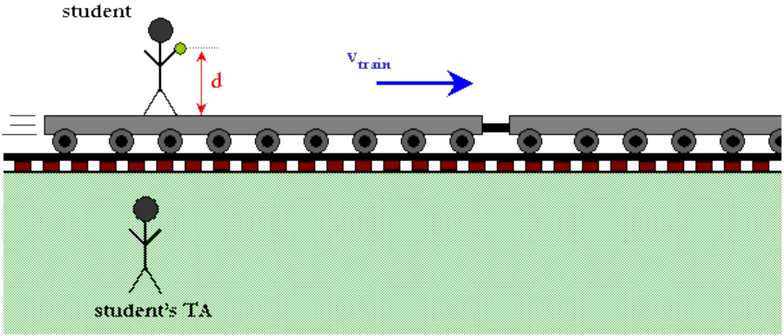
Special Relativity

- Formalized by A. Einstein in 1905
- Based on two "postulates"
- Suggests very nontrivial results (time dilation, length contraction) based entirely on theory
- Experimentally well tested

Special Relativity – Recap

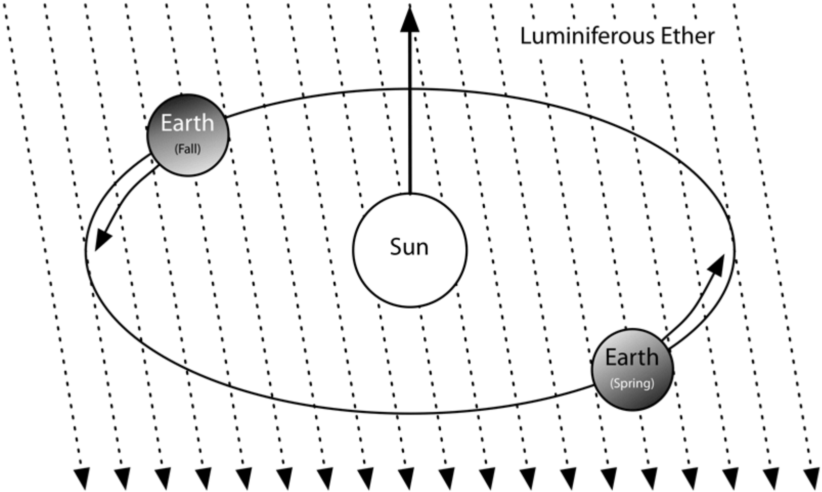
- This is only a very brief recapitulation of special relativity
- In order to get to general relativity, I will assume you already have mastery of special relativity from a prior course
- Here I will only review the basics of special relativity, mostly to set up the notation I need for general relativity

Special Relativity – Recap



Ball velocity from TA's perspective: $v = v_{\rm ball} + v_{\rm train}$

The Ether



Idea: measure earth's velocity relative to ether

The Michelson-Moreley Experiment

- 1887 – Michelson and Moreley set up to measure earth's speed relative to ether
- They used very precise tool – a light interferometer
- The experiment failed: they could not detect any velocity at all
- It was a shock to the physics community

Albert Einstein

Einstein 1893

- Born March 14, 1879 in Ulm, Germany
- Math Prodigy
- Got his teaching degree from ETH Zurich in 1900
- Didn't find work as a teacher; took job at patent office in Bern
- Published 4 ground-breaking papers in 1905; got PhD same year
- Emigrated to US in 1933

Einstein's Postulates of Special Relativity

- The speed of light is constant
- Laws of nature are independent from translational motion and rotations
    - No preferred frame (no ether)
    - Naturally explains failure of Michelson/Moreley experiment
- Provides formal framework for modern high energy physics
- Important consequences: time delay, length contraction, space and time must be treated together (not separately)

4-vectors

- We call (usual) vectors in space three-vectors, e.g. $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- It's common to use index notation for vectors, e.g. x_i, with $i = 1, 2, 3$:

$$x_1 = x\,, \quad x_2 = y\,, \quad x_3 = z\,.$$

- Treating space and time as one entity, we are lead to consider 4-vectors

$$x^\mu = \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix}\,,$$

where $c = 299792458$ m/s is the speed of light in vacuum.
- We use greek indices for 4-vectors, e.g. $\mu = 0, 1, 2, 3$

Lorentz Transformations

- Consider two different inertial frames, with coordinates x^μ, x'^μ
- Assume that at $t = t' = 0$, the origin of the two coordinate systems coincide
- Let the two inertial frames be in constant motion wrt each other along the z-direction with velocity v
- As a consequence of these, we must have

$$x' = x\,, \quad y' = y\,. \tag{1.1}$$

Lorentz Transformations

- Distance L traveled by light in both frames obeys

$$L^2 = x^2 + y^2 + z^2 = c^2 t^2\,, \quad L'^2 = x'^2 + y'^2 + z'^2 = c'^2 t'^2\,. \quad (1.2)$$

- Now use postulates of special relativity that speed of light is constant in any frame: $c' = c$
- Equating distance traveled by light and using (1.1) gives

$$c^2 t^2 - z^2 = c^2 t'^2 - z'^2\,. \quad (1.3)$$

- Constant motion suggestes **linear relations**

$$z' = A(z - vt)\,, \quad t' = Bt + Dz\,, \quad (1.4)$$

with A, B, D constants.

Lorentz Transformations

- We have three unknowns A, B, D and three equations (1.3), (1.4)
- Can solve for A, B, D (homework!)
- Find the transformation laws

$$z' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(z - vt)\,, \quad t' = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(t - \frac{v}{c^2}z\right)\,. \tag{1.5}$$

- It is customary to introduce abbreviations

$$\beta \equiv \frac{v}{c}\,, \quad \gamma \equiv \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1.6}$$

so that

$$z' = \gamma(z - \beta c\, t)\,, \quad c\, t' = \gamma(c\, t - \beta z)\,. \tag{1.7}$$

Lorentz Transformations

- The transformation rules (1.3), (1.7) are examples of so-called Lorentz transformations
- We can represent Lorentz transformations in matrix form

$$\begin{pmatrix} c\,t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c\,t \\ x \\ y \\ z \end{pmatrix} \qquad (1.8)$$

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- We call $\Lambda^\mu{}_\nu$ the *Lorentz transformation matrix*

Lorentz Transformations

- In 4-vector notation, this becomes

$$x'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu x^\nu \,. \tag{1.11}$$

- Note that the matrix multiplication with the 4-vector x^ν is just a sum over one of the indices (exercise!)
- Eq. (1.11) is an extremely compact notation of (1.10)
- We want to be even more efficient in writing and omit the sum symbol from (1.11) and use the so-called **Einstein sum-convention: indices that are repeated twice are summed over**
- As a consequence, we get for a Lorentz transformation

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \,. \tag{1.12}$$