Special Relativity Recap III

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Special Relativity Recap – continued

- Last lectures we started a review of special relativity
- We considered the transformation of the Euclidean metric tensor under rotations
- We will continue this review of special relativity in this lecture

Euclidean Space

- Last lecture, we considered rotations for a 3-vector given by $x_i = R_{ij} x'_j$
- We found the transformation rule for the Euclidean metric tensor $g_{ij} = \delta_{ij}$ under rotations:

$$g'_{ij} = R_{im} R_{in} g_{mn} \tag{3.1}$$

- Let us now generalize this discussion from rotations to transformations in special relativity

Minkowski spacetime

- Consider two coordinate systems with coordinates x^μ, x'^μ, moving wrt each other at a constant speed v in the z-direction
- According to the postulates of special relativity, the speed of light is constant in all inertial frames
- This means the distance traveled by light in two coordinate systems is the same, e.g.

$$0 = x^2 + y^2 + z^2 - c^2 t^2 = 0 = -c^2 t'^2 + x'^2 + y'^2 + z'^2 \,. \qquad (3.2)$$

- As a consequence

$$s^2 = |\vec{x}|^2 - c^2 t^2 = s'^2 = |\vec{x}'|^2 - c^2 t'^2 \qquad (3.3)$$

Minkowski spacetime

- We want to minimize writing, so use *natural units* where $c = 1$ from now on
- We can *formally* consider s^2 to be a generalization of the length of a vector
- We call it the norm of a 4-vector, and write

$$s^2 = |\vec{x}|^2 - t^2 = x^\mu x_\mu \,, \quad x^\mu = \begin{pmatrix} t \\ \vec{x} \end{pmatrix} . \tag{3.4}$$

- It's not quite a norm, because of the minus sign in front of t
- In order to get the minus sign right, we demand

$$x_\mu = \begin{pmatrix} -t \\ \vec{x} \end{pmatrix} . \tag{3.5}$$

Minkowski metric

- If there is a norm, we can associate a metric tensor
- Writing

$$s^2 = x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu = |\vec{x}|^2 - t^2\,, \tag{3.6}$$

we find that the metric tensor for *space-time* must be given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \tag{3.7}$$

- We call this metric tensor the *Minkowski metric*
- Note that the Minkowski metric is very similar to the Kronecker symbol – except for the minus sign!

Minkowski metric

- We have defined

$$x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu \,, \tag{3.8}$$

so therefore $x_\mu = g_{\mu\nu} x^\nu$

- **Because of the minus sign in $g_{\mu\nu}$, the position of the index (up, down) is important!** In particular

$$x^\mu \neq x_\mu \tag{3.9}$$

- To lower an index in relativity, we always use the metric tensor, e.g.

$$a_\mu = g_{\mu\nu} a^\nu \,, \quad a_\mu{}^{\nu\rho} = g_{\mu\alpha} a^{\alpha\nu\rho} \,, \quad \text{etc.} \tag{3.10}$$

Minkowski metric

- Writing
$$x^\mu x_\mu = g^{\mu\alpha} x_\alpha x_\mu = |\vec{x}|^2 - t^2\,, \qquad (3.11)$$
and comparing coefficients, we can also get the metric tensor with upstairs indices $g^{\mu\alpha}$
- We find
$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,. \qquad (3.12)$$
- As a consequence, we can also use the metric tensor to **raise** indices whenever need
- A useful relation that is easy to prove (exercise!) is
$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu\,, \qquad (3.13)$$
where we use δ^μ_ν to denote the 4-dimensional Kronecker symbol

Transformations of the Metric Tensor

- In Eq. (3.3), we argued that the norm of 4-vector is invariant in special relativity
- Let us prove this now more generally
- Consider the "line element"

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2 \tag{3.14}$$

where you may think of dx as the infinitesimal distance in x between two points

- Now write

$$ds^2 = dx^\mu dx_\mu = dx^\mu dx^\nu g_{\mu\nu}\,, \tag{3.15}$$

with $g_{\mu\nu}$ the Minkowski metric

Transformations of the Metric Tensor

- In lecture 1, we recalled that special relativity transformations are generated by the Lorentz transformation matrix $\Lambda^{\mu}{}_{\nu}$, cf. Eq. (1.10)
- In index notation,

$$dx^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu\prime} \,, \tag{3.16}$$

- Therefore

$$ds^2 = dx^{\mu} dx_{\mu} = dx^{\mu} dx^{\nu} g_{\mu\nu} = dx^{\prime\alpha} dx^{\prime\beta} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} g_{\mu\nu} \equiv dx^{\prime\alpha} dx^{\prime\beta} g^{\prime}_{\alpha\beta} \tag{3.17}$$

- We find the transformation rule for the metric tensor in special relativity:

$$g^{\prime}_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} g_{\mu\nu} \,. \tag{3.18}$$

Transformations of the Metric Tensor

- Let us do an example for the transformed metric tensor, using (1.10)

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{3.19}$$

- Writing (3.18) in matrix multiplication form we get

$$g'_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}g_{\mu\nu} = \Lambda^{\mu}{}_{\alpha}g_{\mu\nu}\Lambda^{\nu}{}_{\beta} = \left(\Lambda^{T}\right)_{\alpha}{}^{\mu} g_{\mu\nu}\Lambda^{\nu}{}_{\beta} = \mathbf{\Lambda}^{T}\cdot\mathbf{g}\cdot\mathbf{\Lambda} \tag{3.20}$$

- We have

$$\mathbf{g}\cdot\mathbf{\Lambda} = \begin{pmatrix} -\gamma & \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.21}$$

Transformations of the Metric Tensor

- Next calculate

$$\mathbf{\Lambda}^T \mathbf{g} \cdot \mathbf{\Lambda} = \begin{pmatrix} -\gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & \gamma^2(1-\beta^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.22}$$

- Using $\gamma^2 = \frac{1}{1-\beta^2}$ this directly leads to

$$g'_{\alpha\beta} = g_{\alpha\beta}\,. \tag{3.23}$$

- **The Minkowski metric is invariant under Lorentz transformations**
- As a consequence, we have

$$ds'^2 = ds^2\,. \tag{3.24}$$