Special Relativity Recap IV

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Special Relativity Recap – continued

- Last lectures we started a review of special relativity
- We showed that the line element is invariant under Lorentz transformations
- We will continue the review of special relativity in this lecture

Coordinate Transformations in Special Relativity

Some nomenclature:

- Vectors: Things that carry one index, e.g. A^μ
- Scalars: Things that don't carry any index, e.g. $A^\mu A_\mu$
- Tensors: Things that carry more than one index
- E.g. Rank 2 tensor $g_{\mu\nu}$ has 2 indices
- Rank 4 tensor $R_{\mu\nu\alpha\beta}$ has 4 indices
- Etc.

Vectors

Properties of Vectors:

- For any 4-vector A^μ, can write down Lorentz transformation as

$$A^\mu \to A'^\mu = \Lambda^\mu{}_\nu A^\nu\,; \quad A_\mu \to A'_\mu = \Lambda_\mu{}^\nu A_\nu\,. \tag{4.1}$$

- Example for vectors: coordinate difference $dx^\mu = \begin{pmatrix} dt \\ d\vec{x} \end{pmatrix}$
- Another example: 4-gradient: $\nabla_\mu = \frac{\partial}{\partial x^\mu}$
- Another example: 4-momentum $P^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$

Scalars

Properties of Scalars:

- Any Lorentz scalar is **invariant** under Lorentz transformations
- For instance, under $dx^\mu \to \Lambda^\mu{}_\nu dx'^\nu$ we have

$$ds^2 = g_{\mu\nu}\Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta dx'^\alpha dx'^\beta = g_{\alpha\beta} dx'^\alpha dx'^\beta = ds'^2 \,. \qquad (4.2)$$

- Importantly, we can build scalars out of vectors, e.g.

$$P^\mu P_\mu = -E^2 + \vec{p}^2 \qquad (4.3)$$

is a Lorentz scalar, and hence $P'^\mu P'_\mu = P^\mu P_\mu$

Invariant Mass

- The object $P^\mu P_\mu$ is known as (minus) the **invariant mass square**
- To see why, use the fact that it is invariant under Lorentz transformations; so we can pick a convenient Lorentz frame in which to evaluate it
- Let's pick the rest frame: $\vec{p} = 0$, so that $E = m$
- We find

$$P^\mu P_\mu = -m^2 \tag{4.4}$$

  and is **the same** (invariant) in any Lorentz frame

Tensors

Properties of tensors:

- For the Lorentz transformation of a tensor, put a transformation matrix Λ for every index
- For instance, a rank 2 tensor transforms as

$$g_{\mu\nu} \to g'_{\mu\nu} = \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta g_{\alpha\beta}\,; \quad g^{\mu\nu} \to g'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta} \tag{4.5}$$

- Note: the metric tensor $g_{\mu\nu}$ happens to be unchanged (invariant) under Lorentz transformations. **This is not the case for general tensors!!!!**
- For instance, the Maxwell field strength tensor $F^{\mu\nu}$ or the energy-momentum tensor $T^{\mu\nu}$ do transform non-trivially under Lorentz transformations

Tensors

Properties of tensors:

- Higher-rank tensor transform the same way as 2-tensors
- For instance, the rank 3 tensor $A^{\alpha\beta\gamma}$ transforms as

$$A^{\alpha\beta\gamma} \rightarrow A'^{\alpha\beta\gamma} = \Lambda^{\alpha}{}_{\rho}\Lambda^{\beta}{}_{\sigma}\Lambda^{\gamma}{}_{\tau}A^{\rho\sigma\tau} \,. \tag{4.6}$$

A note of caution

- Beware of objects that **look like**, but are no vectors
- E.g. an object A^μ that carries an index, but does not transform as a vector,

$$A'^\mu \neq \Lambda^\mu{}_\nu A^\nu \tag{4.7}$$

- (Admittedly stupid) example: the object $A^\mu = \begin{pmatrix} E \\ \vec{x} \end{pmatrix}$
- Note: this is less an issue for special relativity than for general relativity, but you should be aware that not every object with 4 entries is a Lorentz 4-vector