Classical Field Theory

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Spring 2021

Outline

- We reviewed classical Lagrangian mechanics for a point particle in the last lecture
- In this lecture, we want to generalize this treatment to classical **field theory**

Relativistic Classical Field Theory

- Lagrangian mechanics is for a point particle, which is localized at a point
- Field theory is for a field that need not be localized, but can extend to all of space
- A real-world example for a field is the temperature field $T(t, \vec{x})$, which gives the temperature T at a position $\vec{x}$ and time t. The temperature is an example for a (non-relativistic) scalar field
- An example for a (non-relativistic) vector field would be the wind field $\vec{v}(t, \vec{x})$, which depends on position $\vec{x}$ and time t but has three components instead of just one
- In the following we restrict ourselves to field theories that are **relativistic** (though it is perfectly possible to consider non-relativistic field theories)

Field Theories

- For the 1D point particle, we had a single coordinate as a function of time, e.g. $x(t)$
- Now consider the point particle in two dimensions. Using index notation with $i = 1, 2$, it has coordinates $x_1(t)$ and $x_2(t)$
- Now consider **many** dimensions, $i = 1, 2, 3, 4, \ldots, N$ with $N \gg 1$; we can write the particle's coordinates as $x_i(t)$ or alternatively as $\phi(t, x_i) = x_i(t)$; taking the **continuum limit** $N \to \infty$ then gives $\lim_{N\to\infty} \phi(t, x_i) \to \phi(t, x)$, a 1+1-dimensional **scalar field**
- We can generalize this to higher-dimensional fields, e.g. through $\phi(t, \mathbf{x})$, in particular 4-dimensional field $\phi(x^\mu)$; or simply denote as $\phi(x)$

Relativistic Classical Field Theory

- A relativistic field theory has the property that its classical action S is invariant under the symmetry group of special relativity (Lorentz transformations plus translations)
- In special relativity, time and space are not independent, so writing the action as $S = \int dtL$ is not a good starting point since the volume element dt does not transform properly
- What does transform properly under the symmetries of special relativity is the volume element of space-time $dtd^3x \equiv d^4x$ such that the action for a classical field theory can be written as

$$S = \int d^4x\mathcal{L}\,, \tag{7.1}$$

where $\mathcal{L}$ is the Lagrangian **density**.

Relativistic Classical Field Theory

- In order to set up a proper relativistic field theory, we have to declare the **content** of the field theory
- Do we have scalar fields (like temperature), vector fields (like wind) or tensor fields?
- Arguably the simplest case is to start with a scalar field, which we denote by $\phi(x)$, and which corresponds to a single degree of freedom at every point in space-time x^μ
- Given the field content, we can construct a Lagrangian density $\mathcal{L}$ from $\phi(x)$ and its derivatives
- At this point, the only rule is that $\mathcal{L}$ must be a scalar under special relativity transformations

Relativistic Classical Field Theory

- If $\mathcal{L}$ contains combinations of $\phi(x)$ and $\partial_\mu\phi(x)$, allowed terms are

$$\mathcal{L} \propto \phi^2(x), \ln\phi(x), e^{-\phi(x)}, \partial_\mu\phi(x)\partial^\mu\phi(x), \partial_\mu\partial_\nu\partial_\rho\partial^\mu\partial^\nu\partial^\rho\phi \ldots \quad (7.2)$$

- On the other hand, disallowed terms would be

$$\mathcal{L} \propto \partial_\mu\phi(x), \partial_\mu\partial^\mu\phi\partial^\rho\phi \ldots \quad (7.3)$$

- To simplify the discussion, we will first limit the discussion to theories with Lagrangians of the form

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - V(\phi)\,, \quad (7.4)$$

where $V(\phi)$ can be arbitrary at this point

Relativistic Classical Field Theory

- For Lagrangian densities of the form (7.4), the action is

$$S = -\int d^4x \left[\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + V(\phi)\right] \tag{7.5}$$

- Demanding that $0 \stackrel{!}{=} \delta S$, one finds the equations of motion for the field ϕ as

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = 0 = -\frac{dV(\phi)}{d\phi} - \partial_\mu\partial^\mu\phi\,. \tag{7.6}$$

- In the following, I sometimes denote the operator

$$\partial_\mu\partial^\mu \equiv \Box\,, \tag{7.7}$$

which is called "Quabla" operator in a wordplay on $\nabla = \partial_i\partial_i$ ("nabla")

The Complex Scalar Field

- We can consider more complicated field theories
- For instance, consider the case when $\phi(x)$ is **complex**
- We can again construct a field theory be considering a real action S built out of Lorentz-invariant building blocks
- For instance, with ϕ^* denoting the complex conjugate of ϕ, we write

$$S = -\int d^4x \left[\partial_\mu \phi \partial^\mu \phi^* + V(\phi\phi^*)\right] , \tag{7.8}$$

  which is real and Lorentz-invariant
- The classical potential V is arbitrary, but it is particular convenient to treat polynomial potentials, e.g.

$$V = m^2 \phi\phi^* + 4\lambda \left(\phi\phi^*\right)^2 , \tag{7.9}$$

  with λ a **coupling constant**

The Complex Scalar Field

- Note that in addition to invariance under Lorentz transformations, the action (7.8) has an additional symmetry
- That is, S is invariant under the transformation

$$\phi(x) \to e^{i\alpha}\phi(x)\,, \quad \left(\phi^*(x) \to e^{-i\alpha}\phi^*(x)\right)\,, \tag{7.10}$$

  with arbitrary (but constant) α
- This is **not a coordinate transformation** (but a so-called gauge-transformation)
- Nevertheless, it has physical implications on the symmetries of the field theory