Special Relativity Recap II

paul.romatschke@colorado.edu

Spring 2021

Special Relativity Recap – continued

- Last lecture we started a review of special relativity
- In particular, we reviewed index notation
- We will continue this review of special relativity in this lecture

More on index notation

- In the last lecture, we introduced index notation
- E.g. for ordinary 3-vectors, we have $x_i = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$
- We can write products of vectors using index notation as

$$u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \tag{2.1}$$

- We can represent matrices using two indices, e.g.

$$a_{ij} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{2.2}$$

More on index notation

- Changing the order of the indices of a rank 2 matrix corresponds to transposition
- For instance,

$$\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad \mathbf{a}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \tag{2.3}$$

so that

$$a_{ij}^T = a_{ji} \,. \tag{2.4}$$

More on index notation

- Products of matrices and vectors are given by

$$\mathbf{a}\cdot\mathbf{v} = a_{ij}v_j = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{pmatrix} \tag{2.5}$$

- Note that the order of the product is important, e.g. in general

$$\mathbf{a}\cdot\mathbf{v} \neq \mathbf{v}\cdot\mathbf{a}\,. \tag{2.6}$$

- In index notation, the order of the product is specified uniquely by the index that appears twice
- In index notation, we have

$$b_i = a_{ij}v_j = v_j a_{ij}\,, \quad c_i = v_j a_{ji} = a_{ji}v_j = a^T_{ij}v_j \tag{2.7}$$

More on index notation

- In index notation, the order of the objects does not matter, e.g. $a_{ij}v_j = v_j a_{ij}$
- However, the order of the *indices* matters, e.g. $a_{ij}v_j \neq a_{ji}v_j$
- Indices appearing twice *next to each other* can be interpreted as products, e.g.

$$a_{ij}v_j = \mathbf{a}\cdot\mathbf{v}\,. \tag{2.8}$$

- Warning: Indices appearing twice *somewhere* do in general *not* correspond to vector products, e.g.

$$a_{ij}v_i \neq \mathbf{a}\cdot\mathbf{v} \tag{2.9}$$

- However, you may recognize such expressions as products by re-ordering the symbols, e.g.

$$a_{ij}v_i = v_i a_{ij} = \mathbf{v}\cdot\mathbf{a}\,. \tag{2.10}$$

Norm of a Vector

- Norm of 3-vectors

$$|\vec{x}|^2 = x^2 + y^2 + z^2 \tag{2.11}$$

- Can write this in index notation

$$|\vec{x}|^2 = x^i x^i = x^i \delta_{ij} x^j \tag{2.12}$$

- Here δ_{ij} is the Kronecker symbol:

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.13}$$

Metric Tensor

- We can view the Kronecker symbol as the metric tensor of Euclidean space
- There is a formal definition for the metric tensor
- In practice, however, all you need to know is that the metric tensor is how you calculate a dot product of vectors
- In 3-dimensional Euclidean space

$$\mathbf{u} \cdot \mathbf{v} = u^i v^j \delta_{ij} \tag{2.14}$$

Transformation of the Metric Tensor

- The length of a vector is expressed through the metric tensor
- For a 3-vector, writing the metric tensor as g_{ij} we have

$$L^2 = |\mathbf{x}|^2 = x_i g_{ij} x_j \tag{2.15}$$

- For Euclidean 3-space, we know that $g_{ij} = \delta_{ij}$.
- Rotations are encoded through a rotation matrix R_{ij} such that

$$x_i = R_{ij} x'_j \tag{2.16}$$

- In the rotated coordinate system, assuming a metric tensor g'_{ij}, we have

$$L'^2 = x'_i x'_i = x'_i g'_{ij} x'_j \tag{2.17}$$

Transformation of the Metric Tensor

- We can use the transformation rule $x_i = R_{ij} x'_j$ to re-write the length of the vector

$$L^2 = x_i g_{ij} x_j = R_{im} x'_m g_{ij} R_{jn} x'_n = x'_m R_{im} g_{ij} R_{jn} x'_n \equiv x'_m g'_{mn} x'_n = L'^2 \tag{2.18}$$

- We therefore get a transformation rule for the metric tensor:

$$g'_{mn} = R_{im} g_{ij} R_{jn} = R_{im} R_{jn} g_{ij} \,. \tag{2.19}$$

- For rotations, it is easy to show that (homework!)

$$g_{ij} = g'_{ij} = \delta_{ij} \,. \tag{2.20}$$

- **The Euclidean metric is invariant under rotations**
- The length of a 3-vector is invariant under rotations of the coordinate system $L^2 = L'^2$