Classical Lagrangian Mechanics

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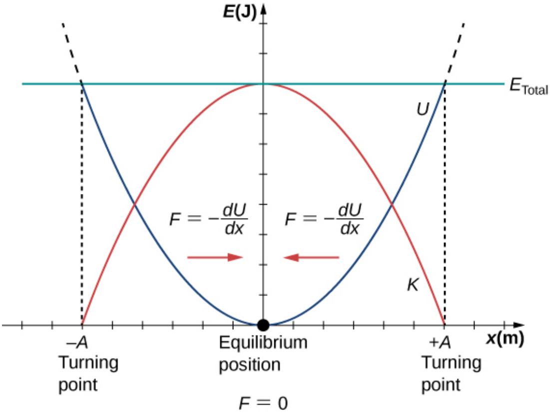
Outline

- The last lectures contained a brief review of SR concepts relevant for GR
- In addition to SR, we also need to review classical field theory
- To this end, we start by reviewing classical Lagrangian mechanics in this lecture

Point Particle in a Potential



- Point particle in potential $U(x)$
- Force on particle is $ma = F = -\frac{dU}{dx}$
- Writing $\dot{x} = \frac{dx}{dt} = v$, the **equations of motion** become

$$m\ddot{x} = -\frac{dU}{dx}\,. \qquad (6.1)$$

Lagrangian Formulation

- The **Lagrangian** L is defined as kinetic energy minus potential energy
- For the problem at hand, it is given by

$$L = \frac{1}{2}mv^2 - U(x) = \frac{1}{2}m\dot{x}^2 - U(x) \equiv L(x,\dot{x})\,. \tag{6.2}$$

- From the Lagrangian, we can defined the **action** S for the system

$$S = \int dt L\,. \tag{6.3}$$

- Claim: the minimum of S under coordinate changes corresponds to the equations of motion

Extremum of the Action

- Extremum means first order variation has to vanish:

$$\begin{aligned} 0 \stackrel{!}{=} \delta S &= \int dt \delta L(x,\dot{x}) = \int dt \left[\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}\right]\,, \\ &= \int dt \left[\frac{\partial L}{\partial x}\delta x + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right) - \delta x \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]\,, \\ &= \int dt \delta x \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right] + \text{boundary terms}\,, \end{aligned} \tag{6.4}$$

- Since any variation $\delta x(t)$ is allowed, the expression in brackets must vanish identically
- This leads to the Lagrangian equations of motion

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 = -\frac{dU}{dx} - m\ddot{x}\,. \tag{6.5}$$

Invariance under Transformations

- What if S is **invariant** under coordinate changes ?
- Consider S with boundary terms:

$$S = \int_0^\tau dt L(x, \dot{x})\,. \tag{6.6}$$

- Variation of the action leads to

$$\delta S = \int_0^\tau dt \left[\delta x \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right)\right] \tag{6.7}$$

- First term vanishes for all δx (EoM)
- Invariance for **some** δx means

$$0 = \delta S = \int_0^\tau dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\delta x\right)\,. \tag{6.8}$$

Invariance under Transformations

- Example: let's consider invariance of S for **constant** δx ("translations")
- We get

$$0 = \delta S = \delta x \int_0^{\tau} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) . \tag{6.9}$$

- For arbitrary τ, the only way for $\delta S = 0$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 , \quad \text{or} \quad \frac{\partial L}{\partial \dot{x}} = \text{const.} \tag{6.10}$$

- We get **conservation law**

Invariance under Transformations

- The invariance of S under **particular** coordinate changes (e.g. translations) gives a **particular** conserved quantity
- Let's do an example
- Consider a **free** particle Lagrangian (6.2) with $U(x) = 0$
- (Note: if $U(x) \neq 0$, we do not have invariance because $U(x + \delta x) \neq U(x)$)
- Based on (6.10) we expect this symmetry to give us a conserved quantity

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}\,. \tag{6.11}$$

- We can recognize the conserved quantity as the particle momentum

Noether Theorem

Emmy Noether (1882-1935)

- **Noether Theorem**: Every symmetry corresponds to a conserved quantity (1918)
- Holds true also on the quantum level; outsize importance in modern physics
- Trivia: Noether second female to hold PhD in math (1907, advisor Gordan); tried to get habilitation in Göttingen (supporters: Hilbert, Klein), but Prussian minister refused; taught as Hilbert's assistant